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A study of penetration at the bottom of a stellar convective envelope and its scaling relationships

Harinder P. Singh[1,2,3], **Ian W. Roxburgh**[2], **and Kwing L. Chan**[1]

[1] Department of Mathematics, The Hong Kong University of Science & Technology, Kowloon, Hong Kong
[2] Astronomy Unit, Queen Mary & Westfield College, University of London, Mile End Road, E1 4NS London, UK
[3] Department of Physics, Sri Venkateswara College, University of Delhi, 110 021 New Delhi, India

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Abstract. A number of scaling relationships have been proposed by several authors relating the penetration depth (Δ_d) at the bottom of a convective region to the velocity of the penetrating motions and the input flux (F_b). These may be expressed as $\Delta_d \propto V_{zo}^{3/2}$ for the case where the radiative conductivity varies smoothly from the unstable to the stable zone; V_{zo} being the vertical velocity at the bottom of the convection zone. When the conductivity varies stepwise from one zone to another, it has been suggested that $\Delta_d \propto (V_{zo}^3/F_b)$ for adiabatic penetration and $\Delta_d \propto F_b^{1/2}$ for non-adiabatic penetration. In this paper, we study the general behaviour of motions penetrating into the stable region at the bottom of a convective envelope by numerically solving the full set of Navier-Stokes equations in three dimensions. We compute a series of models which allow us to examine the scaling relationships between the penetration distance, the input flux and the vertical velocity.

Key words: convection – hydrodynamics – stars: interiors

1. Introduction

A large number of theoretical investigations have been made in the last thirty years to evaluate convective penetration or overshoot in stellar interiors. Because of the complexity of the involved physical processes (see Roxburgh 1997), there has neither been a completely satisfactory treatment of the issue, nor is there a consensus on the magnitude of the overshoot, be it from a convective envelope inwards or a convective core outwards. Even for the best studied case of the Sun, for which high quality oscillation data are available, helioseismic inversion techniques give differing upper limits on the size of overshoot at the bottom of the solar convection zone (Roxburgh 1996). On the other hand, two and three-dimensional simulations of compressible convection, with a view to study overshoot, have recently been attempted (Roxburgh and Simmons 1993; Hurlburt et al. 1994; Singh et al. 1994, 1995, 1996; see also Nordlund & Stein 1996; Muthsam et al. 1995). Compared to convection zone of the sun, the convection zones in these simulations are shallower with low density contrasts. As $F/\rho \propto {V_z}^3$ in a convection zone, where F is the total flux, such configurations generate comparatively large flow velocities and hence tend to over-estimate the extent of penetration or overshoot. Neglect of rotational effects could also be responsible. Since we are still far away from having the desired computing power that will allow us to simulate the solar convection zone, the remedy is to find some scaling relationships between physical quantities of interest which could then be applied to the Sun.

Numerical simulations of turbulent compressible convection have been attempted by various groups. The first attempt to find scaling relationships within a stellar-type convection zone was made by Chan & Sofia (1989). They found scaling relationships between the rms fluctuations of pressure, density, specific entropy, temperature, and velocity. These scaling relationships have been successfully employed to compute equilibrium models of the Sun and α Centauri by replacing the mixing length theory of convection (Lydon et al. 1992, 1993). The relations were further examined by Singh & Chan (1993) and Chan & Sofia (1996).

Simulations of penetrative convection have been performed in two dimensions with different objectives by Roxburgh & Simmons (1993), Hurlburt et al. (1994), and Freytag et al. (1996). Hurlburt et al. gave scaling laws relating the penetration distance with the relative stability of the unstable to the stable zone. Three- dimensional simulations have been performed to study the behaviour of penetrative convection above and below a convection zone by Singh et al. (1994, 1995). A number of models were computed in each case and it was observed that the penetration distance above a convection zone was proportional to the maximum vertical velocity of the convective motions and this distance increased with the factor F_b/ρ_{czt}, where F_b denotes the input flux and ρ_{czt} the density at the top of the convection zone. The penetration distance below a convection zone was found to decrease as the stability of the lower stable region was increased, a behaviour which was also observed in the two-dimensional simulations of Hurlburt et al.

Of particular interest, in the context of the present study, are the calculations of Schmitt et al. (1984), Zahn (1991) and Hurlburt et al. (1994). Schmitt et al. adopted a semitheoretical approach, derived originally for the Earth's atmosphere, to study

Send offprint requests to: H.P. Singh, (hps@ttdsvc.ernet.in)

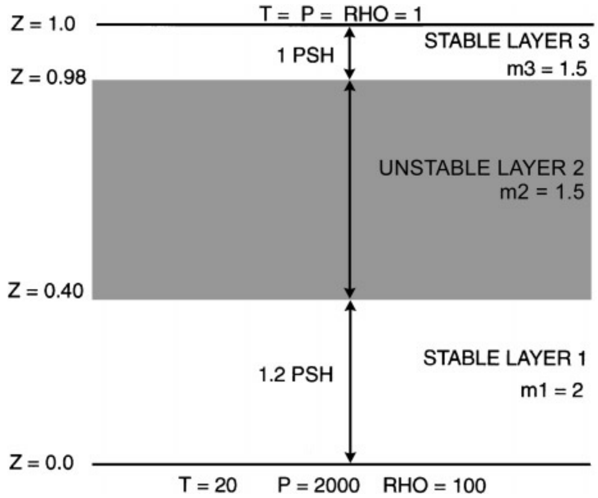

Fig. 1. Some geometrical and physical parameters of the configuration

convective penetration in stars. They considered the equations of motions for buoyant plumes and applied them to the penetration problem in the Sun. The equations were solved numerically to obtain a simple scaling relation for penetration in terms of the initial vertical velocity V_{zo} (velocity at the bottom of the convection zone) and a filling factor f (the fraction of the area occupied by the plumes) as $V_{zo}^{3/2}\,f^{1/2}$. If initial velocities predicted by the mixing length theory are used, their results predict an almost adiabatically stratified region, a few-tenths of a pressure scale height in extent, below the solar convection zone. Zahn (1991) treated the problem analytically, basing it on scaling arguments and explained why the depth of the nearly adiabatic penetration region should scale as $V_{zo}^{3/2}$ provided that the radiative conductivity varies smoothly from the unstable to the stable zone. Hurlburt et al. (1994) performed two dimensional numerical simulations of convective penetration and studied a number of cases by changing the polytropic index and hence the relative stability of the lower stable layer as compared to the unstable zone. It was shown analytically and confirmed by simulations that for nearly adiabatic penetration, $\Delta_d \propto (V_{zo}^3/F_b)$ while for non-adiabatic penetration, $\Delta_d \propto K^{1/2} \propto F_b^{1/2}$ where K is the piece-wise radiative conductivity.

In this paper we study the behaviour of penetrative convection below a deep stellar- type convective envelope by computing a series of models with various input flux values. We also examine the scaling relationships between penetration distance Δ_d, input flux F_b and the vertical velocity V_z of the flows as given by Hurlburt et al. (1994). In the next section, we provide essential features of our models and the parameters of the simulation. In Sect. 3, we report detailed results of our computations in terms of time and space-averaged quantities. Important conclusions based on the results are presented in Sect. 4.

2. Model parameters

A detailed discussion of the input physics and the numerical parameters of the simulation has been given in Singh et al. (1995) (hereafter referred to as SRC95). We provide here some necessary details of the present set of calculations.

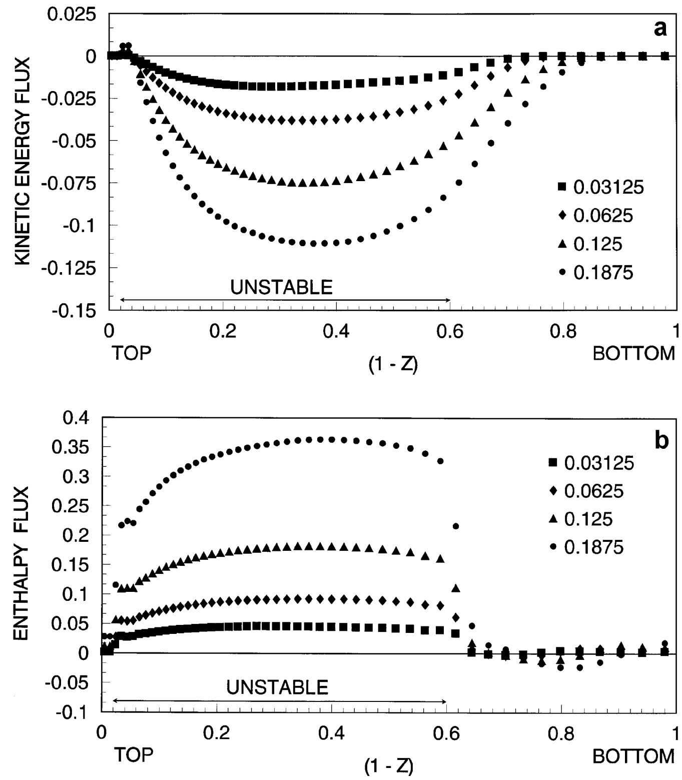

Fig. 2a and b. The depth distributions of time and space (*horizontal*) averaged **a** kinetic energy, and **b** enthalpy fluxes for various cases

The fluid is an ideal gas with $\gamma = 5/3$ and is contained in a rectangular box with the two equal horizontal sides 1.5 times the height (z). A mesh of 35 points each in the two horizontal directions and 51 points in the vertical occupies this rectangular domain. While the mesh points in the horizontal directions are equally spaced, the spacing in the vertical direction decreases smoothly with height. The top and bottom of the box are impenetrable and stress free while the sides are periodic. The temperature at the top is kept constant while a fixed energy flux F_b is imposed at the bottom.

The three (stable-unstable-stable) layer configuration for the simulation is generated by controlling the conductivities of the diffusive flux. The initial distribution of the gas is polytropic and the polytropic indices from the top down are 1.5, 1.5, and 2. In Fig. 1, we have shown the location of the three layers along with the temperature, pressure, and density contrasts between the top and the bottom. The variables are made dimensionless by scaling so that the initial temperature, pressure, density, and the total depth of the domain all equal 1 at the top. The velocities are scaled to $(p_t/\rho_t)^{1/2}$ and the scaling for various fluxes including the input flux is $(p_t\, V_t)$. The subscript ‘t’ refers to the initial values of the quantities at the top of the domain. In all the cases that we study for the penetration at the bottom of the unstable layer, the lower stable layer extends to a height of 0.4 from the bottom and contains around 1.2 pressure scale heights (p.s.h.). The unstable region is embedded between the two stable layers and contains 5.4 p.s.h. A thin stable layer at the top separates the convectively unstable region from the top boundary.

The energy budget of the system is described in terms of a total flux made up of four fluxes: the enthalpy flux (F_e), the flux of kinetic energy (F_k), the diffusive flux (F_d), and the viscous flux (F_v). The three- layer configuration is evolved with time till the the sum of these four fluxes is almost equal to the input flux F_b at all depths. Once the fluid is thermally relaxed, the equations are integrated for some more time and the combined temporal and spatial (horizontal) means of various physical quantities are computed. We have considered four separate configurations in this manner, corresponding to four different values of the input flux F_b. All other parameters are the same for the four models (SRC95).

3. Results

Fig. 2a shows the distribution of the average flux of kinetic energy, $(1/2)\rho V^2 V_z$, with depth for the four cases corresponding to four different flux values. The unstable region is marked by a line extending from a depth of 0.02 to 0.6 from the top. For all the flux values, the kinetic energy flux (F_k) in this region remains negative and drops to zero not at the (unstable- lower stable layer) interface, but a considerable distance into the lower stable layer. The peak F_k value corresponds to -0.111 for the case of input flux F_b equal to 0.1875. The peak F_k for input flux 0.03125 is -0.018. Hence, while the input flux has changed by a factor of 6, from 0.1875 to 0.03125, peak kinetic energy flux has fallen by an almost similar amount, by a factor of 6.17. In all the four cases, this peak kinetic energy flux F_k is around 60% of the input flux F_b. The enthalpy flux F_e is positive in the unstable region (Fig. 2b).

As has been done in some previous studies (Hurlburt et al. 1994; SRC95; Muthsam et al. 1995), we use the profile of the flux of kinetic energy to estimate the extent of penetration into the stable layer at the bottom of the convectively unstable region. We define the penetration distance (Δ_d) to be the length of the region between the interface of the unstable- lower stable layer and the depth where the kinetic energy flux has fallen to -10^{-3}. To achieve this, we perform polynomial interpolation to obtain the values of F_k between the grid points. The penetration distances for our four cases are given in Table 1. The result is also plotted in Fig. 3. It may be seen that the penetration distance increases almost linearly with the input flux F_b. Δ_d is 0.275 corresponding to 0.88 p.s.h. for the case with the largest input flux of 0.1875. It has decreased by a factor of 2.2 to 0.40 p.s.h. while the input flux has decreased by a factor of 6 (to 0.03125).

For the case with the highest input flux $F_b = 0.1875$, the penetration distance is 0.275 while the extent of the lower stable layer is 0.4. It is possible that the lower solid boundary is affecting the flows in this case leading to an underestimation of the penetration distance for this flux value.

Since the computation of penetration distance in the above manner may seem arbitrary, we employ one more criterion, which has been used in some earlier studies (Hurlburt et al. 1994; SRC95) and is also based on the flux of kinetic energy.

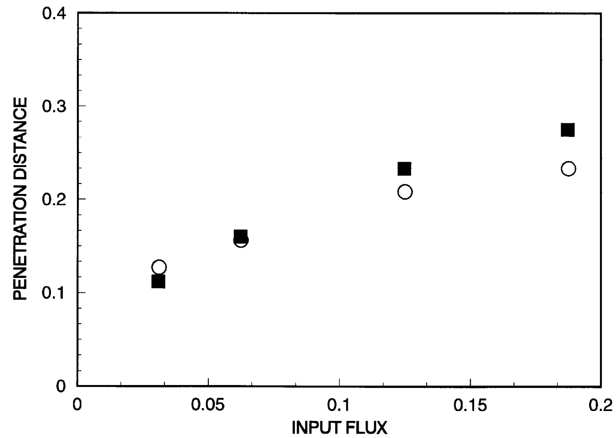

Fig. 3. The extent of penetration for different input fluxes F_b. The circles correspond to penetration depths where the kinetic energy flux is 5% of its value at the bottom of the convection zone

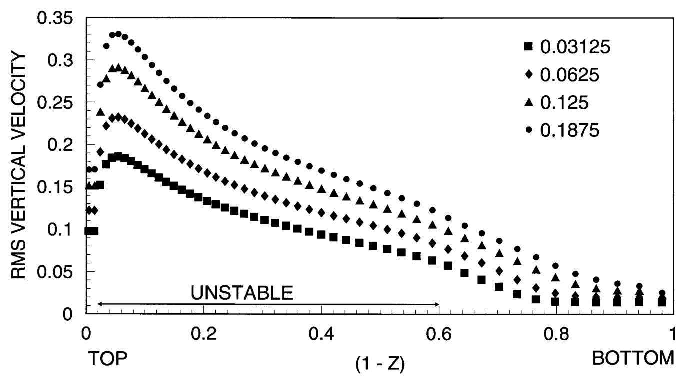

Fig. 4. The rms vertical velocities (V_z'') plotted against depth for the four cases

We compute the penetration depth to where F_k is 5% of its value at the interface of the unstable- lower stable layer. The values obtained with this definition of penetration distance are given in parentheses in Table 1. For the input flux 0.1875, we find $\Delta_d = 0.233$ which is about 15% lower than the one found with our earlier criterion. For the lowest flux of 0.03125, Δ_d is 0.127 instead of 0.112.

In order to study the relationship between the penetration distance, input flux and the vertical velocity, we have computed the average root- mean- square (rms) vertical velocities (V_z'') of the four models. The distributions of V_z'' over the vertical domain for the four input fluxes are shown in Fig. 4. In all the cases the velocities peak at a depth of around 0.055 from the top and fall to about 35% of their peak value at the depth 0.6, the bottom of the convective layer. The maximum values of the rms vertical velocities ($V_z''(max)$) and their values at the bottom of the convection zone (V_{zo}'') are given in Table 1. We have also calculated the quantities $\Delta_d\, F_b/V_{zo}''^3$ and $\Delta_d/F_b^{1/2}$. It may be seen that from Table 1 that for the four configurations under consideration here, $\Delta_d = 0.64\, F_b^{1/2}$ is a fairly good approximation as far as the relationship between the penetration distance and the input flux is concerned. Hurlburt et al. (1994)

Table 1. Penetration below the convective region and its scaling with input flux and the rms vertical velocity

Input Flux F_b	$V_z''(max)$	V_{zo}''	Penetration Δ_d	Penetration Δ_p (in p.s.h.)	$\frac{\Delta_d F_b}{V_{zo}''^3}$	$\frac{\Delta_d}{F_b^{1/2}}$	$\frac{\Delta_d}{V_{zo}''^{3/2}}$
0.1875	0.330	0.119	0.275 (0.233)*	0.88 (0.77)	30.6 (25.9)	0.64 (0.54)	6.70 (5.68)
0.125	0.291	0.103	0.233 (0.208)	0.77 (0.70)	26.7 (23.8)	0.66 (0.59)	7.05 (6.29)
0.0625	0.232	0.081	0.160 (0.156)	0.55 (0.54)	18.8 (18.4)	0.64 (0.62)	6.94 (6.77)
0.03125	0.186	0.062	0.112 (0.127)	0.40 (0.45)	14.7 (16.7)	0.63 (0.72)	7.26 (8.23)

* the numbers in parentheses correspond to depth where F_k has fallen to 5% of its value at the bottom of the convectively unstable layer

have earlier proposed the relationship $\Delta_d \propto F_b^{1/2}$ for the case when penetration is non-adiabatic. In Fig. 5, we have plotted the superadiabatic gradient with depth for the case when input flux is F_b is 0.0625. It is clear from the drop in the superadiabatic gradient at the bottom of the unstable zone that the penetration proceeds non-adiabatically.

In the last column of Table 1, we have calculated the quantity $\Delta_d/V_{zo}''^{3/2}$ for the four models. In choosing to calculate this quantity, we are guided by the treatment of Schmitt et al. (1984) and Zahn (1991). The relation obtained by Schmitt et al. (1984) using the phenomenological plume equations may be written as:

$$\Delta_d \approx V_{zo}^{3/2}\, f^{1/2} \,. \tag{1}$$

Zahn (1991) used some simplified modeling and obtained the following result for the case of a smooth conductivity profile with depth:

$$\frac{\Delta_d}{H_p} = V_{zo}^{3/2}\,(cf)^{1/2}\left[\frac{3}{2}\, g\, Q\, K\, \chi_p\, \nabla_{ad}\right]^{-1/2} , \tag{2}$$

where H_p is the pressure scale height, g is the local gravity, Q is the expansion coefficient at constant pressure, $\chi_p = (\partial \ln K_R/\partial \ln P)_{ad}$, K is the thermal diffusivity, and $\nabla_{ad} = (\partial \ln T/\partial \ln P)_{ad}$. The quantity c measures the asymmetry of the flow and is related to the triple moment of the velocity as

$$cf = \frac{\overline{V_z(V_x^2+V_z^2)}}{(\overline{V_z^2})^{3/2}} \,.$$

As may be noticed, Eq. (2) is in agreement with Eq. (1). In our case, we find (cf. Table 1) that $\Delta_d = C\, V_{zo}''^{3/2}$, where $C \approx 7 \pm 0.3$.

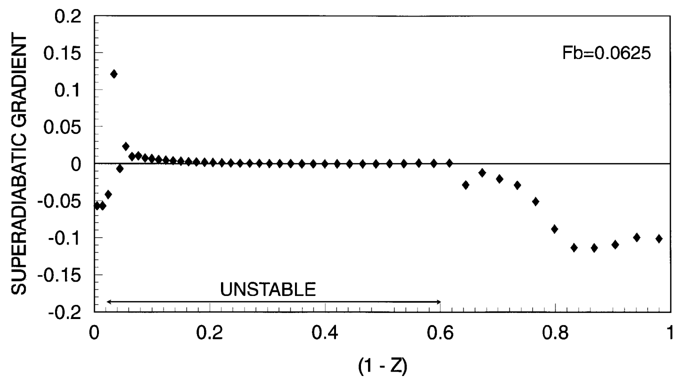

Fig. 5. The superadiabatic gradient plotted against depth for input flux 0.0625. The kink near the interface of the unstable- lower stable layer may be due to low resolution there

4. Conclusions

We have simulated the behaviour of turbulent compressible convection penetrating into stable layer at the bottom of a stellar-type convective envelope. Four models were computed corresponding to four different values of the input flux F_b, which is imposed at the lower boundary of the numerical box. We have seen that a larger input flux implies more vigourous convection with larger velocities. The depth of the penetration region (Δ_d), computed from the profile of kinetic energy flux F_k, has been found to increase with the increase in the input flux. It has been shown that $\Delta_d \propto F_b^{1/2}$ for the simulated cases. We have also examined the relationship between the penetration distance and the root-mean-square vertical velocity. Although we have used a conductivity profile which is piece-wise in the different layers, we find that, $\Delta_d \propto V_{zo}^{3/2}$, at least for the four models under consideration. It is desirable to study this relationship more carefully for a wide range of configurations and with a continuous temperature dependent conductivity to enable the examination of Eq. (2) in greater detail.

The above two relationships ($\Delta_d \propto F_b^{1/2}$ and $\Delta_d \propto V_{zo}^{3/2}$) imply that $F_b \propto V_{zo}^3$. One can explain this behaviour by the argument that within a convection zone, we have $F_b/\rho \propto V_{zo}^3$. It seems that this relationship is valid at the interface of the unstable-lower stable layer, i.e., $F_b = C_1\, \rho\, V_{zo}^3$. Substituting the value of the mean density at the interface (which is $\simeq 50$ for all four cases) into the above expression yields a value of $C_1 = 2.3$ to 2.6 for the four cases.

We have also observed (cf. Fig. 3) that for the larger flux value (0.1875), our configuration slightly underestimates the penetration depth. It is desirable, therefore, to have a broader lower stable layer for large flux values so that the boundary effects are minimal. Such computations will, however, be much more expensive to perform. We hope to address some of these issues in future.

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