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ASTRONOMY AND ASTROPHYSICS

# Acoustic wave propagation in the solar atmosphere

## III. Analytic solutions for adiabatic wave excitations

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**Abstract.** The response of an isothermal solar type atmosphere to adiabatic excitations by various small amplitude acoustic disturbances is studied analytically. Both continuous and pulse excitations are discussed. It is shown that wavetrains of random pulses may be responsible for the excitation of the 3-min solar oscillations.

**Key words:** Sun: photosphere – Sun: oscillations – Sun: chromosphere – acoustic wave – shock waves – hydrodynamics

### 1. Introduction

The oscillations observed on the Sun appear to have different types of origins. There is strong observational and theoretical evidence that acoustic waves trapped inside the Sun are responsible for the observed 5-minute oscillations (e.g., Ulrich 1970; Leibacher & Stein 1971). Propagating acoustic waves generated by turbulent motions in the solar convection zone are now thought to be primarily sources of heating in the solar chromospheric regions where surface magnetic fields are weak (e.g. Buchholz, Ulmschneider & Cuntz 1998). While some of the observed chromospheric 3-min oscillations appear to be due to cavity modes, trapped by the outwardly rising chromospheric and transition-layer temperature gradient, theoretical results indicate that an important part of these oscillations is due to the response of the solar chromosphere at its natural frequency to propagating acoustic waves (Fleck & Schmitz 1991). It is the purpose of the present work to investigate this latter property in greater detail.

Fundamental contributions on the propagation of acoustic waves in an unbounded isothermal atmosphere were made by Lamb (1908, 1932) who showed that the propagation of these waves can be affected by the existence of the atmospheric density and pressure gradients. As a result, the so-called *acoustic cutoff frequency*, defined as the ratio of the sound speed to twice the pressure scale height, can be introduced. Lamb showed that this cutoff is the natural frequency of the atmosphere, which means that any acoustic disturbances imposed on the atmosphere will trigger an atmospheric response at this frequency. He was also first to demonstrate that acoustic waves with frequencies lower than the acoustic cutoff frequency are always evanescent and cannot transfer energy to higher atmospheric layers. Numerous analytical and numerical investigations have followed Lamb's work, concentrating on wave trapping inside the Sun and in the solar chromosphere, on dynamic response of the solar chromosphere to various excitations, and on the problem of chromospheric and coronal heating (see Narain & Ulmschneider 1996, for an extensive review and references).

In recent years, further attention has been given to Lamb's original work. Fleck & Schmitz (1991) have demonstrated that the observed chromospheric 3-min oscillations can be explained as the response of the solar chromosphere at its natural (acoustic cutoff) frequency to propagating acoustic waves. In several papers, Fleck & Schmitz (1991, 1993) and Schmitz & Fleck (1995) have improved Lamb's work by considering a bounded isothermal atmosphere and imposing acoustic disturbances by specifying boundary conditions at a given atmospheric height. Kalkofen et al. (1994) have extended this further and for the case of a monochromatic acoustic wave excitation they were able to describe analytically the temporal development of the generated atmospheric oscillations by obtaining an asymptotic solution. Sutmann & Ulmschneider (1995a, b) have numerically investigated the excitation of linear and nonlinear atmospheric oscillations in both isothermal and non-isothermal solar atmosphere models. For most recent work on nonlinear excitations of atmospheric oscillations see Carlsson & Stein (1997), Theurer et al. (1997a, b). The main result of the previous work is that by switching on a disturbance in an atmosphere which previously was at rest, atmospheric oscillations with the acoustic cutoff frequency are excited by the incoming wave disturbance and that these generated oscillations show a temporal decay.

It is the purpose of this paper to formulate a general method which allows to determine analytically oscillation states of a bounded isothermal atmosphere for various types of excitation. Thus, from mathematical point of view, we solve an inhomogeneous boundary value problem for the Klein-Gordon equation for acoustic waves by using Laplace transformation (Sect. 2). Physically this means that we investigate the generation of forced and free atmospheric oscillations driven by

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various acoustic disturbances imposed on the atmosphere at its lower boundary.

We consider the excitation by the following acoustic disturbances: monochromatic acoustic waves (Sect. 3), a spectrum of partial waves (Sect. 4), a $\delta$-function pulse (Sect. 5), a sinusoidal pulse with a given period (Sect. 6), and a wavetrain of pulses with randomly chosen amplitudes and periods (Sect. 7). Some applications of the obtained analytical results to the Sun are discussed in Sect. 8. Our conclusions are given in Sect. 9.

## 2. Basic formulation and mathematical solutions

Let us consider the propagation of linear acoustic waves in an isothermal atmosphere. For simplicity but without any loss of generality, we assume that the waves propagate only in the upward vertical direction (+z-direction), and that gravity is given by $\boldsymbol{g} = -g\hat{z}$. We assume that a one-dimensional treatment can be used. The problem is described mathematically by a set of hydrodynamic equations (see, for example, Landau & Lifshitz 1987) governing the evolution of the wave variables: pressure $p$, density $\rho$, and velocity $u$, in time and space. Since we consider an isothermal atmosphere, both the background pressure $p_0$, and density $\rho_0$, are functions of height $z$ and the temperature $T_o$, is constant. The latter means that the sound speed $c_S = \gamma p_0/\rho_0$ with $\gamma$ being the ratio of specific heats, is also constant in the entire atmosphere and, as a result, the conditions for the propagating and evanescent wave solutions are well-known (see Lamb 1908). We begin with a brief discussion of the acoustic wave equation and then show how to solve it for given initial and boundary conditions by using the Laplace transform method.

From the linearized hydrodynamic equations one can derive for acoustic waves a wave equation for, let say, the velocity $u$

$$\frac{\partial^2 u}{\partial t^2} - c_S^2 \frac{\partial^2 u}{\partial z^2} + \gamma g \frac{\partial u}{\partial z} = 0. \qquad (1)$$

This wave equation was first obtained by Lamb (1908) who showed that it fully describes the propagation of linear acoustic waves in an isothermal atmosphere. The fact that the wave equation is obtained for the velocity $u$ is not significant here as the form of this equation for the other wave variables is the same, because the sound speed $c_S$ is independent of height. In order to write the acoustic wave equation in its normal form, we introduce the following transformation:

$$u(t, z) = v(t, z) e^{\lambda z}, \qquad (2)$$

where the parameter $\lambda$ will be chosen in such a way that the resulting wave equation does not have first derivatives in $v$. The procedure is well-known (see, for example, Kahn 1990) and in the considered case leads to the so-called Klein-Gordon equation (e.g., Morse & Feshbach 1953; Courant & Hilbert 1962). Discussion of the Klein-Gordon equation for different types of waves that might be important in the solar atmosphere is given by Roberts (1982), Musielak, Fontenla & Moore (1992), Stark & Musielak (1993), and Musielak & Moore (1995).

We now substitute the above transformation into Eq. (1) and obtain

$$\frac{\partial^2 v}{\partial t^2} - c_S^2 \left( \frac{\partial^2 v}{\partial z^2} + 2\lambda \frac{\partial v}{\partial z} + \lambda^2 v \right) + \gamma g \left( \lambda v + \frac{\partial v}{\partial z} \right) = 0 \, , \qquad (3)$$

from which we find the condition on $\lambda$

$$-2\lambda c_S^2 + \gamma g = 0, \qquad (4)$$

or

$$\lambda = \frac{\gamma g}{2 c_S^2}. \qquad (5)$$

Using this condition, we get

$$\frac{\partial^2 v}{\partial t^2} - c_S^2 \frac{\partial^2 v}{\partial z^2} + \omega_A^2 v = 0, \qquad (6)$$

where

$$\omega_A^2 \equiv \left( \frac{2\pi}{P_A} \right)^2 = \left( \frac{c_S}{2H} \right)^2 = \left( \frac{\gamma g}{2 c_S} \right)^2, \qquad (7)$$

is the acoustic cutoff frequency, $P_A$ is the cutoff period, first introduced by Lamb (1908), and $H = c_S^2/\gamma g$ is the pressure scale height. After solving the wave equation (6) by Fourier transforms in time and space, the dispersion relation is derived and, as shown by Lamb, acoustic waves with frequencies $\omega > \omega_A$ are always propagating waves in an isothermal atmosphere, however, waves with frequencies $\omega \leq \omega_A$ are always evanescent. The solutions found by Lamb describe the behavior of freely propagating acoustic waves in an unbounded isothermal atmosphere without any wave sources.

The approach pursued in our present paper is different, namely, we introduce a wave source in an isothermal atmosphere by specifying the boundary condition at $z = 0$. This means that the atmosphere is bounded, even if formally it extends to infinity for positive values of $z$. In addition, we assume that initially, at time $t = 0$, the atmosphere is at rest and that there are no wave motions or derivatives of wave motions anywhere between $z > 0$ and $z = \infty$. Actually because it would take infinite time for our introduced wave disturbance to reach $z = \infty$, there never are wave motions at $z = \infty$. Thus mathematically, the initial and boundary conditions can be expressed as follows

$$\lim_{t \to 0} v(t, z) = 0, \qquad \lim_{t \to 0, z \neq 0} \frac{\partial v}{\partial t} = 0. \qquad (8)$$

$$\lim_{z \to 0} v(t, z) = A_0(t), \qquad \lim_{z \to \infty} v(t, z) = 0. \qquad (9)$$

$A_0(t)$ describes the velocity imposed on the atmosphere at the height $z = 0$. This means that we have posed the problem of a *homogeneous Klein-Gordon equation* for acoustic waves (Eq. 6) with *inhomogeneous boundary conditions* (Eq. 9). There are several different ways of solving this problem (e.g., Straton 1941; Budak et al. 1962; Davies 1978; Edwards & Penny 1989).

In this paper, we solve the problem by using a Laplace transformation in time. It must be noted, however, that according to the theory of partial differential equations, the problem can be reformulated to obtain an inhomogeneous Klein-Gordon equation with homogeneous boundary conditions, and then solved it by using the method of eigenfunction expansions (e.g., Haberman 1987). Let $w(s,z)$ be the Laplace transform of $v(t,z)$

$$w(s,z) \;=\; \mathcal{L}\{v(t,z)\} \;=\; \int_0^\infty v(t,z)\, e^{-st}\, dt. \tag{10}$$

Applying it to Eq. (6), one finds

$$\frac{\partial^2 w(s,z)}{\partial z^2} - \left(\frac{s^2+\omega_s^2}{c_S^2}\right) w(s,z) = 0. \tag{11}$$

It is easy to check that this equation has a general solution given in the form

$$w(s,z) = a_0(s)\, e^{-\sqrt{s^2+\omega_A^2}\; z/c_S} + a_1(s)\, e^{\sqrt{s^2+\omega_A^2}\; z/c_S}, \tag{12}$$

where $a_0$ and $a_1$ are to be determined by the initial and boundary conditions. The imposed boundary conditions require that there are no wave motions at $z = \infty$. The first term on the RHS of Eq. (12) obeys the condition because it decays exponentially with increasing values of $z$, however, the second term on the RHS of this equation contradicts our boundary conditions because it increases exponentially with height $z$. Clearly, the second term cannot be a physically acceptable solution to our problem as it would require an infinite amount of energy. In order to eliminate this part of the solution we take $a_1 = 0$.

To find the inverse Laplace transform of $w(z,s)$, we write the exponential factor in Eq. (12) in the following form (see Appendix A)

$$e^{-\sqrt{s^2+\omega_A^2}\; z/c_S} = e^{-sz/c_S} - \frac{\omega_A z}{c_S}\int_{z/c_S}^\infty \frac{J_1(\omega_A\sqrt{t^2-(z/c_S)^2})}{\sqrt{t^2-(z/c_S)^2}} e^{-st} dt\ . \tag{13}$$

We now define

$$W(t,z) \;\equiv\; -\frac{\omega_A z}{c_S}\, \frac{J_1(\omega_A\sqrt{t^2-(z/c_S)^2})}{\sqrt{t^2-(z/c_S)^2}} \times\mathcal{H}(t-z/c_S)\ , \tag{14}$$

where $\mathcal{H}(t-z/c_S)$ is the Heaviside step-function and its value is 0 for $t < z/c_S$ and 1 for all values of $t > z/c_S$, while $\mathcal{H}(t-z/c_S) = 0.5$ if $t = z/c_S$. This allows writing Eq. (12) as

$$\begin{aligned} w(s,z) &= a_0(s)\, e^{-\sqrt{s^2+\omega_A^2}\; z/c_S} \\ &= a_0(s)\, e^{-sz/c_S} + a_0(s)\, \mathcal{L}\{W(t,z)\} \\ &= a_0(s)\, e^{-sz/c_S} + \mathcal{L}\{A_0(t)\}\mathcal{L}\{W(t,z)\}. \end{aligned} \tag{15}$$

Here we have chosen $a_0(s)$ to be the Laplace transform of the specified velocity function $A_0(t)$. The first term on the RHS in this equation can be rewritten by using the *second-shifting theorem*

$$a_0(s)\, e^{-sz/c_S} = \mathcal{L}\{A_0(t-z/c_S)\, \mathcal{H}(t-z/c_S)\}. \tag{16}$$

Then, the *convolution theorem* can be used to rewrite the second term on the RHS in Eq. (15). This gives

$$\mathcal{L}\{A_0(t)\}\, \mathcal{L}\{W(t,z)\} = \mathcal{L}\{\int_0^t A_0(t-\tau)\, W(\tau,z)\, d\tau\ \}\ . \tag{17}$$

Substituting now Eqs. (16) and (17) into Eq. (15), and taking the inverse Laplace transform, we finally obtain

$$v(t,z) \;=\; A_0(t-z/c_S)\, \mathcal{H}(t-z/c_S) + \int_0^t A_0(t-\tau)\, W(\tau,z)\, d\tau, \tag{18}$$

where $W(\tau,z)$ is given by Eq. (14) after $t$ is replaced by $\tau$. The derived expression is the final solution of the Klein-Gordon equation (6) subject to the initial and boundary conditions specified by Eqs. (8) and (9). We will use this solution in the remaining part of this paper to investigate the response of the background atmosphere to different boundary conditions introduced at $z = 0$ and determined by $A_0(t)$.

However, before the solution given by Eq. (18) is formally used, it must be noted that the behavior of the velocity $u(z,t)$ is described by the acoustic wave equation (Eq. 1) and that the relationship between the velocity $v(z,t)$ and $u(z,t)$ is given by

$$u(z,t) = v(z,t)\, e^{\omega_A z/c_S}. \tag{19}$$

This means that the velocity $u(z,t)$ always increases with height, or in the extreme case $\omega_A \to 0$ remains constant. This obviously does not contradict our boundary conditions as they are imposed only on the velocity $v(z,t)$. In the following, we consider *five* different boundary conditions corresponding to five different wave excitation cases and for each case we present the full analytical solution for the velocity $v(z,t)$. Finally, for full analytical solutions of the velocity $u(z,t)$, Eq. (19) must be used.

## 3. Excitation by monochromatic acoustic waves

We first consider a source of monochromatic acoustic waves (for example a moving piston) to be located at the lower boundary ($z = 0$) of a semi-infinitely extended isothermal atmosphere. We assume that the waves are generated continuously by the source and that they have the frequency $\omega$ and amplitude $V_0$. According to Eq. (9), the described physical situation requires

$$A_0(t) \;=\; V_0\, e^{-i\omega t}. \tag{20}$$

Using this condition, we can write the general solution given by Eq. (18) in the following form:

$$v(t,z) = V_0 \left[e^{-i\omega(t-z/c_S)}\, \mathcal{H}(t-z/c_S) - I_{31} \;+\; I_{32}\right], \tag{21}$$

where

$$I_{31} \;\equiv\; C_0\int_0^\infty \frac{J_1(\omega_A\sqrt{\tau^2-(z/c_S)^2})}{\sqrt{\tau^2-(z/c_S)^2}} \times\mathcal{H}(\tau-z/c_S)e^{i\omega\tau}d\tau \tag{22}$$

and

$$I_{32} \equiv C_0 \int_t^{\infty} \frac{J_1(\omega_A \sqrt{\tau^2 - (z/c_S)^2})}{\sqrt{\tau^2 - (z/c_S)^2}} \times \mathcal{H}(\tau - z/c_S) e^{i\omega\tau} d\tau \quad (23)$$

with

$$C_0 \equiv \frac{\omega_A z}{c_S} e^{-i\omega t}. \quad (24)$$

Note that the integral in Eq. (18) is replaced in Eq. (21) by two integrals with different limits of integration. The integral $I_{31}$ can be evaluated analytically by using Eqs. (13) and (14), the result is

$$I_{31} = \left[ e^{i\omega z/c_S} - e^{-i\sqrt{\omega^2-\omega_A^2}\, z/c_S} \right] e^{-i\omega t}. \quad (25)$$

However, the integral $I_{32}$ cannot be evaluated analytically unless $\tau >> z/c_S$, which is equivalent to $t >> z/c_S$ as $\tau$ varies from $t$ to $\infty$ in this integral. Making this assumption and taking an expression for the Bessel function $J_1$ that holds for large arguments, we evaluate the second integral asymptotically by retaining only the first order term and neglecting all higher order terms (see Appendix B for details). This gives

$$I_{32} = \sqrt{\frac{2\omega_A}{\pi}} \frac{1}{\omega^2 - \omega_A^2} \frac{z}{c_S} \frac{1}{t^{3/2}} \times \left[ \omega_A \sin\left(\omega_A t - \frac{3\pi}{4}\right) + i\omega \cos\left(\omega_A t - \frac{3\pi}{4}\right) \right], \quad (26)$$

Substituting Eqs. (25) and (26) into Eq. (21), we find

$$v(t,z) = V_0\, e^{-i\left(\omega t - \sqrt{\omega^2-\omega_A^2}\, z/c_S\right)} + V_0 \sqrt{\frac{2\omega_A}{\pi}} \frac{1}{\omega^2 - \omega_A^2} \frac{z}{c_S} \frac{1}{t^{3/2}} \times \left[ \omega_A \sin\left(\omega_A t - \frac{3\pi}{4}\right) + i\omega \cos\left(\omega_A t - \frac{3\pi}{4}\right) \right], \quad (27)$$

which is our final expression describing the response of the atmosphere to the presence of monochromatic acoustic waves. This formula has been originally derived by Kalkofen et al. (1994) who also extensively discussed its physical meaning. In their discussion, however, they did not address the case when $\omega \to \omega_A$. In the following, we briefly summarize their discussion and explain the behavior of oscillations when they are driven with frequencies near $\omega_A$ and with $\omega = \omega_A$.

As shown in Appendix B, Eq. (27) is valid only for $\omega \neq \omega_A$. Its validity has been also confirmed by Sutmann & Ulmschneider (1995a) who performed numerical studies of the response of the solar atmosphere to the presence of continuously generated acoustic waves with frequencies below the acoustic cutoff frequency and found a good agreement between analytical and numerical results in the limit of $t >> z/c_S$. According to Eq. (27), the behavior of the velocity $v(t,z)$ in the atmosphere perturbed by monochromatic acoustic waves of a given frequency $\omega$ is described by a superposition of two atmospheric oscillations: the so-called *forced atmospheric oscillations* with the wave frequency $\omega$ and *free atmospheric oscillations* with the acoustic cutoff frequency $\omega_A$. The forced oscillations with the driving frequency $\omega$ represent propagating acoustic waves if $\omega > \omega_A$ and evanescent acoustic waves if $\omega \leq \omega_A$. Since the background atmosphere is isothermal, the behavior of these waves in both frequency regimes is well-known and it was described by Lamb (1908). Now, the free atmospheric oscillations are described by the second term on the RHS of Eq. (27) and it is seen that these oscillations decay in time as $t^{-3/2}$ at any given height and their amplitude increases linearly with height. In the following, we investigate the behavior of these free oscillations when they are driven with $\omega = \omega_A$.

According to the results presented in Appendix B, the corresponding formula for $v(t,z)$ is

$$v(t,z) = V_0\, e^{-i\omega_A t} + V_0 \sqrt{\frac{2\omega_A}{\pi}} \frac{z}{c_S} \frac{1}{t^{1/2}} \times \left[ \cos\left(\omega_A t - \frac{3\pi}{4}\right) - i \sin\left(\omega_A t - \frac{3\pi}{4}\right) \right] + V_0 \sqrt{\frac{2\omega_A}{\pi}} \frac{1}{4\omega_A} \frac{z}{c_S} \frac{1}{t^{3/2}} \times \left[ i\cos\left(\omega_A t - \frac{3\pi}{4}\right) - \sin\left(\omega_A t - \frac{3\pi}{4}\right) \right]. \quad (28)$$

The first term in this formula describes periodic and non-decaying oscillations in time that take place at any height in the atmosphere. The second and third term shows that the velocity decays in time as $t^{-1/2}$ and $t^{-3/2}$, respectively. Since for large $t$, the third term is always much smaller than the second one, it can be formally neglected. Based on this result, one may conclude that the free atmospheric oscillations driven with $\omega = \omega_A$ decay in time as $t^{-1/2}$. However, this is not the case here ! As shown in Appendix B, it is necessary to take $t \to \infty$ when $\omega \to \omega_A$. Thus, in the limit of $\omega = \omega_A$, the second term in Eq. (28) also becomes zero. Since the remaining first term in Eq. (28) describes the forced atmospheric oscillations driven with the natural frequency $\omega_A$, and since the oscillations with the frequency $\omega_A$ are called the free atmospheric oscillations, we conclude that in the case of $\omega = \omega_A$ the forced and free atmospheric oscillations are the same and, according to Eq. (28), they do not decay in time. This is an interesting result as it shows that only free atmospheric oscillations driven by acoustic waves with frequencies different than the natural cutoff frequency are decaying in time as $t^{-3/2}$, however, the oscillations do not decay in time if the driving frequency is exactly the same as the natural frequency. Obviously, the main physical reason that these oscillations do not decay in time is that they are driven continuously by the wave source.

We would like to close this section by addressing the problem of nomenclature used in some papers devoted to studies of the response of the atmosphere to different perturbations. The free atmospheric oscillations with the acoustic cutoff frequency $\omega_A$ have also been called *resonance* oscillations (e.g., Fleck & Schmitz 1991; Schmitz & Fleck 1995; Sutmann & Ulmschneider 1995a,b), where the word “resonance” was used in a sense of "echo". Since in typical resonance phenomena $v(t,z)$ would be expected to become unbounded when $\omega$ approaches $\omega_A$, and since according to Eq. (28) the latter never occurs, we follow Kalkofen et al. (1994) and call these oscillations *free atmospheric oscillations*.

## 4. Excitation by a spectrum of waves

The results presented in the previous section can now be extended to a spectrum of waves. We consider a linear superposition of sinusoidal partial waves with different amplitudes, frequencies and phases, and assume that the spectrum of these waves is generated by a wave source located at the lower boundary ($z=0$) of a semi-infinite isothermal atmosphere. In this case the lower boundary condition $A_0(t)$ can be written as

$$A_0(t) = \sum_{n=1}^{N} V_n\, e^{-i(\omega_n t + \varphi_n)}, \tag{29}$$

where $V_n$ and $\omega_n$ are the velocity amplitudes and frequencies of the partial waves, respectively, and $\varphi_n$ are arbitrary constant phases. With this boundary condition, Eq. (18) gives

$$\begin{aligned} v(t,z) \;&=\; \sum_{n=1}^{N} V_n \epsilon^{-i\varphi_n}\, e^{-i\omega_n (t - z/c_S)}\, \mathcal{H}(t - z/c_S) \\ &\quad - \sum_{n=1}^{N} V_n\, e^{-i\varphi_n}\, (I_{41} \;-\; I_{42})\,, \end{aligned} \tag{30}$$

where

$$\begin{aligned} I_{41} \;&\equiv\; C_n \int_0^\infty \frac{J_1(\omega_A \sqrt{\tau^2 - (z/c_S)^2})}{\sqrt{\tau^2 - (z/c_S)^2}} \\ &\quad \times \mathcal{H}(\tau - z/c_S) e^{i\omega_n \tau} d\tau \end{aligned} \tag{31}$$

and

$$\begin{aligned} I_{42} \;&\equiv\; C_n \int_t^\infty \frac{J_1(\omega_A \sqrt{\tau^2 - (z/c_S)^2})}{\sqrt{\tau^2 - (z/c_S)^2}} \\ &\quad \times \mathcal{H}(\tau - z/c_S) e^{i\omega_n \tau} d\tau \end{aligned} \tag{32}$$

where

$$C_n \;\equiv\; \frac{\omega_A z}{c_S}\, e^{-i\omega_n t}. \tag{33}$$

Note that these integrals are very similar to those derived in the previous section (see Eqs. 22 and 23). Therefore, the same procedure of evaluating them (see Appendix B) can be used. The result is

$$I_{41} = \left[ e^{i\omega_n z/c_S} \;-\; e^{-i\sqrt{\omega_n^2 - \omega_A^2}\; z/c_S} \right]\, e^{-i\omega_n t}, \tag{34}$$

and

$$\begin{aligned} I_{42} \;&=\; \sqrt{\frac{2\omega_A}{\pi}}\, \frac{1}{\omega_n^2 - \omega_A^2}\, \frac{z}{c_S}\, \frac{1}{t^{3/2}} \\ &\quad \times \left[ \omega_A \sin\left(\omega_A t - \frac{3\pi}{4}\right) + i\omega_n \cos\left(\omega_A t - \frac{3\pi}{4}\right) \right] . \end{aligned} \tag{35}$$

Using Eqs. (34) and (35), we may write Eq. (30) in the following form

$$\begin{aligned} v(t,z) \;&=\; \sum_{n=1}^{N} V_n\, e^{-i\varphi_n} e^{-i\left(\omega_n t - \sqrt{\omega_n^2 - \omega_A^2}\; z/c_S\right)} \\ &\quad + \sum_{n=1}^{N} V_n\, e^{-i\varphi_n}\, \sqrt{\frac{2\omega_A}{\pi}}\, \frac{1}{\omega_n^2 - \omega_A^2}\, \frac{z}{c_S}\, \frac{1}{t^{3/2}} \\ &\quad \times \left[ \omega_A \sin\left(\omega_A t - \frac{3\pi}{4}\right) + i\omega_n \cos\left(\omega_A t - \frac{3\pi}{4}\right) \right]. \end{aligned} \tag{36}$$

Since the considered wave spectrum is a superposition of many partial waves and since each partial wave can be formally treated as a monochromatic wave, the interpretation of the obtained results is very similar to that given at the end of Sect. 3. Namely, the first sum on the RHS of Eq. (36) describes forced atmospheric oscillations and shows that all partial waves with frequencies $\omega_n > \omega_A$ are propagating waves in the atmosphere, however, partial waves with $\omega_n < \omega_A$ are evanescent. In both cases, the oscillations are periodic and non-decaying in time. The amplitude of the propagating waves stays constant in the atmosphere, however, for the evanescent waves the amplitude decreases exponentially with height. The other terms on the RHS of the above equation describe free atmospheric oscillations with the acoustic cutoff frequency, $\omega_A$. It must be noted that each partial wave excites its own free oscillations, which superpose and have amplitudes that decay in time in the same manner as that found for monochromatic waves. As discussed in the previous section, if $\omega_n = \omega_A$, then the free and forced oscillations are the same and they do not decay in time. Some application of these results to the Sun can be found in Sect. 8.

## 5. Excitation by a $\delta$–function pulse

After considering the continuous excitation of acoustic waves by a source located at the lower boundary of a semi-infinite isothermal atmosphere, we now assume that the wave source generates one pulse that has a $\delta$–function shape. The required boundary condition is

$$A_0(t) = V_0'\, \delta(t). \tag{37}$$

Since the $\delta$-function has a non-dimensional argument, $V_0'$ has a different dimension than $V_0$ and $V_n$ used in Sects. 3 and 4, respectively. To make it consistent with the previously used quantities, we introduce the dimensionless time $t' = \omega_A t/2\pi = t/P_A$, where $P_A$ is the cutoff period (see Eq. 7). Because $\delta(t) = \delta(P_A t')$, we can write

$$A_0(t') = V_0\, \delta(t'), \tag{38}$$

where now $V_0 = V_0'/P_A$ has the same dimension (cm/s) as $V_0$ and $V_n$ used in Eqs. (27) and (36), respectively. Using Eqs. (18) and (38), we obtain

$$\begin{aligned} v(t',z) \;&=\; V_0\, \delta(t' - z/c_S P_A)\, \mathcal{H}(t' - z/c_S P_A) \\ &\quad + V_0 \int_0^{t'} \delta(t' - \tau')\, W(\tau', z)\, d\tau'\ , \end{aligned} \tag{39}$$

with $\tau' = \tau/P_A$. Taking $t' > z/c_S P_A$ and using $t = t' P_A$, we have

$$v(t,z) \ = \ -\pi \ V_0 \ \frac{z}{c_S} \ \frac{J_1(\omega_A \sqrt{t^2-(z/c_S)^2})}{\sqrt{t^2-(z/c_S)^2}}. \qquad (40)$$

In the limit of $t >> z/c_S$, the Bessel function $J_1$ can be expanded for large arguments and Eq. (40) can be written in the following form:

$$v(t,z) \ = \ -V_0 \sqrt{\frac{2\pi}{\omega_A}} \ \frac{z}{c_S} \ \frac{1}{t^{3/2}} \ \cos\left(\omega_A t - \frac{3\pi}{4}\right). \qquad (41)$$

The obtained result shows that the $\delta$–function pulse excites only free atmospheric oscillations that decay in time in the same manner ($t^{-3/2}$) as free oscillations generated by monochromatic waves and by a spectrum of waves. Since the excitation is not continuous and since we consider the situation when the pulse has already passed through the atmosphere ($t >> z/c_S$), forced atmospheric oscillations are not present in Eq. (41).

Our result is different from that found by Kalkofen et al. (1994) who showed that free atmospheric oscillations generated by a $\delta$-function pulse should decay in time as $t^{-1/2}$. As already recognized by Sutmann & Ulmschneider (1995a) and Schmitz & Fleck (1995), the $\delta(z)$-function pulse prescribed by Kalkofen et al. means that they solved an inhomogeneous *initial* value problem. Since here we are solving an inhomogeneous *boundary* value problem (see Eq. 9) for the Klein-Gordon equation (see Eq. 6), a $\delta(t)$-function pulse must be imposed on the atmosphere. From a physical point of view, the inhomogeneous initial value problem solved by Kalkofen et al. can only be used to describe free oscillations in an *unbounded* isothermal atmosphere, however, in order to describe free atmospheric oscillations in a *bounded* isothermal atmosphere, the inhomogeneous boundary value problem has to be solved. Thus, the result given by Eq. (41) cannot be directly compared to that obtained by Kalkofen et al. (seee their Eq. 13). To clarify the issue further, it has to be emphasized that Kalkofen et al. formally solved two physically different problems, namely, the inhomogeneous initial value problem for the excitation by the $\delta(z)$-function pulse and the inhomogeneous boundary value problem for the excitation by monochromatic acoustic waves. Therefore, these two different cases should not be compared.

## 6. Excitation by a sinusoidal pulse

As next case, we consider a pulse generated by a sinusoidal wave with period $P = 2\pi/\omega$. The generation process is stopped after one wave period. The lower boundary condition $A_0(t)$ describing the pulse is given by

$$A_0(t) = V_0 \ [\mathcal{H}(t) - \mathcal{H}(t-P)] \ e^{-i\omega t}. \qquad (42)$$

Before we substitute $A_0(t)$ into Eq. (18), let us first consider

$$A_0(t - z/c_S) \ = \ V_0 \ [\mathcal{H}(t-z/c_S) \ - \ \mathcal{H}(t-P-z/c_S)] \times e^{-i\omega(t-z/c_S)} \ . \qquad (43)$$

Since we are primarily interested in times $t >> z/c$ (see the last three sections), which implies that $t > (P + z/c_S)$ and that both Heaviside functions in Eq. (43) become unity and cancel each other. Thus $A_0(t) = 0$. As a result, the solution given by Eq. (18) can be written as

$$v(t,z) \ = \ V_0 \ \underbrace{\int_0^t \mathcal{H}(t-\tau) \ e^{-i\omega(t-\tau)} \ W(\tau,z) d\tau}_{I_{61}} \qquad (44)$$
$$- V_0 \ \underbrace{\int_0^t \mathcal{H}(t-P-\tau) \ e^{-i\omega(t-\tau)} \ W(\tau,z) \ d\tau}_{I_{62}} .$$

The first integral $I_{61} = 0$ because for all values of the variable $\tau$ in the interval $(0,t)$ we have $\mathcal{H}(t-\tau) = 0$; note that for $\tau = t$ the Heaviside-step function is non-zero, $\mathcal{H}(0) = 0.5$, but in this case the integral vanishes anyway as $\tau$ is reduced to just one point.

Thus, the only integral to be considered is $I_{62}$. We change the lower limit of integration by removing the Heaviside–step function and obtain

$$I_{62} \ = \ -\int_{t-P}^{t} e^{-i\omega(t-\tau)} \ W(\tau,z) \ d\tau, \qquad (45)$$

which can be also written as

$$I_{62} \ = \ -\int_{t-P}^{\infty} e^{-i\omega(t-\tau)} \ W(\tau,z) \ d\tau + \int_{t}^{\infty} e^{-i\omega(t-\tau)} \ W(\tau,z) \ d\tau \ . \qquad (46)$$

This integral can again be evaluated analytically for large times, $t >> z/c_S$, by using the results presented in Appendix B. We find

$$\int_{t-P}^{\infty} e^{-i\omega(t-\tau)} \ W(\tau,z) \ d\tau \ = \ \sqrt{\frac{2\omega_A}{\pi}} \ \frac{V_0}{\omega^2-\omega_A^2} \ \frac{z}{c_S} \times \frac{1}{(t-P)^{3/2}} \left[\omega_A \sin\left(\omega_A(t-P) - \frac{3\pi}{4}\right) + i\omega \cos\left(\omega_A(t-P) - \frac{3\pi}{4}\right)\right], \qquad (47)$$

with $e^{i\omega P} = e^{2\pi i} = 1$, and

$$\int_{t}^{\infty} e^{-i\omega(t-\tau)} \ W(\tau,z) \ d\tau \ = \ \sqrt{\frac{2\omega_A}{\pi}} \ \frac{V_0}{\omega^2-\omega_A^2} \ \frac{z}{c_S} \ \frac{1}{t^{3/2}} \times \left[\omega_A \sin\left(\omega_A t - \frac{3\pi}{4}\right) + i\omega \cos\left(\omega_A t - \frac{3\pi}{4}\right)\right]. \qquad (48)$$

For large times $t >> P$ one can expand

$$\frac{1}{(t-P)^{3/2}} \ \approx \ \left(1 + \frac{3}{2}\frac{P}{t}\right) \ \frac{1}{t^{3/2}}. \qquad (49)$$

Taking only the first term in this expansion (see Appendix B), we obtain

$$v(t,z) \ = \ V_0 \sqrt{\frac{2\omega_A}{\pi}} \ \frac{1}{\omega^2-\omega_A^2} \ \frac{z}{c_S} \ \frac{2}{t^{3/2}}$$

$$\times \left\{ \omega_A \left[ \sin\left( \omega_A (t-P) - \frac{3\pi}{4} \right) - \sin\left( \omega_A t - \frac{3\pi}{4} \right) \right] \right.$$
$$\left. + i\omega \left[ \cos\left( \omega_A (t-P) - \frac{3\pi}{4} \right) - \cos\left( \omega_A t - \frac{3\pi}{4} \right) \right] \right\}. \quad (50)$$

Using trigonometric relationships for difference between two sine and two cosine functions of different arguments, we can rewrite the above equation in the following form:

$$v(t,z) \;=\; V_0 \sqrt{\frac{2\omega_A}{\pi}} \, \frac{1}{\omega^2 - \omega_A^2} \, \frac{z}{c_S} \, \frac{2}{t^{3/2}}$$
$$\times \left[ \omega_A \cos\left( \omega_A (t - \frac{P}{2}) - \frac{3\pi}{4} \right) \, \sin\left( \frac{\omega_A P}{2} \right) \right.$$
$$\left. - i\omega \sin\left( \omega_A (t - \frac{P}{2}) - \frac{3\pi}{4} \right) \, \sin\left( \frac{\omega_A P}{2} \right) \right]. \quad (51)$$

As in the case of the $\delta(t)-$pulse excitation considered in the previous section, there are only free oscillations present in the atmosphere after long times ($t >> z/c_S$) and they decay in time as $t^{-3/2}$. The main difference is that the velocity $v(t,z)$ is a function of the pulse frequency $\omega$ (with $\omega = 2\pi/P$) and, as a result, the factor $1/(\omega^2 - \omega_A^2)$ is present in Eq. (51). An interesting case is when $\omega \to \omega_A$, which corresponds to $P \to P_A$. As shown in Appendix B and discussed in Sect. 2, in this case, the limit $t \to \infty$ must be enforced, which means that no free oscillations exist in the atmosphere.

## 7. Excitation by a wavetrain of random pulses

Finally, we consider the response of the atmosphere to a wavetrain of sinusoidal pulses with randomly chosen amplitudes and periods. The wavetrain is introduced at the lower boundary of the atmosphere where after each passed wave period a new period and a new amplitude are stochastically chosen. This boundary condition requires $A_0(t)$ to be given in the following form:

$$A_0(t) \;=\; \sum_{n=1}^{\infty} V_n \, e^{-i\omega_n t} \; [\mathcal{H}(t - t_{n-1}) - \mathcal{H}(t - t_n)] \,, \quad (52)$$

where $V_n$ and $\omega_n$ are randomly chosen wave amplitudes and periods, respectively. In addition, we have

$$t_n \;=\; \sum_{i=0}^{n} T_i \qquad , \qquad t_{n-1} \;=\; \sum_{i=0}^{n-1} T_i, \quad (53)$$

and

$$T_0 \;=\; 0 \qquad , \qquad T_i \;=\; \frac{2\pi}{\omega_i}. \quad (54)$$

Using Eq. (18), one obtains the following expression for the velocity distribution in the atmosphere

$$v(z,t) \;=\; \sum_{n=1}^{\infty} V_n \, e^{-i\omega_n (t - z/c_S)}$$
$$\times \left( \mathcal{H}(t - t_{n-1} - z/c_S) - \mathcal{H}(t - t_n - z/c_S) \right)$$
$$+ \sum_{n=1}^{\infty} V_n \; \underbrace{\int_0^t e^{-i\omega_n (t-\tau)} \, \mathcal{H}(t - t_{n-1} - \tau) \, W(\tau, z) \, d\tau}_{I_{71}}$$
$$- \sum_{n=1}^{\infty} V_n \; \underbrace{\int_0^t e^{-i\omega_n (t-\tau)} \, \mathcal{H}(t - t_n - \tau) \, W(\tau, z) \, d\tau}_{I_{72}}. \quad (55)$$

Let us first consider the integrals $I_{71}$ and $I_{72}$ for times $t > t_n$. The integration limits can be changed according to the following rule:

$$I_{71} \;-\; I_{72} \;=\; \int_0^{t-t_{n-1}} \;+\; \int_{t-t_{n-1}}^{t} \;-\; \int_0^{t-t_n} \;-\; \int_{t-t_n}^{t}$$
$$= \int_0^{t-t_{n-1}} + \int_{t-t_{n-1}}^{\infty} - \int_t^{\infty} - \int_0^{t-t_n} - \int_{t-t_n}^{\infty} + \int_t^{\infty}. \quad (56)$$

Since both integrals from $t$ to $\infty$ have different signs they cancel each other. In addition, the integrals from $0$ to $t - t_n$ and from $0$ to $t_{n-1}$ will disappear because of the properties of the Heaviside step functions (see discussion in the previous section). Thus we have

$$I_{71} \;-\; I_{72} \;=\; \int_{t-t_{n-1}}^{\infty} \;-\; \int_{t-t_n}^{\infty}. \quad (57)$$

These integrals can again be evaluated asymptotically by using the procedure outlined in Appendix B. Note that the asymptotic expansion is valid only for large times $t >> z/c$ which means that the generating wavetrain has already passed height $z$ in the atmosphere.

Thus, in the asymptotic limit, we obtain

$$I_{71} \;=\; \sqrt{\frac{2\omega_A}{\pi}} \, \frac{z}{c_S} \, \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \omega_A^2} \, \frac{1}{(t - t_{n-1})^{3/2}}$$
$$\times \left[ \omega_A \sin\left( \omega_A (t - t_{n-1}) - \frac{3\pi}{4} \right) \right.$$
$$\left. + i\omega_n \cos\left( \omega_A (t - t_{n-1}) - \frac{3\pi}{4} \right) \right], \quad (58)$$

and

$$I_{72} = \sqrt{\frac{2\omega_A}{\pi}} \, \frac{z}{c_S} \, \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \omega_A^2} \, \frac{1}{(t - t_n)^{3/2}} \quad (59)$$
$$\times \left[ \omega_A \sin\left( \omega_A (t - t_n) - \frac{3\pi}{4} \right) + i\omega_n \cos\left( \omega_A (t - t_n) \frac{3\pi}{4} \right) \right]$$

For large times $t >> t_{n-1}$ and $t >> t_n$ one can expand

$$\frac{1}{(t - t_{n-1})^{3/2}} \;\approx\; \frac{1}{t^{3/2}} \, \left( 1 + \frac{3}{2} \, \frac{t_{n-1}}{t} \right), \quad (60)$$

and

$$\frac{1}{(t - t_n)^{3/2}} \;\approx\; \frac{1}{t^{3/2}} \, \left( 1 + \frac{3}{2} \, \frac{t_n}{t} \right). \quad (61)$$

Taking only the first term in each expansion (see Appendix B), and using trigonometric relationships for difference between two sine and two cosine functions of different arguments, we write the final expression describing the behavior of the velocity in the atmosphere

$$v(t,z) \;=\; \sqrt{\frac{2\omega_A}{\pi}}\;\frac{z}{c_S}\;\sum_{n=1}^{N} V_n\;\frac{1}{\omega_n^2-\omega_A^2}\;\frac{2}{t^{3/2}}$$

$$\times\left[\omega_A\cos(\omega_A t-\varphi_1)\;-\;i\omega_n\sin(\omega_A t-\varphi_1)\,\right]\sin\varphi_2, \qquad (62)$$

where $N$ is chosen in such a way that $t > t_N$, and

$$\varphi_1 \;=\; \frac{1}{2}\omega_A(t_n+t_{n-1})+\frac{3\pi}{4}, \qquad (63)$$

$$\varphi_2 \;=\; \frac{1}{2}\omega_A\Delta t_n, \qquad (64)$$

with $\Delta t_n = t_n - t_{n-1}$.

The obtained result is similar to that found for the excitation by a sinusoidal pulse (see Eq. 51) because it shows that only free atmospheric oscillations are described by Eq. (62). This is a rather surprising result because the excitation is continuous (see Eq. 52) and, therefore, one would expect the presence of terms in Eq. (62) that also describe the forced atmospheric oscillations (see, for example, Eqs. 27 and 36). To explain this result, we recall that during our derivation of Eq. (62) the summation from $n = 1$ to $\infty$ in Eq. (52) which describes a continuous wavetrain was replaced by the summation from $n = 1$ to $N$, with $N$ being chosen in such a way that $t > t_N$. This means that at the time $t > t_N$ the excitation is stopped after $N$ number of pulses are generated. This wavetrain with finite number of pulses propagates through the atmosphere and excites both forced and free oscillations. For the time, however, when the analytical solution is valid ($t >> t_{n-1}, t >> t_n$ and $t >> z/c_S$), the wavetrain and the resulting forced atmospheric oscillations are no longer present in the atmosphere. Hence, the analytical solution describes only the free atmospheric oscillations that had been generated by the finite wavetrain that already passed through the atmosphere.

As shown by Eq. (62), the induced free atmospheric oscillations decay in time as $t^{-3/2}$, which is the same dependence as found for all other excitation mechanisms discussed in this paper. All pulses with $\omega_n \neq \omega_A$ have the same temporal behavior. However, according to the results presented in the previous section, pulses generated with $\omega_n = \omega_A$ do not contribute to the oscillations and they require a special analytical treatment (see Appendix B). Some application of these results to the Sun can be found in the next section.

## 8. Applications to the Sun

We consider an isothermal atmosphere with solar parameters, gravity $g = 2.376 \cdot 10^4$ cm/s$^2$ and temperature $T_o = 5000$ K, extending from $z = 0$ to $z = 2000$ km, which corresponds to approximately 17 scale heights. Taking $\gamma = 5/3$, we find $c_S = 7.3$ km/s and the cutoff frequency $\omega_A = 0.0312\ s^{-1}$. Since the excitation by monochromatic acoustic waves and $\delta$-function pulses have been extensively discussed in the literature (e.g., Fleck & Schmitz 1991, 1993; Kalkofen et al. 1994; Schmitz & Fleck 1995; Sutmann & Ulmschneider 1995a,b), here a special emphasis is given to the excitation by a spectrum of partial waves, by a sinusoidal pulse and by a wavetrain of random pulses. We show only the results for free atmospheric oscillations by plotting the variation of the real part of the velocity $v(t,z)$ with time at two different atmospheric heights, $z = 500$ and $2000$ km; note that in all calculations the conditions $t >> z/c_S$ and $t >> 1/|\omega_A \pm \omega|$ are always fulfilled. In case of the excitation by the spectrum of partial waves and by the wavetrain of random pulses, we also show the time evolution of the perturbation at the lower atmospheric boundary $z = 0$.

### 8.1. Spectrum of partial waves

Let us consider the free atmospheric oscillations described by Eq. (36). To plot the variation of $v(t,z)$ with time in the atmosphere, we have to prescribe the spectrum of partial waves that enters the atmosphere at the lower boundary $z = 0$. According to Eq. (29), the spectrum of partial waves is specified when the wave amplitudes $V_n$, wave frequencies $\omega_n$ and wave phases $\varphi_n$ are known. To specify $V_n$, $\omega_n$ and $\varphi_n$, we follow the procedure originally introduced by Huang, Musielak & Ulmschneider (1994) and recently modified by Ulmschneider & Musielak (1998), who investigated the generation of nonlinear magnetic tube waves in the solar atmosphere by a spectrum of $N = 100$ partial waves. The basic steps of this procedure are outlined in Appendix C. Taking the real part of Eq. (29), we may write

$$A_0(t) = \sum_{n=1}^{N} V_n\cos(\omega_n t+\varphi_n), \qquad (65)$$

where $V_n$ is determined by the turbulent energy spectrum (see Appendix C), $\omega_n$ are numbers ranging from $0.0065$ to $0.2985$ Hz with the step $\Delta\omega_n = \omega_A$, and $\varphi_n = 2\pi r_n$ is an arbitrary but constant phase angle with $r_n$ being a random number in the interval [0, 1]. Note that none of the considered partial waves has its frequency equal to the natural frequency $\omega_A$.

The time evolution of $A_0(t)$ entering the atmosphere at the height $z = 0$ is shown in Fig. 1. The results are obtained for a prescribed rms turbulent velocity $u_t = 1$ km/s (see Appendix C) and they show a very stochastic variation of $A_0$ with time. In Fig. 2, we present the $t^{-3/2}$ decay in time of the free atmospheric oscillations at two different heights in the atmosphere; note that only the real part of the term describing these oscillations in Eq. (36) is plotted. It is seen that the amplitude of these oscillations increases linearly with height, as required by Eq. (36), and that the calculations start at later times for higher values of $z$ to satisfy the condition $t >> z/c_S$. Comparing these results to those previously obtained by Kalkofen et al. (1994) and Sutmann & Ulmschneider (1995a,b), we find that the response of the atmosphere to the presence of the spectrum of partial waves is the same as in the case of the excitation by monochromatic acoustic waves and $\delta(t)$-function pulses.

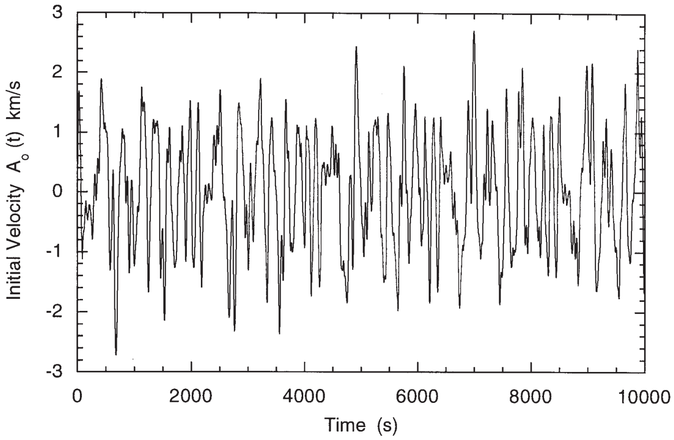


**Fig. 1.** Time evolution of the real part of the velocity fluctuation $A_0(t)$, computed from Eq. (29) and applied at the lower boundary $z = 0$ in the atmosphere.

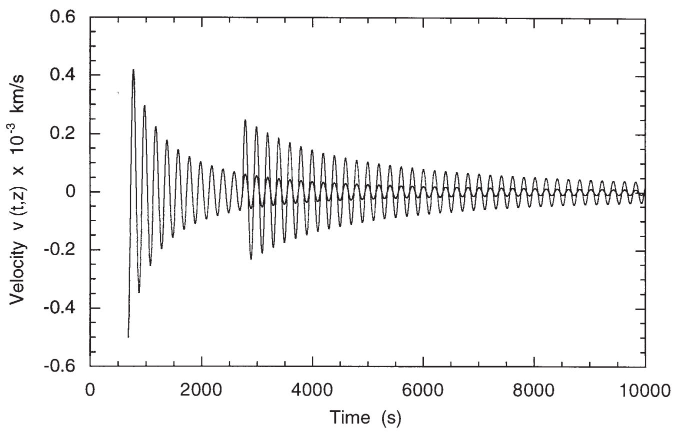


**Fig. 2.** Time evolution of the free atmospheric oscillations generated by a spectrum of 100 partial waves with random phases. We plot the real part of $v(t,z)$ at two different atmospheric heights $z = 500$ and 2000 km.

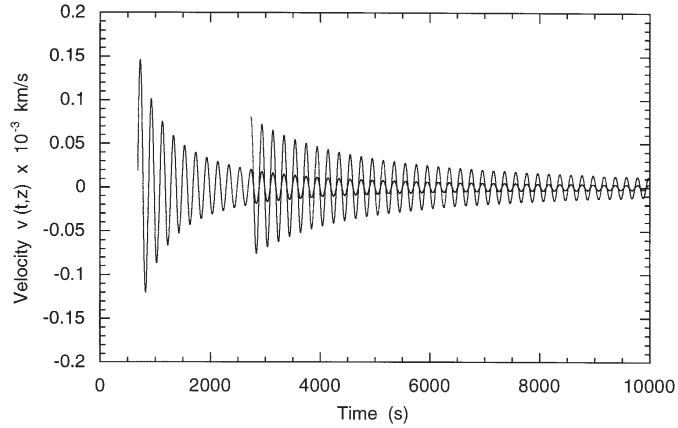


**Fig. 3.** Time evolution of the free atmospheric oscillations generated by a sinusoidal pulse with period $P = 50$ s. We plot the real part of $v(t,z)$ at two different atmospheric heights $z = 500$ and 2000 km.

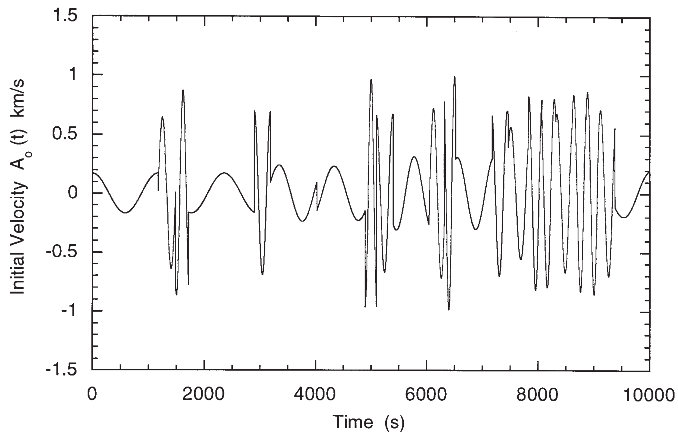


**Fig. 4.** Time evolution of the real part of $A_0(t)$ computed by Eq. (52) at the lower boundary $z = 0$ of the atmosphere.

### *8.2. Sinusoidal pulse*

As the next case, we consider a sinusoidal pulse with frequency $\omega = 2\pi/P$, where $P \neq P_A$, and $V_0 = 1$ km/s as described by Eq. (42) and introduced at height $z = 0$. The response of the atmosphere to this pulse is plotted in Fig. 3; note that only the real part of $v(t,z)$. The results are shown at two different atmospheric heights, namely, $z = 500$ and 2000 km, and the $t^{-3/2}$ dependence is clearly seen.

By comparing these results with those given in Fig. 2, we see that in both cases the time behavior is very similar despite the fact that quite different forms of the initial disturbances are imposed on the atmosphere. However, some differences in amplitudes of the induced free atmospheric oscillations can be seen, namely, the amplitudes of the oscillations generated by the spectrum of partial waves are more than twice times higher than those generated by the sinusoidal pulse. The main reason for this effect is that amplitudes of some partial waves are higher (see Fig. 1) than the amplitude of the sinusoidal pulse which is normalized to unity.

### *8.3. Wavetrain of finite number of random pulses*

Finally, we consider a wavetrain of pulses where both amplitudes and frequencies are chosen randomly. The form of this wavetrain at the atmospheric height $z = 0$ is shown in Fig. 4; note that only the real part of Eq. (52) is plotted. As described in Sect. 7, the wavetrain has a finite number of pulses (determined by $t > t_N$) and a new pulse, with new period and amplitude, is introduced after each passed wave period. The shape of all introduced pulses is sinusoidal. Pulses with $\omega_n = \omega_A$ are not included in these calculations.

Since the wavetrain is finite, it propagates through the atmosphere in a finite time and after that time only free atmospheric oscillations are present (see Eq. 62). The results are plotted in Fig. 5, which shows that each pulse obeys the $t^{-3/2}$ rule of decaying in time. It is seen that the free oscillations are sustained in the atmosphere, at least during the time of 10000 s adopted for our calculations. Clearly, if the random pulses were contin-

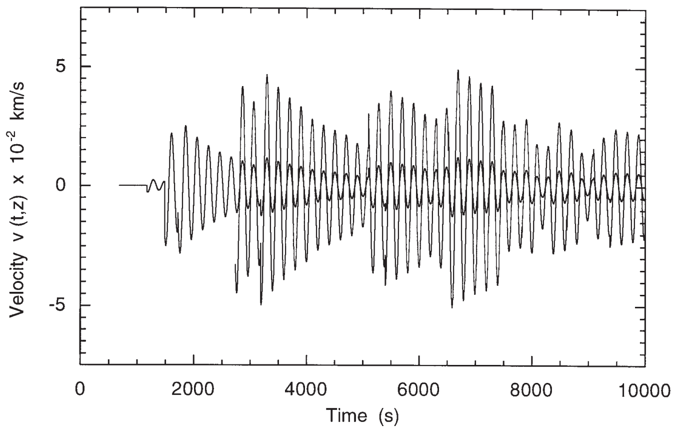


**Fig. 5.** Time evolution of the free atmospheric oscillations generated by a wavetrain of finite number of random pulses. We plot the real part of $v(t,z)$ at two different atmospheric heights $z = 500$ and 2000 km.

uously generated, which seemed to be the most realistic case for the solar atmosphere, then the free atmospheric oscillations would be permanently present in the atmosphere despite their $t^{-3/2}$ decaying time. Because the frequency of the free atmospheric oscillations is the natural frequency of the atmosphere ($\omega_A$), wavetrains of random pulses which continuously propagate through the solar chromosphere may be responsible for the observed 3-min oscillations.

## 9. Conclusions

The main conclusions of this paper can be summarized as follows:

1. Upon the excitation of an isothermal atmosphere with prescribed acoustic disturbances one always obtains free atmospheric oscillations with the acoustic cutoff frequency.
2. The free atmospheric oscillations decay in time as $t^{-3/2}$ if the frequency $\omega$ of the driving waves is not equal to the natural frequency $\omega_A$. This time dependence holds for all considered mechanisms of excitation, namely, monochromatic acoustic waves, a spectrum of partial waves, $\delta$-function pulses, sinusoidal pulses and for a wavetrain with finite number of random pulses.
3. In case of the continuous excitation the forced atmospheric oscillations are always present and after a certain time they become dominant in the atmosphere. This time depends on the excitation frequency and for evanescent waves it is often very large.
4. In the limit $\omega \to \omega_A$, the derived analytical results are correct only if the limit $t \to \infty$ is enforced. For the special case of the continuous excitation with $\omega = \omega_A$, the resulting free atmospheric oscillations are the same as the forced atmospheric oscillations and they are not decaying in time.
5. Applying our analytical results to the Sun, we find that in the case of the excitation by wavetrains of random pulses, atmospheric oscillations are a persistent feature in the solar chromosphere. Such wave trains may be responsible for the observed 3-min chromospheric oscillations.

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## Appendix A: integral representation of the exponential factor

As indicated in the main text, the exponential factor of Eq. (12) can be represented by another exponential factor and an integral that involves the Bessel function $J_1$ (see Eq. 13). To formally derive this relationship, we begin with the following integral

$$I \equiv \int_0^\infty \frac{x J_0(\lambda x)}{\sqrt{x^2+a^2}}\, e^{ik\sqrt{x^2+a^2}}\, dx. \tag{A1}$$

Using *Tables of Summable Series and Integrals Involving Bessel Functions* (see Wheelon 1968), the integral can be evaluated analytically and the result is

$$I = \frac{e^{-a\sqrt{\lambda^2-k^2}}}{\sqrt{\lambda^2-k^2}}, \tag{A2}$$

valid for $\lambda^2 > k^2$ and $a > 0$. Defining $t = \sqrt{x^2+a^2}$ and $s = -ik$, we can rewrite the integral as

$$I = \int_a^\infty J_0(\lambda\sqrt{t^2-a^2})\, e^{-st}\, dt = \frac{e^{-a\sqrt{\lambda^2+s^2}}}{\sqrt{\lambda^2+s^2}}. \tag{A3}$$

After differentiation of both sides in Eq. (A3) with respect to a, we obtain

$$\frac{dI}{da} = -\lambda a \int_a^\infty \frac{J_0'(\lambda\sqrt{t^2-a^2})}{\sqrt{t^2-a^2}}\, e^{-st}\, dt - e^{-sa}$$
$$= -e^{-a\sqrt{\lambda^2+s^2}}. \tag{A4}$$

Since $J_0'(x) = -J_1(x)$ and taking $a = z/c_S$ and $\lambda = \omega_A$, we may write

$$e^{-\sqrt{s^2+\omega_A^2}\, z/c_S} = e^{-sz/c_S} - \frac{\omega_A z}{c_S}\int_{z/c_S}^\infty \frac{J_1(\omega_A\sqrt{t^2-(z/c_S)^2})}{\sqrt{t^2-(z/c_S)^2}} e^{-st} dt, \tag{A5}$$

which is the same as Eq. (13) in the main text.

## Appendix B: asymptotic evaluation of integrals

In the following we show how to evaluate asymptotically the integral $I_{32}$ from the main text. This result can also be used to evaluate asymptotically the integrals $I_{42}$, $I_{62}$ and $I_{72}$, because their form is similar. According to Eq. (23), the integral $I_{32}$ is of the form

$$I = -\int_t^\infty W(\tau,z) e^{-i\omega(t-\tau)} d\tau$$

$$= C_0 \int_t^\infty \frac{J_1(\omega_A \sqrt{\tau^2 - (z/c_S)^2})}{\sqrt{\tau^2 - (z/c_S)^2}} \mathcal{H}(\tau - z/c_S) e^{i\omega\tau} d\tau, \quad \text{(B1)}$$

where

$$C_0 \;=\; \frac{\omega_A \; z}{c_S} \; e^{-i\omega t}. \quad \text{(B2)}$$

Assuming that $\tau >> z/c_S$, which is equivalent to $t >> z/c_S$ because $\tau$ changes from $t$ to $\infty$, we may approximate the integral $I$ as

$$I \;\approx\; C_0 \int_t^\infty \frac{J_1(\omega_A \tau)}{\tau} \; e^{i\omega\tau} \; d\tau. \quad \text{(B3)}$$

Since $\tau$ is large, the Bessel function $J_1$ can be expanded for large arguments (Bronshtein & Semendyayev 1985) according to the following formula:

$$J_1(\tau) \;\approx\; \sqrt{\frac{2}{\pi \omega_A \tau}} \left[ \cos \left( \omega_A \tau - \frac{3\pi}{4} \right) + O\left( \frac{1}{\tau} \right) \right]. \quad \text{(B4)}$$

This gives

$$I \;=\; C_0 \sqrt{\frac{2}{\pi \omega_A}} \int_t^\infty \cos \left( \omega_A \tau - \frac{3\pi}{4} \right) \; e^{-i\omega\tau} \; \frac{d\tau}{\tau^{3/2}}. \quad \text{(B5)}$$

To evaluate this integral, we use the exponential representation of the cosine function in Eq. (B5) and write

$$I \;=\; \frac{1}{2} \sqrt{\frac{2\omega_A}{\pi}} \; \frac{z}{c_S} \left[ e^{-i(\omega t + 3\pi/4)} \int_t^\infty \tau^{-3/2} \; e^{i(\omega_A + \omega)\tau} \; d\tau \right.$$

$$\left. + \; e^{-i(\omega t - 3\pi/4)} \int_t^\infty \tau^{-3/2} \; e^{-i(\omega_A - \omega)\tau} \; d\tau \right]. \quad \text{(B6)}$$

Now, we define

$$I_1 \;=\; \int_t^\infty \tau^{-3/2} \; e^{i(\omega_A + \omega)\tau} \; d\tau, \quad \text{(B7)}$$

and

$$I_2 \;=\; \int_t^\infty \tau^{-3/2} \; e^{-i(\omega_A - \omega)\tau} \; d\tau. \quad \text{(B8)}$$

As the next step, we integrate both integrals by parts. The obtained results are similar for both integrals, so we present only the result for $I_1$

$$I_1 = \frac{-1}{i(\omega_A + \omega) t^{3/2}} \; e^{i(\omega_A + \omega)t} + \frac{3}{2} \int_t^\infty \tau^{-5/2} \; e^{i(\omega_A + \omega)\tau} d\tau$$

$$= \; \frac{-1}{i(\omega_A + \omega) t^{3/2}} \; e^{i(\omega_A + \omega)t} \left[ 1 \; + \; \frac{3}{2} \; \frac{1}{i(\omega_A + \omega)t} \right]$$

$$- \; \frac{15}{4} \int_t^\infty \tau^{-7/2} \; e^{i(\omega_A + \omega)\tau} \; d\tau. \quad \text{(B9)}$$

The integration by parts can be continued and as a result the following series is obtained: $1 + 3 \times [2i(\omega_A + \omega)t]^{-1} - 15 \times [4(\omega_A + \omega)t]^{-2}$+*higher order terms*. For the integral $I_2$, we find: $1 - 3 \times [2i(\omega_A - \omega)t]^{-1} - 15 \times [4(\omega_A - \omega)t]^{-2}$+*higher order terms*, where each term is multiplied by $[\exp(-i(\omega_A - \omega)t)/i(\omega_A - \omega)t^{3/2}]$. If $\omega$ is not too close to $\omega_A$, then both series converge very quickly for sufficiently large values of $t$ and the only important term is the first one. Since $\omega = \omega_A$ are the poles of the second series, it is then obvious that $t \to \infty$ if $\omega \to \omega_A$; note that the first term in this series is also affected by this limit because it has the term $(\omega_A - \omega)$ in the denominator. The special case of $\omega = \omega_A$ is treated separately at the end of this section.

All results presented in this paper are obtained by considering **only** the first term in the above series. This assumption can be justified by the fact that only large values of $t$ are considered, namely, $t >> z/c_S$ and $t >> 1/|\omega_A \pm \omega|$. It must be noted, however, that both series are obtained by using only the principal asymptotic form for the expansion of the Bessel function $J_1$ (see Eq. B4). Clearly, the form of the second, third and higher order terms in these series will be different if $J_1$ is expanded by using Hankel's asymptotic form (see Abramowitz & Stegun 1980); the latter would be necessary if these higher order terms would also be taken into account.

After taking only the first term in each of the above series, we can write

$$I_1 \;=\; -\frac{1}{t^{3/2}} \; \frac{e^{i(\omega_A + \omega)t}}{i(\omega_A + \omega)}, \quad \text{(B10)}$$

and

$$I_2 \;=\; \frac{1}{t^{3/2}} \; \frac{e^{-i(\omega_A - \omega)t}}{i(\omega_A - \omega)}. \quad \text{(B11)}$$

After combining Eqs. (B10) and (B11) with Eq. (B6), we obtain

$$I \;=\; \sqrt{\frac{2\omega_A}{\pi}} \; \frac{1}{\omega^2 - \omega_A^2} \; \frac{z}{c_S} \; \frac{1}{t^{3/2}}$$

$$\times \left[ \omega_A \sin \left( \omega_A t - \frac{3\pi}{4} \right) + i\omega \cos \left( \omega_A t - \frac{3\pi}{4} \right) \right]. \quad \text{(B12)}$$

This is our final expression for the integral $I_{32}$ (see Eq. 26 in the main text). The result will be also used to evaluate asymptotically the integrals $I_{42}$, $I_{62}$ and $I_{72}$ because they are of the same form as the integral $I_{32}$.

We now consider the special case when $\omega = \omega_A$. The integrals $I_1$ and $I_2$ can be written as

$$I_1 \;=\; \int_t^\infty \tau^{-3/2} \; e^{2i\omega_A \tau} \; d\tau, \quad \text{(B13)}$$

and

$$I_2 \;=\; \int_t^\infty \tau^{-3/2} \; d\tau. \quad \text{(B14)}$$

This gives

$$I_1 \;=\; -\frac{1}{t^{3/2}} \; \frac{e^{2i\omega_A t}}{2i\omega_A}, \quad \text{(B15)}$$

and

$$I_2 \;=\; \frac{2}{t^{1/2}}. \quad \text{(B16)}$$

Combining Eqs. (B15) and (B16), and using (B6) get

$$I = \sqrt{\frac{2\omega_A}{\pi}} \frac{z}{c_S} \frac{1}{t^{1/2}}$$
$$\times \left[\cos\left(\omega_A t - \frac{3\pi}{4}\right) - i\sin\left(\omega_A t - \frac{3\pi}{4}\right)\right]$$
$$+\sqrt{\frac{2\omega_A}{\pi}} \frac{1}{4\omega_A} \frac{z}{c_S} \frac{1}{t^{3/2}}$$
$$\times \left[i\cos\left(\omega_A t - \frac{3\pi}{4}\right) - \sin\left(\omega_A t - \frac{3\pi}{4}\right)\right]. \quad \text{(B17)}$$

This result was used to obtain Eq. (28) in the main text.

## Appendix C: evaluation of amplitudes of partial waves

According to Eq. (67) in the main text, we have

$$A_0(t) = \sum_{n=1}^{N} V_n \cos(\omega_n t + \varphi_n), \quad \text{(C1)}$$

where $V_n, \omega_n$ and $\varphi_n$ are the amplitudes, frequencies and phases of partial waves. The procedure of choosing $\omega_n$ and $\varphi_n$ is described in Sect. 8. Here, however, we show how to determine the amplitudes $V_n$ (see Huang, Musielak & Ulmschneider 1995). Time averaging $A_0^2(t)$, we find

$$\overline{A_0^2} = \frac{1}{2}\sum_{n=1}^{N} V_n^2 \ = \ u_t^2 \ , \quad \text{(C2)}$$

where $u_t$ is the rms turbulent velocity. As described by Huang et al. (see also Musielak et al. 1995), the turbulent energy spectrum, $E(k)$, and the frequency factor, $\Delta(\omega/ku_k)$ are normalized to

$$\frac{3}{2}u_t^2 = \int_0^\infty d\omega \int_0^\infty dk\ E(k)\Delta\left(\frac{\omega}{ku_k}\right) = \int_0^\infty E'(\omega)d\omega\ . \quad \text{(C3)}$$

Now, we decompose $u_t$ into the spectral components by writing

$$\frac{3}{2}u_t^2 = \frac{3}{4}\sum_{n=1}^{N} V_n^2 = \int_0^\infty E'(\omega)d\omega = \sum_{n=1}^{N} E'(\omega_n)\Delta\omega\ , \quad \text{(C4)}$$

from which we have

$$V_n = \sqrt{\frac{4}{3}E'(\omega_n)\Delta\omega}\ , \quad \text{(C5)}$$

with

$$E'(\omega_n) = \int_0^\infty E(k)\Delta\left(\frac{\omega_n}{ku_k}\right) dk\ . \quad \text{(C6)}$$

To specify the turbulent energy spectra appropriate for the solar convection zone, where the spectrum of partial waves is generated, we follow Musielak et al. (1994). These authors argue on the basis of observations and numerical simulations of the solar convection that a realistic turbulent energy spectrum should reasonably well be described by an extended Kolmogorov spectrum $E(k)$ and a modified Gaussian frequency factor $\Delta(\frac{\omega}{ku_k})$. The extended Kolmogorov spatial spectrum is given by

$$E(k) = \begin{cases} 0 & 0 < k < 0.2k_t \\ a\frac{u_t^2}{k_t}\left(\frac{k}{k_t}\right) & 0.2k_t \le k < k_t \\ a\frac{u_t^2}{k_t}\left(\frac{k}{k_t}\right)^{-5/3} & k_t \le k \le k_d \end{cases}, \quad \text{(C7)}$$

where the factor $a = 0.758$ is determined by the normalization condition

$$\int_0^\infty E(k)dk = \frac{3}{2}u_t^2\ . \quad \text{(C8)}$$

The modified Gaussian frequency factor is described by

$$\Delta\left(\frac{\omega}{ku_k}\right) = \frac{4}{\sqrt{\pi}}\frac{\omega^2}{|ku_k|^3}e^{-(\frac{\omega}{ku_k})^2}\ , \quad \text{(C9)}$$

where $u_k$ is computed from

$$u_k = \left[\int_k^{2k} E(k')dk'\right]^{1/2}\ . \quad \text{(C10)}$$

Here $k_t = 2\pi/H$, with $H$ being the density (pressure) scale height.

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