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Hydrodynamic approach to cosmic ray propagation

II. Nonlinear test particle picture in a shocked background

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Abstract. In this article, cosmic ray acceleration is studied in a nonlinear test particle picture in the hydrodynamic approach. The model includes cosmic rays and two oppositely propagating Alfvén waves. It is a nonlinear model because the interactions between the cosmic rays, waves and background plasma depend on the waves themselves. The background is a simple fast shock. Besides the parameters characterising the nonlinear test particle picture, two more parameters are required for the shock, namely, the compression ratio and the location of the shock. In a uniform super-Alfvénic background cosmic rays can be accelerated. When the downstream of the background undergoes a fast shock transition, the cosmic ray may be accelerated further. We find that the cosmic ray energy flux far downstream remains unchanged when a shock is introduced. We define the efficiency of shock acceleration in terms of the difference in far downstream cosmic ray pressure between a uniform background and a shocked background with the same far upstream pressure. Depending on the parameters, the efficiency can take positive and negative values. It is possible that the cosmic ray pressure far downstream of the shock may be less than the pressure without the shock. We also comment on the role of the shock in accelerating particles.

Key words: diffusion – shock waves – acceleration of particles – ISM: cosmic rays

1. Introduction

Cosmic rays are energetic charged particles. They are coupled to a highly conducting plasma via hydromagnetic waves or irregularities by e.g. gyro-resonant scattering. As a result they are advected and diffused in the plasma. Diffusion in momentum space or stochastic acceleration occurs if waves of different phase speeds are present. Advection, spatial diffusion and stochastic acceleration all depend on the strength of coupling between the cosmic rays and waves, which depends on the waves themselves. When cosmic rays stream through the magnetized plasma, they generate hydromagnetic waves by the so called cosmic ray streaming instability. These waves serve as the scattering centres for the cosmic rays. Together with the backreaction of the cosmic rays and waves on the thermal plasma, a self-consistent model for the cosmic-ray–plasma system can be established.

One of the major applications of the theory of cosmic ray propagaion is cosmic ray acceleration. Among all possible mechanisms, shock acceleration is probably the best-known urrent idea is that most of the cosmic rays (say, from 10^9 to 10^{15} eV/nucleon) in our Galaxy are generated by supernova remnant shocks. The "effficiency" of shock acceleration is rather high (at least for plane shocks). The plasma flow may then be seriously modified by the cosmic rays. For various aspects of shock acceleration and cosmic ray modification, the reader is referred to reviews by Drury (1983), Blandford & Eichler (1987), Berezhko & Krymskii (1988), Jones & Ellison (1991) and Ko (1995).

To study the dynamics or structure of the cosmic-ray–plasma system, the odynamic approach is a fairly good approximation (see e.g., Drury & Völk 1981; Axford et al. 1982; McKenzie & Völk 1982; Ko 1992). In this approach all ingredients are treated as fluids. For instance, the model by Ko (1992) comprises four fluids: thermal plasma, cosmic rays and two oppositely propagating Alfvén waves. Cosmic rays and Alfvén waves are considered as massless fluids but with significant pressures or energy densities. In general, there are some prescribed parameters known as the closure parameters, such as the ratio of the ic ray energy density to its pressure (or polytropic index), the coupling strength between waves and cosmic rays, etc. (see e.g., Duffy et al. 1994; Jiang et al. 1996, Paper I of the series and will be refered to JCK hereafter).

Of course, the hydrodynamic approach has its limitations. For comments on the approach, the reader is referred to, e.g., Heavens (1984), Achterberg et al. (1984), Jones & Kang (1990), Jones & Ellison (1991), and Ko (1995). We should point out that the origin of some of its problems was elucidated by Malkov (1997a, b). Basically, the hydrodnamic approach is a singular limiting case of the more vigorous distribution function approach, in which cosmic rays are described by a phase space distribution function while the plasma is still in fluid description.

The steady state of one-dimensional cosmic-ray–modified shocks had been studied quite thoroughly in the hydrodynamic

Send offprint requests to: Chung-Ming Ko approach. Drury & Völk (1981) and Axford et al. (1982) worked out in detail the structure of the shock without waves. Still considering a system without waves, Ko et al. (1997) studied the structure and efficiency of the shock with a simple model of injection. McKenzie & Völk (1982) worked out the structure of the shock with backward propagating Alfvén waves. Stochastic acceleration is absent in all of these systems.

To study stochastic acceleration, both forward and backward propagating waves are required. Ko (1992) considered a four-fluid model which included two oppositely propagating Alfvén waves. However, we must be cautious about the notion of two oppositely propagating waves. When the two waves interact strongly, the system will evolve into the strong turbulence regime. The model of Ko (1992) may then be questionable, because his model was based on a propagation equation derived from quasi-linear theory (which is valid only in the weak turbulence regime). To investigate the range of validity, we should examine the time scales of turbulence cascade, of wave generation (say by cosmic-ray streaming instability), and of cosmic ray acceleration (say by stochastic acceleration). This is outside the scope of the present discussion and we adopt the model of Ko (1992) for simplicity.

The four-fluid model is fairly complex and a comprehensive discussion on the full 3Dsystem has yet to come. However, some simplified versions dedicated to stochastic acceleration has been investigated. If spatial homogeneity is assumed, it can be shown that the system is always driven towards a state with only one wave as expected (see Ko 1992). Nevertheless, we have to point out that such uni-directional wave systems are susceptible to cosmic-ray driven magneto-acoustic instability (McKenzie & Webb 1984; Zank 1989; Ko & Jeng 1994). JCK negected the backreaction of the cosmic ray and the two Alfvén waves on the thermal plasma flow, and analysed the so called nonlinear test particle picture. They found that in some cases stochastic acceleration is dominant, while in some other cases it may just act as a trigger or catalyst and the work done by the background plasma flow contributes most to the acceleration of cosmic ray (see end of Sect. 2).

There are several energy exchange mechanisms in the cosmic-ray plasma system: work done by the plasma flow, the cosmic-ray streaming instability, or the stochastic acceleration. If there is a shock in the flow, shock acceleration is also expected. We would like to compare all these mechanisms, and hopefully find the efficiency of shock acceleration in the presence of waves and stochastic acceleration. Using the distribution function approach, Campeanu & Schlickeiser (1992), Ostrowski & Schlickeiser (1993) and Ostrowski (1994) obtained some interesting results on the stochastic acceleration at shock waves. In this article, we extend the nonlinear test particle picture of JCK to shocked background. The nonlinear test particle picture is briefly reviewed in Sect. 2. In Sect. 3, we describe the model with shocked background and the appropriate boundary conditions at the shock. Some results are discussed in Sect. 4, in particular shock acceleration and its efficiency. Sect. 5 provides a summary.

Before we go on, let us define some parameters for later use.

$P_{\rm c}$	cosmic ray pressure
$F_{\rm c}$	cosmic ray energy flux
$P_{\rm w}^+$	forward propagating Alfvén wave pressure
$F_{\rm w}^+$	forward propagating Alfvén wave energy flux
$P_{\rm w}^-$	backward propagating Alfvén wave pressure
$F_{\rm w}^-$	backward propagating Alfvén wave energy flux
U	thermal plasma flow speed
$V_{\rm A}$	Alfvén speed
$M_{\rm A}$	Alfvén Mach number
$W_{\rm A}$	wave-action integral
S	streaming integral
q	compression ratio of the shock
σ	location parameter of the shock

The speeds and pressures are normalized to arbitrary constants $U_\circ$, $P_\circ$, while energy fluxes and the two integrals are normalized to $U_\circ P_\circ$. The upstream and downstream quantities are labelled by subscripts $_1$ and $_2$, respectively.

Moreover, we will touch upon six systems:

$\mathcal{F}_{\rm u}$	uniform background with forward wave only
$\mathcal{B}_{\rm u}$	uniform background with backward wave only
$\mathcal{T}_{\rm u}$	uniform background with both waves
$\mathcal{F}_{\rm s}$	upstream of shock with forward wave only
$\mathcal{B}_{\rm s}$	upstream of shock with backward wave only
$\mathcal{T}_{\rm s}$	upstream of shock with both waves

2. The nonlinear test particle picture

The interaction of cosmic rays and magnetized plasma is mediated by the magnetic field. Cosmic rays are scattered by the hydromagnetic waves or turbulence in the plasma. While they advect with and diffuse through the plasma, they excite hydromagnetic waves via cosmic-ray streaming instability. In some idealized versions this cosmic-ray plasma system can be made self-consistent. In a hydrodynamic approach, the system comprises four fluids: thermal plasma, cosmic rays and two oppositely propagating Alfvén waves (Ko 1992). The cosmic rays and the waves are treated as massless fluids but with significant pressures or energy densities. The thermal plasma carries all the inertia of the system. The system is governed by the overall mass, momentum and energy equations, the cosmic ray energy equation, the wave energy exchange equations and the magnetic induction equation. As one can be imagine, it is a formidable task to carry out a complete analysis on the full system.

Concentrating on the behaviour of cosmic rays and waves, JCK proposed the *nonlinear test particle picture*, in which the energy density of the thermal plasma is considered to be much larger than the energy densities of the cosmic rays and waves. In other words the thermal plasma acts like a *thermal bath*, and quantities pertaining to the plasma (and the background magnetic field), are prescribed quantities (or functions). The governing equations of the model are presented in Appendix A.

JCK studied the following simple example thoroughly: a one-dimensional steady state system with uniform background (the flow and magnetic field are parallel to each other), i.e., U, $V_{\rm A}$ and $M_{\rm A} = U/V_{\rm A}$ as well are prescribed constants. In this simplified model there are two useful integrals: the *wave-action integral* $W_{\rm A}$ and the *streaming integral* S,

$$W_{\rm A} = F_{\rm c} + 2(M_{\rm A}+1)^2 V_{\rm A} P_{\rm w}^+ - 2(M_{\rm A}-1)^2 V_{\rm A} P_{\rm w}^- \,, \quad (1)$$

$$S = V_{\rm A} P_{\rm c} + 2(M_{\rm A}+1) V_{\rm A} P_{\rm w}^+ - 2(M_{\rm A}-1) V_{\rm A} P_{\rm w}^- \,. \quad (2)$$

We note that solutions for this simplified model can be conveniently classified by the two parameters $M_{\rm A}$ and $W_{\rm A}/S$.

After some manipulation, the model can be summarized by the two wave energy exchange equations,

$$\frac{\eta}{V_{\rm A}} \frac{{\rm d}P_{\rm w}^\pm}{{\rm d}x} = \mp \frac{P_{\rm w}^\pm (D_{\rm c} \pm 2\beta P_{\rm c} P_{\rm w}^\mp)}{2(M_{\rm A} \pm 1)(P_{\rm w}^+ + P_{\rm w}^-)} \,, \quad (3)$$

in which $P_{\rm c}$ and $D_{\rm c}$ are given by

$$V_{\rm A} P_{\rm c} = S - 2(M_{\rm A}+1) V_{\rm A} P_{\rm w}^+ + 2(M_{\rm A}-1) V_{\rm A} P_{\rm w}^- \,, \quad (4)$$

$$D_{\rm c} = \frac{\eta}{V_{\rm A}} \frac{{\rm d}P_{\rm c}}{{\rm d}x} = \gamma_{\rm c} P_{\rm c} \big[(M_{\rm A}+1) P_{\rm w}^+ + (M_{\rm A}-1) P_{\rm w}^-\big] - (\gamma_{\rm c}-1)(P_{\rm w}^+ + P_{\rm w}^-) \frac{F_{\rm c}}{V_{\rm A}} \,, \quad (5)$$

and

$$F_{\rm c} = W_{\rm A} - 2(M_{\rm A}+1)^2 V_{\rm A} P_{\rm w}^+ + 2(M_{\rm A}-1)^2 V_{\rm A} P_{\rm w}^- \,. \quad (6)$$

The spatial dimension is normalized to a certain length scale l. The constants $\gamma_{\rm c}$ (adiabatic or polytropic index of cosmic rays) and β are prescribed closure properties of cosmic rays. Note that $\gamma_{\rm c} \in (\frac{4}{3}, \frac{5}{3})$ and $\beta \in (1, \frac{4}{3})$, and for extremely relativistic cosmic rays $\gamma_{\rm c} = \beta = \frac{4}{3}$. η is the normalized inverse coupling parameter (see Appendix A). It depends on (or defines) the length scale and does not affect quantities far upstream and far downstream.

For physically admissible solutions (or simply physical solutions), we demand that they ought to be finite far upstream and far downstream ($x \to \pm\infty$). Of course, all the pressures must be non-negative. It turns out that there are no periodic solutions and thus the far upstream and far downstream states must be uniform. The necessary condition for uniform states is the vanishing of stochastic acceleration, which can be achieved either by $P_{\rm c} = 0$, by $P_{\rm w}^- = 0$ or by $P_{\rm w}^+ = 0$. In analysing Eq. (3) four uniform states or critical points are found. They are:

(I) $P_{\rm c} = 0$,
$P_{\rm w}^- = \big[W_{\rm A} - (M_{\rm A}+1)S\big]/4(M_{\rm A}-1)V_{\rm A}$,
$P_{\rm w}^+ = \big[W_{\rm A} - (M_{\rm A}-1)S\big]/4(M_{\rm A}+1)V_{\rm A}$;
(II) $P_{\rm c} = (\gamma_{\rm c}-1)\big[W_{\rm A} - (M_{\rm A}+1)S\big]/(M_{\rm A}+1)V_{\rm A}$,
$P_{\rm w}^- = 0$,
$P_{\rm w}^+ = -\big[(\gamma_{\rm c}-1)W_{\rm A} - \gamma_{\rm c}(M_{\rm A}+1)S\big]/2(M_{\rm A}+1)^2 V_{\rm A}$;
(III) $P_{\rm c} = -(\gamma_{\rm c}-1)\big[W_{\rm A} - (M_{\rm A}-1)S\big]/2(M_{\rm A}-1)V_{\rm A}$,
$P_{\rm w}^- = \big[(\gamma_{\rm c}-1)W_{\rm A} - \gamma_{\rm c}(M_{\rm A}-1)S\big]/2(M_{\rm A}-1)^2 V_{\rm A}$,
$P_{\rm w}^+ = 0$; and
(IV) $P_{\rm c} = S/V_{\rm A}$, $P_{\rm w}^- = 0$, $P_{\rm w}^+ = 0$.

These uniform states can be divided into divergent and convergent states. Convergent (divergent) means that the neighbouring states approach the uniform state in the downstream (upstream) direction. Non-trivial physical solutions are those connecting a divergent uniform state and a convergent uniform state.

In this article we are interested in super-Alfvénic flows ($M_{\rm A} > 1$). Non-negative pressure of the states above requires that $W_{\rm A}/S > \gamma_{\rm c}(M_{\rm A}-1)/(\gamma_{\rm c}-1)$ and $(M_{\rm A}+1) < W_{\rm A}/S < \gamma_{\rm c}(M_{\rm A}+1)/(\gamma_{\rm c}-1)$. We should point out that this is only a necessary condition for physical solutions. In fact, there is a small parameter region within the aforementioned region where no physical solution is allowed (the reader is referred to JCK for details). In the allowable parameter region, there are two types of physical solutions: one-wave and two-wave. In one-wave systems (i.e., systems $\mathcal{F}_{\rm u}$ and $\mathcal{B}_{\rm u}$ mentioned in Sect. 1) only the forward or backward propagating wave exists and no stochastic acceleration occurs, and the solution goes from state II (forward wave) or state III (backward wave) to state IV. In two-wave systems (i.e., system $\mathcal{T}_{\rm u}$ mentioned in Sect. 1) both forward and backward propagating waves exist and stochastic acceleration occurs, and the solution goes from state I to state IV. In all these systems the final destination is always state IV, i.e., all waves are gone. Moreover, it can be shown that the wave pressures are monotonically decreasing functions of x in all these systems (systems $\mathcal{F}_{\rm u}$ and $\mathcal{B}_{\rm u}$ have close form solutions, see JCK).

Cosmic rays can be accelerated (or decelerated) by three mechanisms (see Appendix A): (i) work done by plasma flow, (ii) streaming instability, and (iii) stochastic acceleration (or second order Fermi effect). (i) and (ii) are facilitated by cosmic ray pressure gradient. (ii) and (iii) involve energy exchange between cosmic rays and waves.

We define the gain (or loss) in cosmic ray pressure and energy flux as $\Delta P_{\rm c} = P_{\rm c}(\infty) - P_{\rm c}(-\infty)$ and $\Delta F_{\rm c} = F_{\rm c}(\infty) - F_{\rm c}(-\infty)$. In the case of one-wave systems (systems $\mathcal{F}_{\rm u}$ and $\mathcal{B}_{\rm u}$), we obtain $\Delta P_{\rm c} = -\big[(\gamma_{\rm c}-1)W_{\rm A} - \gamma_{\rm c}(M_{\rm A} \pm 1)S\big]/(M_{\rm A} \pm 1)V_{\rm A}$, and $\Delta F_{\rm c} = -\big[(\gamma_{\rm c}-1)W_{\rm A} - \gamma_{\rm c}(M_{\rm A} \pm 1)S\big]/V_{\rm A}$, where the upper (lower) sign corresponds to system $\mathcal{F}_{\rm u}$ (system $\mathcal{B}_{\rm u}$). Note that both $\Delta P_{\rm c}$ and $\Delta F_{\rm c}$ are positive (negative) for system $\mathcal{F}_{\rm u}$ (system $\mathcal{B}_{\rm u}$). In the case of two-wave systems (system $\mathcal{T}_{\rm u}$), we obtain $\Delta P_{\rm c} = S/V_{\rm A} > 0$, and $\Delta F_{\rm c} = W_{\rm A} > 0$. Moreover, it is easy to compute the fraction of energy gain (or loss) via work done by plasma flow. In a system $\mathcal{F}_{\rm u}$ or system $\mathcal{B}_{\rm u}$ the fraction is $M_{\rm A}/(M_{\rm A} \pm 1)$, while in a system $\mathcal{T}_{\rm u}$ the fraction is $M_{\rm A} S/W_{\rm A}$. Therefore, in one-wave systems work done by the plasma flow is always dominant over the streaming instability in accelerating (or decelerating) cosmic rays. In two-wave systems stochastic acceleration is often the dominant acceleration mechanism, but in some parameter regime work done by plasma flow can be the major one.

3. Shocked background

We would like to investigate what is the role of shock acceleration in the context of the nonlinear test particle picture. To facilitate our discussion, we take a simple one-dimensional parallel fast shock (magnetic field and flow are parallel) as our working example. The upstream and downstream quantities pertaining to plasma flow and magnetic field are uniform and prescribed. If the compression ratio of the shock is q, then we have

$$q = \frac{U_1}{U_2} = \left(\frac{V_{\rm A1}}{V_{\rm A2}}\right)^2 = \left(\frac{M_{\rm A1}}{M_{\rm A2}}\right)^2 . \tag{7}$$

For a fast shock we require $M_{\rm A1} > M_{\rm A2} > 1$, i.e., $1 < q < M_{\rm A1}^2$. Of course, q is also limited by the adiabatic index of the thermal plasma.

Besides the compression ratio q, the location of the shock is also important in our discussion. For a given set of upstream parameters $M_{\rm A1}$, $W_{\rm A1}$ and S_1 (which is equivalent to a given set of far upstream quantities of cosmic rays and waves), the location of the shock defines the quantities just upstream of the shock.

In a uniform background the wave pressures are monotonically decreasing functions of x for both one-wave and two-wave systems (see JCK). Hence in a shocked background we may describe the location of the shock by the ratio of the pressure just upstream of the shock to the pressure far upstream. Define *location parameters* as $\sigma^\pm = P^\pm_{\rm ws1}/P^\pm_{\rm w}(-\infty)$, where $P^\pm_{\rm ws1}$ are wave pressures just upstream of the shock. $P^\pm_{\rm w}(-\infty)$ are given by the corresponding upstream uniform state (see Sect. 2).

When only the forward (backward) wave exists upstream of the shock, we call the system $\mathcal{F}_{\rm s}$ ($\mathcal{B}_{\rm s}$). When both forward and backward waves exist upstream of the shock, we call the system $\mathcal{T}_{\rm s}$. Note that both forward and backward waves exist downstream of the shock, because both waves are generated by the shock. Hereafter subscript $_{(1)}$ ($_{(2)}$) denotes quantities corresponding to systems $\mathcal{F}_{\rm u}$, $\mathcal{B}_{\rm u}$, $\mathcal{F}_{\rm s}$ and $\mathcal{B}_{\rm s}$ (systems $\mathcal{T}_{\rm u}$ and $\mathcal{T}_{\rm s}$). In systems $\mathcal{F}_{\rm s}$ and $\mathcal{B}_{\rm s}$ only one of the two location parameters is needed, and

$$\begin{aligned} &\sigma^\pm_{(1)} = P^\pm_{\rm ws1}/P^\pm_{\rm w(1)}(-\infty)\,, \qquad \text{with} \\ &P^\pm_{\rm w(1)}(-\infty) = \mp\frac{\left[(\gamma_{\rm c1}-1)W_{\rm A1} - \gamma_{\rm c1}(M_{\rm A1}\pm 1)S_1\right]}{2(M_{\rm A1}\pm 1)^2 V_{\rm A1}}\,. \end{aligned} \tag{8}$$

The range of $\sigma^\pm_{(1)}$ is $(0,1)$, where 0 corresponds to far downstream and 1 corresponds to far upstream. In system $\mathcal{T}_{\rm s}$ both location parameters are needed, and

$$\begin{aligned} &\sigma^\pm_{(2)} = P^\pm_{\rm ws1}/P^\pm_{\rm w(2)}(-\infty)\,, \qquad \text{with} \\ &P^\pm_{\rm w(2)}(-\infty) = \frac{\left[W_{\rm A1} - (M_{\rm A1}\mp 1)S_1\right]}{4(M_{\rm A1}\pm 1)V_{\rm A1}}\,. \end{aligned} \tag{9}$$

The range of each of $\sigma^\pm_{(2)}$ is $(0,1)$, but not the whole region defined by $0 < \sigma^+_{(2)} < 1$ and $0 < \sigma^-_{(2)} < 1$ is allowable. The reason is that $P^\pm_{\rm w}$ do not evolve independently, see Fig. 2d of JCK.

We now turn to the matching (or jumping) conditions at the shock. The matching conditions for plasma quantities are irrelevant because the plasma is considered to be a heat bath. The matching conditions for cosmic rays can be obtained by integrating the cosmic ray energy equation with suitable test functions across the shock (see e.g., Drury 1983; Ko et al. 1997). Without injection at the shock, the conditions are just the continuity of cosmic ray pressure (or energy density) and energy flux,

$$P_{\rm cs1} = P_{\rm cs2}\,, \qquad F_{\rm cs1} = F_{\rm cs2}\,. \tag{10}$$

From the study of Alfvén wave transmission and reflection at a shock (McKenzie & Westphal 1969; McKenzie & Bornatici 1974), we obtain the matching conditions for the wave pressures at a fast shock,

$$\begin{aligned} P^+_{\rm ws2} =\ & \frac{q(1+\sqrt{q})^2(M_{\rm A1}+1)^2}{4(M_{\rm A1}+\sqrt{q})^2}P^+_{\rm ws1} \\ & + \frac{q(1-\sqrt{q})^2(M_{\rm A1}-1)^2}{4(M_{\rm A1}+\sqrt{q})^2}P^-_{\rm ws1}\,, \\ P^-_{\rm ws2} =\ & \frac{q(1-\sqrt{q})^2(M_{\rm A1}+1)^2}{4(M_{\rm A1}-\sqrt{q})^2}P^+_{\rm ws1} \\ & + \frac{q(1+\sqrt{q})^2(M_{\rm A1}-1)^2}{4(M_{\rm A1}-\sqrt{q})^2}P^-_{\rm ws1}\,. \end{aligned} \tag{11}$$

This is different from the model used by Ostrowski (1994). We note that the theory we adopted for Alfvén wave transmission is based on small perturbations (McKenzie & Westphal 1969; McKenzie & Bornatici 1974). Moreover, to derive Eq. (11) we have assumed "incoherent" waves (see Appendix B).

Owing to the shock, the parameters defining the upstream and downstream states ($M_{\rm A}, S, W_{\rm A}$) may not be the same. First, the Alfvén Mach numbers are related by

$$M_{\rm A2} = M_{\rm A1}/\sqrt{q}\,. \tag{12}$$

Using the matching conditions for the cosmic ray and waves (Eqs. (10)–(11)), the difference in streaming integrals is given by

$$\begin{aligned} \frac{S_2}{V_{\rm A2}} - \frac{S_1}{V_{\rm A1}} =\ & \frac{(q-1)}{(M_{\rm A1}^2-q)} \\ & \times\Big[(M_{\rm A1}+1)(2M_{\rm A1}^2 - qM_{\rm A1} - q)P^+_{\rm ws1} \\ & -(M_{\rm A1}-1)(2M_{\rm A1}^2 + qM_{\rm A1} - q)P^-_{\rm ws1}\Big]\,, \end{aligned} \tag{13}$$

where $P^\pm_{\rm ws1} = \sigma^\pm P^\pm_{\rm w}(-\infty)$. Of course, in system $\mathcal{F}_{\rm s}$, $P^-_{\rm ws1} = 0$. Similarly, in system $\mathcal{B}_{\rm s}$, $P^+_{\rm ws1} = 0$.

Finally, a rather unexpected result is obtained. We find that the wave-action integral is continuous across the fast shock, i.e.,

$$W_{\rm A2} = W_{\rm A1}\,. \tag{14}$$

4. Results and discussion

We would like to seek physical solutions of the nonlinear test particle picture in a shocked background. Recall the criteria for physical solutions (Sect. 2 or JCK): the pressures must be finite and non-negative throughout the space. One of the consequences is that the far upstream ($x \to -\infty$) and far downstream ($x \to \infty$) states are uniform. If the background is uniform, then a non-trivial physical solution approaches a uniform state at far upstream and another uniform state at far downstream. Now the background we are considering has a shock. The requirement of a physical solution becomes that the upstream (downstream) solution should approach a uniform state at far upstream (downstream). JCK called such upstream (downstream) solutions semi-physical solutions. Of course, the upstream and downstream solutions must satisfy the matching conditions Eqs. (10)–(11).

4.1. Parameter space

As one can imagine, the set of semi-physcial solutions is much larger than the set of physical solutions (see JCK). However, we would like to restrict ourselves somewhat. We are interested in the role of the shock in cosmic ray acceleration. Thus we would like to compare the results from a system with a shock to a system without a shock, while both systems have the same upstream state. In the upstream region, the parameter region for physical solutions of uniform background is $W_{\rm A1}/S_1 > \gamma_{\rm c1}(M_{\rm A1}-1)/(\gamma_{\rm c1}-1)$ and $(M_{\rm A1}+1) < W_{\rm A1}/S_1 < \gamma_{\rm c1}(M_{\rm A1}+1)/(\gamma_{\rm c1}-1)$ (see Sect. 2). In the downstream region, physical solutions approach uniform state IV (or critical point IV), i.e., all waves vanish eventually. At critical point IV, (i) $P_{\rm c} > 0$ requires $S_2 > 0$, and (ii) convergent state requires $\gamma_{\rm c2}(M_{\rm A2}-1)/(\gamma_{\rm c2}-1) < W_{\rm A2}/S_2 < \gamma_{\rm c2}(M_{\rm A2}+1)/(\gamma_{\rm c2}-1)$.

According to Sect. 3, $M_{\rm A2}$, $W_{\rm A2}$ and S_2 are functions of $M_{\rm A1}$, $W_{\rm A1}$, S_1, q and $\sigma^{\pm}$. Among these five parameters we focus on the shock parameters: the compression ratio q ($1 < q < M_{\rm A1}^2$), and the location parameter $\sigma^{\pm}$ ($0 < \sigma^{\pm} < 1$). The requirements (i) and (ii) give a restriction on the parameter space $(q, \sigma^{\pm})$. However this is a necessary condition only. To find the sufficient (and necessary condition) we have to make sure that the solution of Eq. (3) must go to uniform state IV, when $P_{\rm ws2}^{\pm}$ are used as initial conditions. (i.e., $P_{\rm ws2}^{\pm}$ must lie within the separatrices of the four critical points in the phase space $(P_{\rm w}^{-}, P_{\rm w}^{+})$, see Fig. 2d of JCK).

To illustrate these ideas, we consider a system $\mathcal{F}_{\rm s}$ (i.e., a forward-wave upstream system). Fig. 1 shows the parameter space $(q, \sigma_{(1)}^{+})$ with $M_{\rm A1} = 2$, $W_{\rm A1} = 5$, $S_1 = 1$, $V_{\rm A1} = 1$ and $\gamma_{\rm c1} = \gamma_{\rm c2} = \beta_1 = \beta_2 = \frac{4}{3}$. The region below the solid curve in Fig. 1 is the parameter region for physically admissible solutions; that is, the shock should be far enough downstream and with a small enough compression ratio to ensure physical solutions. The region left (right) of the dashed line in Fig. 1 corresponds to positive (negative) acceleration efficiency, which we turn to now.

4.2. Efficiency

JCK showed that in a uniform super-Alfvénic flow background cosmic rays gain energy in forward-wave systems ($\mathcal{F}_{\rm u}$) and in two-wave systems ($\mathcal{T}_{\rm u}$), while they lose energy in backward-wave systems ($\mathcal{B}_{\rm u}$). We obtain similar results in a fast shock background. In system $\mathcal{F}_{\rm s}$ or system $\mathcal{T}_{\rm s}$ cosmic rays gain energy, while in system $\mathcal{B}_{\rm s}$ they lose energy.

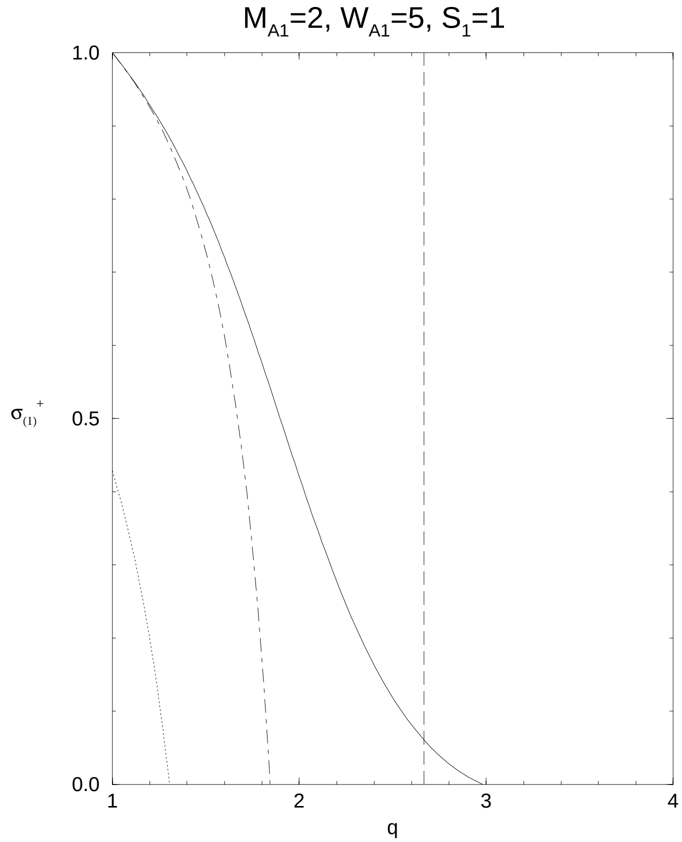

Fig. 1. The physically admissible region for a system $\mathcal{F}_{\rm s}$ (a forward-wave upstream system) in the parameter space $(q, \sigma_{(1)}^{+})$ when the upstream parameters $M_{\rm A1} = 2$, $W_{\rm A1} = 5$, $S_1 = 1$, $V_{\rm A1} = 1$, $\gamma_{\rm c1} = \beta_1 = \frac{4}{3}$, and $\gamma_{\rm c2} = \beta_2 = \frac{4}{3}$ as well. The region below the solid curve allows physical solutions. The acceleration efficiency is positive (negative) on the left (right) of the dashed line. Furthermore, the cosmic ray pressure gradient just downstream of the shock is positive (negative) in the region below (above) the dot-dashed curve. The cosmic ray pressure gradient just downstream of the shock is larger (smaller) than that just upstream of the shock in the region below (above) the dotted curve.

Define the gain (or loss) in cosmic ray pressure and energy flux as $\Delta P_{\rm c} = P_{\rm c}(\infty) - P_{\rm c}(-\infty)$ and $\Delta F_{\rm c} = F_{\rm c}(\infty) - F_{\rm c}(-\infty)$. The gain in system $\mathcal{F}_{\rm s}$ and the loss in system $\mathcal{B}_{\rm s}$ are

$$\Delta_{(1)}^{\pm} P_{\rm c} = \pm 2(M_{\rm A1} \pm 1) P_{\rm w(1)}^{\pm}(-\infty) \times \left[1 + \frac{(q-1)(2M_{\rm A1}^2 \mp q M_{\rm A1} - q)}{2(M_{\rm A1}^2 - q)} \sigma_{(1)}^{\pm}\right], \tag{15}$$

$$\Delta_{(1)}^{\pm} F_{\rm c} = \pm 2(M_{\rm A1} \pm 1)^2 V_{\rm A1} P_{\rm w(1)}^{\pm}(-\infty)\,, \tag{16}$$

where $P^{\pm}_{\rm w(1)}(-\infty)$ is given by Eq. (8). The gain in system $\mathcal{T}_{\rm s}$ is

$$\Delta_{(2)}P_{\rm c} = \frac{S_1}{V_{\rm A1}} + \frac{(q-1)}{(M_{\rm A1}^2-q)} \times\Big[(M_{\rm A1}+1)(2M_{\rm A1}^2-qM_{\rm A1}-q)\sigma^+_{(2)}P^+_{\rm w(2)}(-\infty) -(M_{\rm A1}-1)(2M_{\rm A1}^2+qM_{\rm A1}-q)\sigma^-_{(2)}P^-_{\rm w(2)}(-\infty)\Big]\,, \tag{17}$$

$$\Delta_{(2)}F_{\rm c} = W_{\rm A1}\,, \tag{18}$$

where $P^{\pm}_{\rm w(2)}(-\infty)$ is given by Eq. (9).

When we put $\sigma^{\pm}=0$ in the above equations, we obtain the gain and loss of system $\mathcal{F}_{\rm u}$, $\mathcal{B}_{\rm u}$ and $\mathcal{T}_{\rm u}$ in a uniform background (which is denote by subscript $_0$),

$$\Delta^{\pm}_{0(1)}P_{\rm c} = \pm 2(M_{\rm A1}\pm 1)P^{\pm}_{\rm w(1)}(-\infty)\,, \tag{19}$$

$$\Delta^{\pm}_{0(1)}F_{\rm c} = \pm 2(M_{\rm A1}\pm 1)^2 V_{\rm A1}P^{\pm}_{\rm w(1)}(-\infty)\,, \tag{20}$$

and

$$\Delta_{0(2)}P_{\rm c} = S_1/V_{\rm A1}\,, \tag{21}$$

$$\Delta_{0(2)}F_{\rm c} = W_{\rm A1}\,. \tag{22}$$

It is intriguing to observe that the gain (or loss) in the cosmic ray energy flux is the same with or without a shock, i.e. $\Delta_0 F_{\rm c} = \Delta F_{\rm c}$. (Noted that there was an algebraic mistake in Ko (1997), so a wrong gain in energy flux was obtained.) Therefore, to examine the role of the shock on cosmic ray acceleration, we resort to the change in cosmic ray pressure (or energy density) and not in its energy flux.

Usually the efficiency of acceleration is simply defined by the gain in cosmic ray pressure or flux in a shocked background. We deem that this definition is suitable for the cases where the gain is zero in a uniform background, but it is not appropriate for the cases where the gain is non-zero in a uniform background as in the present discussion. We would like to know what is the change in the gain if a shock is added to a uniform background. We find that the simple idea of a "before-and-after" picture is very handy, and we define the efficiency of shock acceleration as the fractional extra gain in cosmic ray pressure when a shock is added to a uniform background: $\varepsilon_{\rm P} = (\Delta P_{\rm c} - \Delta_0 P_{\rm c})/\Delta_0 P_{\rm c}$. For systems $\mathcal{F}_{\rm s}$ and $\mathcal{B}_{\rm s}$, we obtain

$$\varepsilon^{\pm}_{\rm P(1)} = \frac{(q-1)(2M_{\rm A1}^2 \mp qM_{\rm A1}-q)}{2(M_{\rm A1}^2-q)}\sigma^{\pm}_{(1)}\,, \tag{23}$$

and for system $\mathcal{T}_{\rm s}$,

$$\varepsilon_{\rm P(2)} = \frac{(q-1)}{4(M_{\rm A1}^2-q)S_1} \times\Big\{(2M_{\rm A1}^2-qM_{\rm A1}-q)\big[W_{\rm A1}-(M_{\rm A1}-1)S_1\big]\sigma^+_{(2)} -(2M_{\rm A1}^2+qM_{\rm A1}-q)\big[W_{\rm A1}-(M_{\rm A1}+1)S_1\big]\sigma^-_{(2)}\Big\}\,. \tag{24}$$

Recall that $\Delta^-_{0(1)}P_{\rm c} < 0$ (see Eq. (19)), thus the efficiency of system $\mathcal{B}_{\rm s}$ ($\varepsilon^-_{\rm P(1)}$) should be interpreted as efficiency of deceleration. In fact, it can easily be shown that $\varepsilon^-_{\rm P(1)} \geq 0$ always. However, we are after the efficiency of acceleration of a shock, so we are interested in systems $\mathcal{F}_{\rm s}$ and $\mathcal{T}_{\rm s}$.

To illustrate these ideas, we examine system $\mathcal{F}_{\rm s}$ in detail. Fig. 2 shows a profile of $\varepsilon^+_{\rm P(1)}/\sigma^+_{(1)}$ against q. From Eq. (23) we notice that $\varepsilon^+_{\rm P(1)} > 0$ (< 0) for $q < 2M_{\rm A1}^2/(M_{\rm A1}+1)$ ($> 2M_{\rm A1}^2/(M_{\rm A1}+1)$). We should point out that depending on the value of the location parameter $\sigma^+_{(1)}$, the compression ratio q may not attain a large enough value to have negative acceleration efficiency. In fact, the parameter region for negative efficiency is small in general, see Fig. 1. Moreover, from Eq. (23) we notice that $\varepsilon^+_{\rm P(1)}/\sigma^+(1)$ attains a maximum $\frac{1}{2}(M_{\rm A1}-1)$ at $q = M_{\rm A1}$. Interestingly enough, this is the compression ratio at which the backward wave pressure just downstream of the shock $P^-_{\rm ws2}$ starts to exceed the forward wave pressure just downstream of the shock $P^+_{\rm ws2}$, see Eq. (11). This can be understood as follows. In the downstream region both forward and backward waves tend to zero monotonically as $x\to\infty$. Recall that the upshot of cosmic ray streaming instability is that when forward (backward) wave energy flux decreases, cosmic ray pressure increases (decreases). Therefore, the gain in cosmic ray pressure starts to decline when $P^+_{\rm ws2}$ becomes less than $P^-_{\rm ws2}$.

As a function of both q and $\sigma^+_{(1)}$, the maximum of $\varepsilon^+_{\rm P(1)}$ is located on the solid curve in Fig. 1. In the case of $M_{\rm A1}=2$, $W_{\rm A1}=5$, $S_1=1$, $V_{\rm A1}=1$, $\gamma_{\rm c1}=\gamma_{\rm c2}=\beta_1=\beta 2=\frac{4}{3}$, the maximum of $\varepsilon^+_{\rm P(1)}$ is 2.9 at $(q,\sigma^+_{(1)})=(1.64,0.69)$.

The efficiency $\varepsilon_{\rm P(2)}$ of system $\mathcal{T}_{\rm s}$ exhibits similar behaviour, but the result is complicated because two location parameters are involved. For instance, $\varepsilon_{\rm P(2)}>0$ if

$$\begin{aligned}&(2M_{\rm A1}^2-qM_{\rm A1}-q)\big[W_{\rm A1}-(M_{\rm A1}-1)S_1\big]\sigma^+_{(2)}\\ >\ &(2M_{\rm A1}^2+qM_{\rm A1}-q)\big[W_{\rm A1}-(M_{\rm A1}+1)S_1\big]\sigma^-_{(2)}\,,\end{aligned} \tag{25}$$

and $\varepsilon_{\rm P(2)}$ attains an extremum if

$$\begin{aligned}&(M_{\rm A1}+1)(M_{\rm A1}-q)(2M_{\rm A1}^2-M_{\rm A1}-q)\\ &\quad\times\big[W_{\rm A1}-(M_{\rm A1}-1)S_1\big]\sigma^+_{(2)}\\ =\ &(M_{\rm A1}-1)(M_{\rm A1}+q)(2M_{\rm A1}^2+M_{\rm A1}-q)\\ &\quad\times\big[W_{\rm A1}-(M_{\rm A1}+1)S_1\big]\sigma^-_{(2)}\,.\end{aligned} \tag{26}$$

For the sake of completeness, $\varepsilon^-_{\rm P(1)}$ of system $\mathcal{B}_{\rm s}$ is always positive, because $\varepsilon^-_{\rm P(1)}$ increases monotonically with q (see Eq. (23)).

From the point of view of energy flux, it seems that the gain in cosmic ray pressure should increase whenever a shock is added, because the cosmic ray energy flux remains the same but the plasma flow velocity is reduced. The reason that this conclusion is not correct is that the diffusive flux is not zero even in the region far downstream. Recall that the energy flux can be

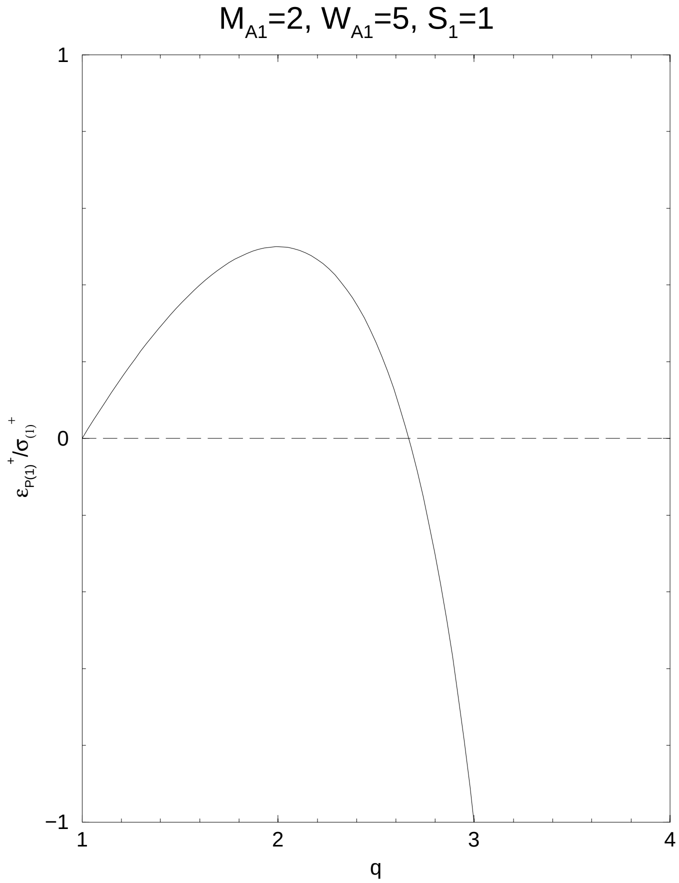

Fig. 2. The efficiency of acceleration of a system $\mathcal{F}_{\rm s}$ is expressed as the ratio of efficiency to shock location parameter $\varepsilon^{+}_{\rm P(1)}/\sigma^{+}_{(1)}$ against the compression ratio q. The parameters used are the same as in Fig. 1: $M_{\rm A1}=2, W_{\rm A1}=5, S_1=1, V_{\rm A1}=1, \gamma_{\rm c1}=\gamma_{\rm c2}=\beta_1=\beta_2=\frac{4}{3}$.

divided into advective ($\gamma_{\rm c}(M_{\rm A}+1)V_{\rm A}P_{\rm c}/(\gamma_{\rm c}-1)$) and diffusive ($-V_{\rm A}D_{\rm c}/[(\gamma_{\rm c}-1)(P^{+}_{\rm w}+P^{-}_{\rm w})]$) components. Depending on how the wave pressures approach zero towards the uniform state IV, the advective flux can be

$$F^{\rm (advec)}_{\rm c}(\infty)=\frac{\gamma_{\rm c}}{(\gamma_{\rm c}-1)}(M_{\rm A}\pm 1)S\,, \tag{27}$$

and the diffusive flux is

$$F^{\rm (diff)}_{\rm c}(\infty)=W_{\rm A}-\frac{\gamma_{\rm c}}{(\gamma_{\rm c}-1)}(M_{\rm A}\pm 1)S\,. \tag{28}$$

The upper (lower) sign corresponds to $P^{-}_{\rm w}/P^{+}_{\rm w}\to 0$ ($P^{+}_{\rm w}/P^{-}_{\rm w}\to 0$) as $x\to\infty$. Their sum $F_{\rm c}(\infty)$ remains constant when a shock is added to the uniform background, but each of them does change. For instance, the change in advective flux at far downstream is the difference between $\gamma_{\rm c2}(M_{\rm A2}\pm 1)S_2/(\gamma_{\rm c2}-1)$ and $\gamma_{\rm c1}(M_{\rm A1}\pm 1)S_2/(\gamma_{\rm c1}-1)$. There are two possible outcomes for system $\mathcal{F}_{\rm s}$ or system $\mathcal{B}_{\rm s}$, and four possible outcomes for system $\mathcal{T}_{\rm s}$.

We should point out that the cosmic ray pressure gradient vanishes far downstream. The diffusive flux is still nonzero solely because the diffusion coefficient tends to zero far downstream in our model. The diffusion coefficient is inversely proportional to the total wave pressure, and the wave pressures vanish far downstream (state IV). If there are some residue scatterings of cosmic rays not produced by waves, then the diffusive flux should be zero.

Recall that three mechanisms (work done by plasma flow, cosmic-ray streaming instability and stochastic acceleration) are responsible for the acceleration (or deceleration) of cosmic rays in this model. Although the gain in cosmic ray flux remains the same when a shock is added to a uniform background flow, the distribution of the three mechanisms may change. The cumulative work done by the thermal plasma flow on the cosmic ray energy flux is given by $\Delta\mathcal{W}_{\rm P}=V_{\rm A2}M_{\rm A2}P_{\rm c}(\infty)-V_{\rm A1}M_{\rm A1}P_{\rm c}(-\infty)$. The contributions from streaming instability and stochastic acceleration have to be found numerically. Define the fractional contribution of work done by plasma on cosmic ray energy flux as $\mathcal{P}_{\rm p}=\Delta\mathcal{W}_{\rm P}/\Delta F_{\rm c}$. After some manipulation,

$$\mathcal{P}^{\pm}_{\rm p(1)}=\frac{M_{\rm A1}}{(M_{\rm A1}\pm 1)}\left[1\mp\frac{(q-1)(M_{\rm A1}\mp 1)\sigma^{\pm}_{(1)}}{2(M^2_{\rm A1}-q)}\right], \tag{29}$$

for systems $\mathcal{F}_{\rm s}$ and $\mathcal{B}_{\rm s}$, and

$$\begin{aligned}\mathcal{P}_{\rm p(2)}=\ &\frac{M_{\rm A1}S_1}{W_{\rm A1}}\Bigg\{1-\frac{(q-1)}{4(M^2_{\rm A1}-q)}\\ &\times\left[\frac{W_{\rm A1}}{S_1}\left[(M_{\rm A1}-1)\sigma^{+}_{(2)}+(M_{\rm A1}+1)\sigma^{-}_{(2)}\right]\right.\\ &\left.-\left[(M_{\rm A1}-1)^2\sigma^{+}_{(2)}+(M_{\rm A1}+1)^2\sigma^{-}_{(2)}\right]\right]\Bigg\}\,,\end{aligned} \tag{30}$$

for system $\mathcal{T}_{\rm s}$. It is clear from these expressions that $\mathcal{P}_{\rm p}$ decreases when the compression ratio of the shock increases and/or the location of the shock is further upstream for systems $\mathcal{F}_{\rm s}$ and $\mathcal{T}_{\rm s}$ (where $\Delta F_{\rm c}>0$). (The opposite conclusion is obtained for system $\mathcal{B}_{\rm s}$ where $\Delta F_{\rm c}<0$.)

When a shock is added to a uniform background, it amplifies and generates waves downstream of the shock (see Eq. (11)). As the compression ratio q increases, the relative importance of the three acceleration mechanisms shifts from work done by the plasma flow (which doesn't involve waves) to streaming instability and stochastic acceleration (which are wave-related mechanisms). The shock does not increase the gain in cosmic ray energy flux, it just re-distributes the relative importance of the three mechanisms.

Finally, we take system $\mathcal{F}_{\rm s}$ as an example, and display its pressure and energy flux profiles in Figs. 3–6. In these figures $M_{\rm A1}=2, W_{\rm A1}=5, S_1=1, V_{\rm A1}=1$ and $\gamma_{\rm c1}=\gamma_{\rm c2}=\beta_1=\beta_2=\frac{4}{3}$, but the compression ratio of the shock q and the location parameter $\sigma^{+}_{(1)}$ are different. The acceleration efficiency $\varepsilon^{+}_{\rm P(1)}$ is positive in Figs. 3 and 4, but it is negative in Fig. 5 (& 6). (See Fig. 2 for the relative position in the allowable region in the parameter space $(q,\sigma^{+}_{(1)})$.) In fact, $\varepsilon^{+}_{\rm P(1)}=0.21$, 0.15 & -0.00327 for Figs. 3–5, respectively. In the case of negative acceleration efficiency, the generation of a backward wave by the shock is so efficient that the dynamics are dominated by the backward wave pressure gradient.

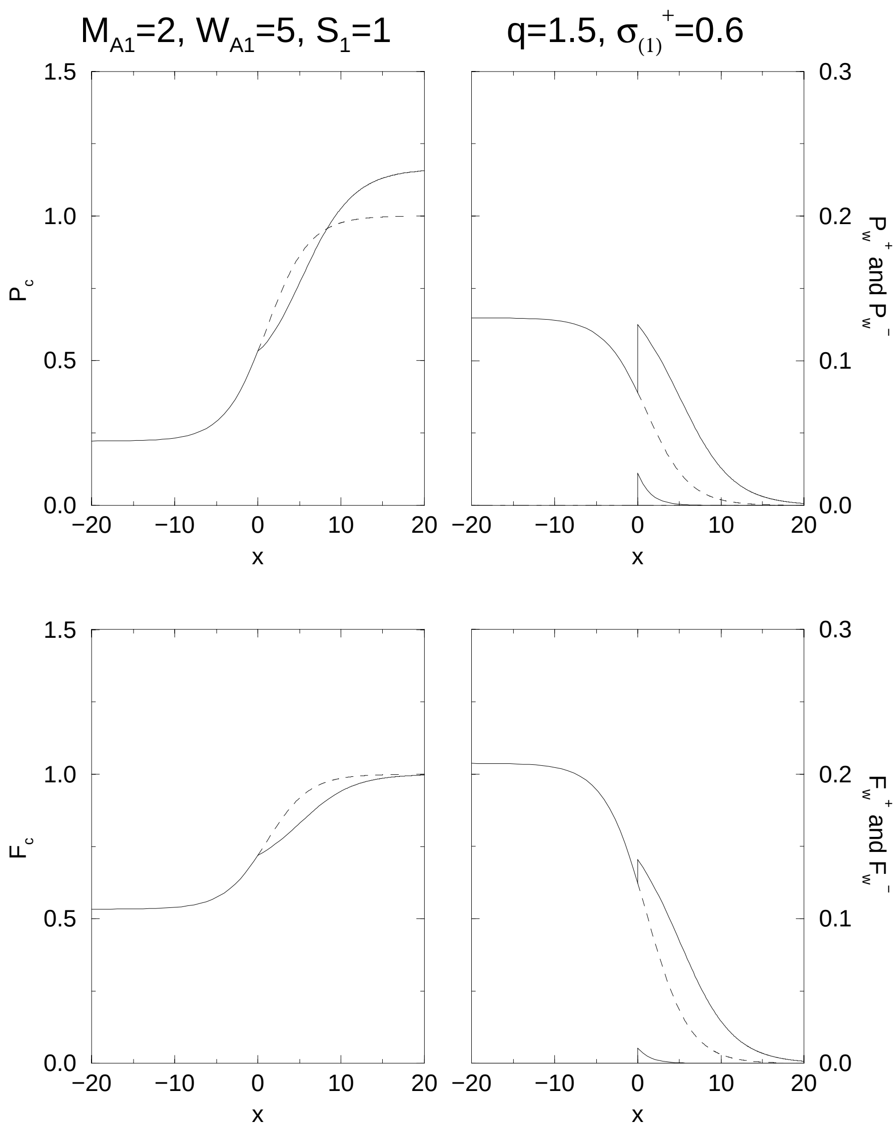

Fig. 3. The pressure and energy flux profiles of cosmic ray and waves when $(q, \sigma^{+}_{(1)}) = (1.5, 0.6)$. Other parameters are the same as in Figs. 1 and 2. The shock is located at $x = 0$. The dashed curves are the profiles when there is no shock. The pressure and energy flux of the forward wave are larger than those of the backward wave.

What has the shock done? As shown in Figs. 3–6, the cosmic ray flux increases more slowly when the shock is added, although it attains the same value far downstream. (In fact, it can be shown that for $\gamma_{c1} = \gamma_{c2} = \beta_1 = \beta_2$ the cosmic ray flux gradient just upstream of the shock is always larger than the gradient just downstream of the shock.) Thus, in a sense, the shock "slows down" the increment of cosmic ray energy flux. This "slowing down" comes from the generation of waves by the shock. The backward wave generated by the shock decays rapidly, and the cosmic ray pressure gradient decreases accordingly (this can be thought of as a consequence of cosmic ray streaming instability). The work done by the plasma flow on the cosmic rays decreases, and the increase in stochastic acceleration cannot cover the loss. Hence the rate of increase of the cosmic ray energy flux is reduced. Alternatively, the process can be understood as the reduction of the advective flux of cosmic ray by the shock. The diffusive flux has to increase accordingly. In the cases of high compression ratio, the reduction is so large that the diffusive flux becomes positive (and the cosmic ray pressure gradient becomes negative) just downstream of the shock, see Figs. 4–6.

5. Summary and concluding remarks

In this article shock acceleration efficiency is investigated in the context of the nonlinear test particle picture developed by Jiang et al. (1996) (see Sect. 2 and Appendix A for a brief description). The model is a simplified version of the hydrodynamic approach to cosmic-ray–plasma systems by Ko (1992). It describes the interaction between cosmic rays and two oppositely propagating Alfvén waves, but the thermal plasma flow is treated as a heat bath.

For simplicity we considered one-dimensional parallel fast shocks with uniform flow upstream and downstream of the shock. Furthermore, we discussed steady state only. The model is governed by two sets of parameters. One set comes from the upstream uniform flow: the Alfvén Mach number M_{A1}, the wave-action integral W_{A1} and the streaming integral S_1. Another set is related to the shock: the compression ratio q

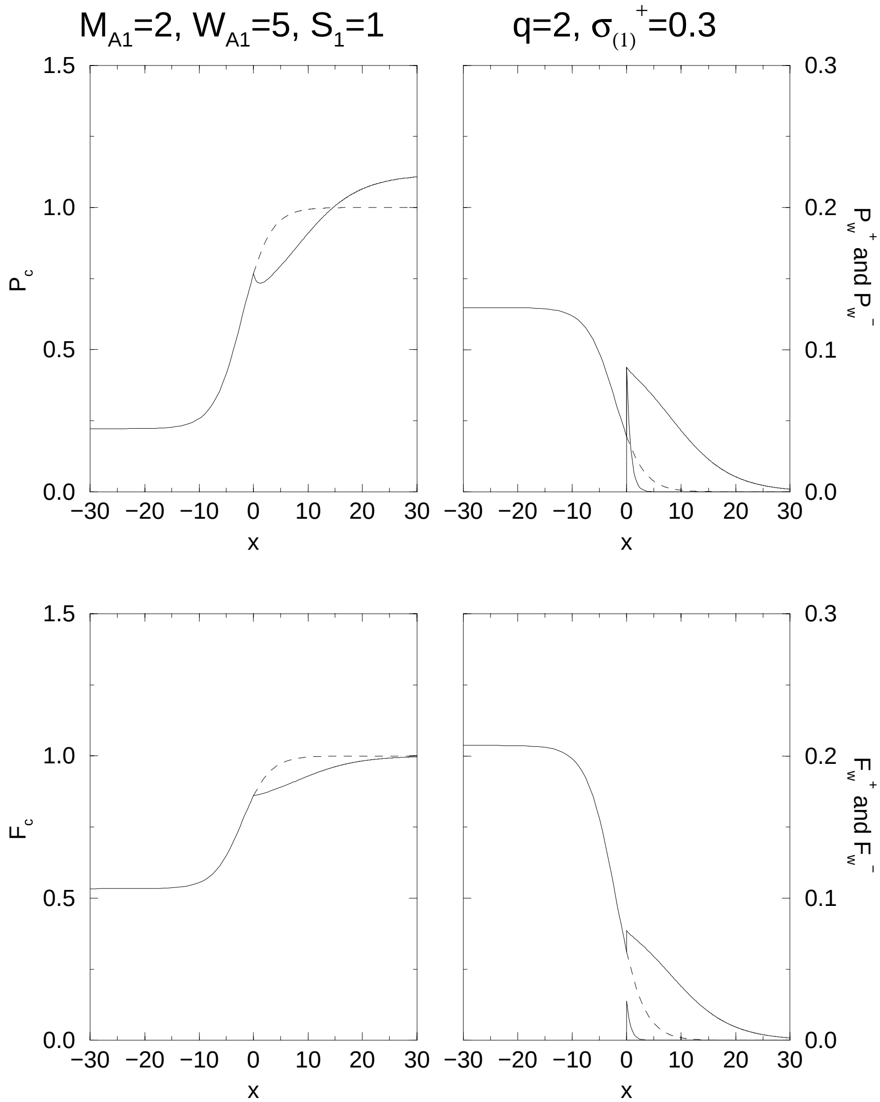

Fig. 4. Same as Fig. 3, except that $(q, \sigma^+_{(1)}) = (2.0, 0.3)$. The spikes correspond to backward waves. Note that the cosmic ray pressure gradient is negative just downstream of the shock.

$(1 < q < M^2_{\rm A1})$ and the location parameter $\sigma^\pm$ $(0 < \sigma^\pm < 1$ where 0 and 1 correspond to far downstream and far upstream, respectively).

Without injection the cosmic ray pressure (or energy density) and energy flux are continuous across the shock (see e.g., Drury 1983; Ko et al. 1997). Adopting the theory of transmission of Alfvén wave through a shock by McKenzie & Westphal 1969 (also McKenzie & Bornatici 1974), the matching conditions for the wave pressures at a fast shock are given by Eq. (11). Note that Ostrowski (1994) used a different model for the Alfvén transmission and obtained different results.

A rather unexpected result is obtained when we applied the matching conditions at the shock for cosmic ray and waves, i.e., Eqs. (10) and (11). The wave-action integral, given by Eq. (1), is continuous. Consequently, for the same far upstream states the cosmic ray energy fluxes far downstream in a uniform background and in a shocked background are the same. In other words, there is no acceleration in terms of energy flux. The shock increases the wave energy fluxes, but "mysteriously" these extra energy fluxes do not contribute to the *total* cosmic ray energy flux far downstream. It seems that they just return back to the plasma flow. We stress that it is the *total* cosmic ray energy flux that does not change. The advective and diffusive components of the energy flux do change indeed. We thus discussed the efficiency of shock acceleration in terms of cosmic ray pressure or energy density.

Recall that JCK showed that cosmic rays could be accelerated in a uniform background (see Sect. 2). Thus, we deem that the efficiency of shock acceleration should be defined in terms of the difference between the acceleration of cosmic ray in a shocked background and a uniform background (a simple "before-and-after" picture). We defined the efficiency $\varepsilon_{\rm P}$ as the fractional extra gain in cosmic ray pressure when a shock is added to a uniform background (see Eqs. (23) and (24)). There are three typical systems (see the end of Sect. 1). In terms of the difference between the pressure far downstream and the pressure far upstream $(\Delta P_{\rm c} = P_{\rm c}(\infty) - P_{\rm c}(-\infty))$, two of them (systems $\mathcal{F}_{\rm s}$ and $\mathcal{T}_{\rm s}$) accelerate cosmic rays, and one of them (system $\mathcal{B}_{\rm s}$) decelerates cosmic rays. However, in terms of $\varepsilon_{\rm P}$,

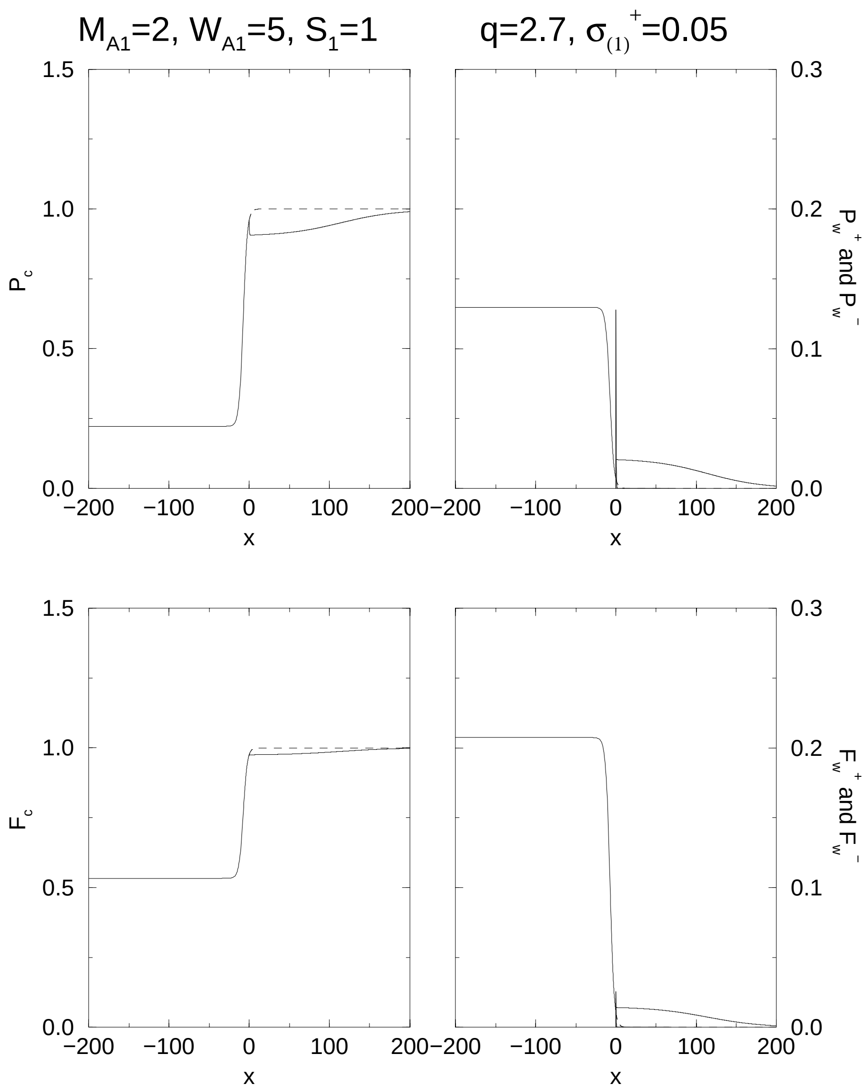

Fig. 5. Same as Fig. 3, except that $(q, \sigma^+_{(1)}) = (2.7, 0.05)$. The spikes correspond to backward waves. Note that the cosmic ray pressure gradient is negative just downstream of the shock. The cosmic ray pressure is indeed reduced when the shock is added, although the reduction is small, see text.

systems $\mathcal{F}_s$ and $\mathcal{T}_s$ may have negative efficiency in some parameter regime.

As an example, Fig. 2 illustrates how the efficiency of shock acceleration varies with the compression ratio of the shock in system $\mathcal{F}_s$. It is clear that the efficiency becomes negative for large q. However, a physical solution restricts the parameter space $(q, \sigma^+_{(1)})$ (see Fig. 1), and q may not be large enough to give negative efficiency (e.g., $\sigma^+_{(1)}$ close to 1, i.e., the shock occurs far upstream). Figs. 3–6 display the profiles of pressure and energy flux for a system that has positive and negative efficiency respectively.

Cosmic rays can be accelerated by three mechanisms, among which two involve waves (streaming instability and stochastic acceleration) and one does not involve waves (work done by plasma flow). Waves are amplified by the shock. As the compression ratio increases, the relative importance of the three mechanisms shifts from the no-wave mechanism to wave-related mechanisms (see Eqs (29) and (30)). We should point out that except for stochastic acceleration, the mechanisms are capable of decelerating cosmic rays.

What role does the shock play in the acceleration of cosmic rays in the nonlinear test particle approximation? Without injection the cosmic ray pressure and energy flux are continuous across the shock. There is no "instantaneous" acceleration at the shock. The energy gain (or loss) far downstream has to be interpreted as the cumulative result of the three aforementioned mechanisms. The same interpretation applies to the gain (or loss) of energy in a uniform background. So what is the difference? In a shocked background, waves are amplified at the shock and the flow velocity is reduced (consider the case where the upstream state is the same as the uniform background). Interestingly enough the gains in cosmic ray energy flux in uniform and shocked background are the same. (This is a result of the wave-action integral being continuous across the shock.) Thus, the shock just re-distributes the relative importance of the three mechanisms. As a result, it reduces the advective flux, increases the diffusive flux, and "slows down" the increase of the total cosmic ray flux, see Figs. 3–6.

In some small parameter regimes, negative efficiency is obtained (see Figs. 1 and 2). We caution that this unexpected out-

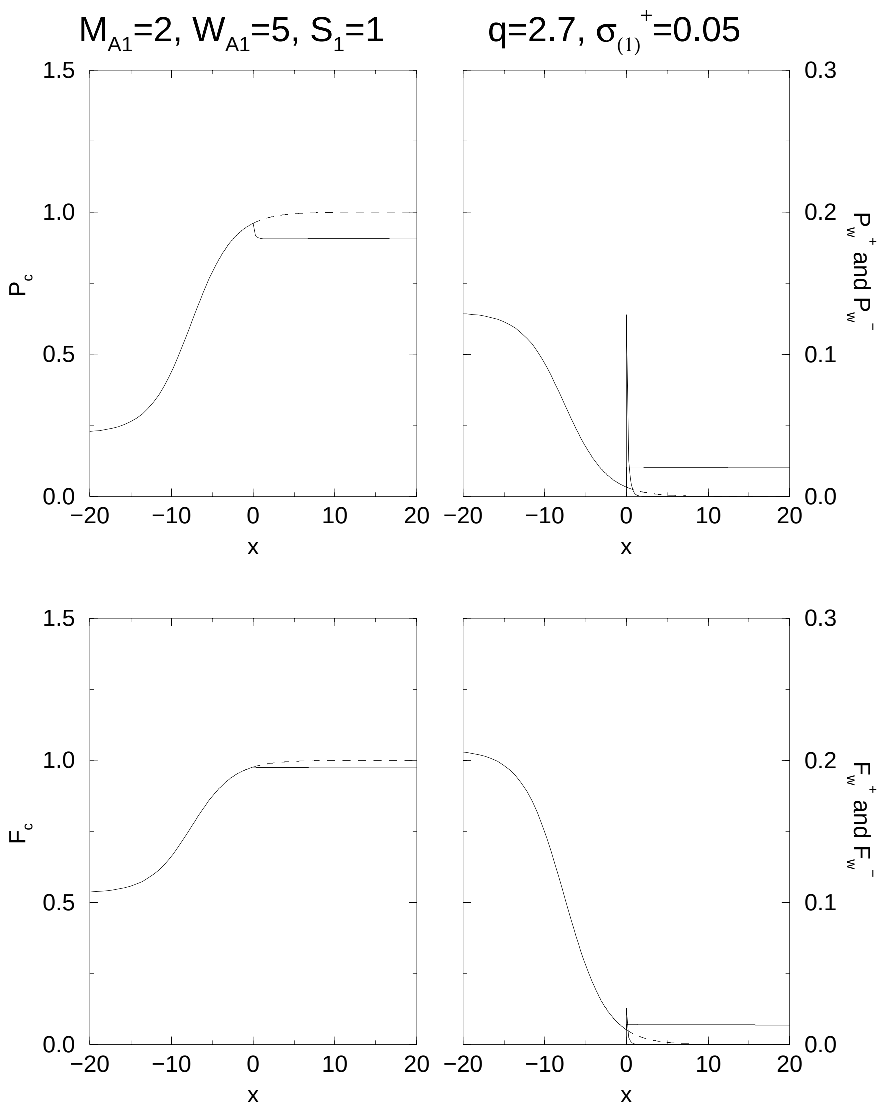

Fig. 6. This figure is an enlargement of the central part of Fig. 5.

come may just be telling us that the test particle picture is no longer valid in those parameter regimes. Fig. 5 or 6 shows that backward wave pressure gradients are very large and they ought to alter the background flow. The "complete" model, which takes into account the backreaction of cosmic rays and waves, may or may not have negative efficiency. Moreover, the region not allowed by the test particle picture in Fig. 1, may become available in the "complete" model. In any case, it is not clear to us how to define an efficiency in the "complete" model using the idea of a "before-and-after" picture similar to the one we have in this article.

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Appendix A: nonlinear test particle picture

The equations governing the one-dimensional steady-state nonlinear test particle picture used in this article are the cosmic ray energy equation and the two wave energy exchange equations (Ko 1992; Jiang et al. 1996),

$$\frac{\mathrm{d}F_{\mathrm{c}}}{\mathrm{d}x} = \left[U + (e_+ - e_-)V_{\mathrm{A}}\right]\frac{\mathrm{d}P_{\mathrm{c}}}{\mathrm{d}x} + \frac{P_{\mathrm{c}}}{\tau}\,, \tag{A1}$$

$$\frac{\mathrm{d}F_{\mathrm{w}}^{\pm}}{\mathrm{d}x} = U\frac{\mathrm{d}P_{\mathrm{w}}^{\pm}}{\mathrm{d}x} \mp e_{\pm}V_{\mathrm{A}}\frac{\mathrm{d}P_{\mathrm{c}}}{\mathrm{d}x} - \frac{P_{\mathrm{c}}}{2\tau}\,, \tag{A2}$$

where the energy fluxes are given by

$$F_{\mathrm{c}} = \left[U + (e'_+ - e'_-)V_{\mathrm{A}}\right](E_{\mathrm{c}} + P_{\mathrm{c}}) - \kappa\frac{\mathrm{d}E_{\mathrm{c}}}{\mathrm{d}x}\,, \tag{A3}$$

$$F_{\mathrm{w}}^{\pm} = (U \pm V_{\mathrm{A}})E_{\mathrm{w}}^{\pm} + UP_{\mathrm{w}}^{\pm}\,. \tag{A4}$$

The first term on the right hand side of Eq. (A1) is the work done per unit time by the plasma flow, the second term is the streaming instability, and the third term is the stochastic acceleration. The first two terms can be positive or negative, but the third term is always positive.

We adopt Ko (1992) and take $E_\mathrm{c} = P_\mathrm{c}/(\gamma_\mathrm{c} - 1)$, $E_\mathrm{w}^\pm = 2P_\mathrm{w}^\pm$, and

$$\kappa = \frac{ac^2}{3\alpha(P_\mathrm{w}^+ + P_\mathrm{w}^-)}\,, \quad \frac{1}{\tau} = 16b\alpha\frac{V_\mathrm{A}^2}{c^2}\frac{P_\mathrm{w}^+ P_\mathrm{w}^-}{(P_\mathrm{w}^+ + P_\mathrm{w}^-)}\,, \qquad \text{(A5)}$$
$$e_\pm = e'_\pm = \frac{P_\mathrm{w}^\pm}{(P_\mathrm{w}^+ + P_\mathrm{w}^-)}\,,$$

where a and b are constants of order unity with $a \le 1$ and $b \ge 1$ but $\frac{3}{4} \le ab \le 1$. For extremely relativistic cosmic rays, $a = b = 1$. The coupling parameter α arises from relating the collision frequency of Alfvén waves resonant scattering to the wave energy densities or pressures. We assume that α is the same for both waves. γ_c and α are known as closure parameters.

If U and V_A are constants, two integrals can be obtained. The wave-action integral

$$W_\mathrm{A} = F_\mathrm{c} + \frac{(U+V_\mathrm{A})^2}{V_\mathrm{A}}E_\mathrm{w}^+ - \frac{(U-V_\mathrm{A})^2}{V_\mathrm{A}}E_\mathrm{w}^-\,, \qquad \text{(A6)}$$

and the streaming integral

$$S = (U+V_\mathrm{A})E_\mathrm{w}^+ - (U-V_\mathrm{A})E_\mathrm{w}^- + V_\mathrm{A}P_\mathrm{c}\,. \qquad \text{(A7)}$$

Normalizing the spatial scale to l, velocities to $U_\circ$, pressures to $P_\circ$, energy fluxes to $U_\circ P_\circ$, W_A and S to $U_\circ P_\circ$, we obtain

$$\frac{\mathrm{d}F_\mathrm{c}}{\mathrm{d}x} = \left[M_\mathrm{A} + (e_+ - e_-)\right]V_\mathrm{A}\frac{\mathrm{d}P_\mathrm{c}}{\mathrm{d}x} + \frac{4\beta V_\mathrm{A}^2}{\eta}\frac{P_\mathrm{w}^+ P_\mathrm{w}^- P_\mathrm{c}}{(P_\mathrm{w}^+ + P_\mathrm{w}^-)}\,, \qquad \text{(A8)}$$

$$\frac{\mathrm{d}F_\mathrm{w}^\pm}{\mathrm{d}x} = M_\mathrm{A}V_\mathrm{A}\frac{\mathrm{d}P_\mathrm{w}^\pm}{\mathrm{d}x} \mp e_\pm V_\mathrm{A}\frac{\mathrm{d}P_\mathrm{c}}{\mathrm{d}x} - \frac{2\beta V_\mathrm{A}^2}{\eta}\frac{P_\mathrm{w}^+ P_\mathrm{w}^- P_\mathrm{c}}{(P_\mathrm{w}^+ + P_\mathrm{w}^-)}\,, \qquad \text{(A9)}$$

$$F_\mathrm{c} = \left\{\left[M_\mathrm{A} + (e_+ - e_-)\right]\frac{\gamma_\mathrm{c}}{(\gamma_\mathrm{c} - 1)}P_\mathrm{c} - \frac{\eta}{(P_\mathrm{w}^+ + P_\mathrm{w}^-)}\frac{D_\mathrm{c}}{(gc-1)}\right\}V_\mathrm{A}\,, \qquad \text{(A10)}$$

$$F_\mathrm{w}^\pm = (3M_\mathrm{A} \pm 2)V_\mathrm{A}P_\mathrm{w}^\pm\,, \qquad \text{(A11)}$$

$$W_\mathrm{A} = F_\mathrm{c} + 2(M_\mathrm{A}+1)^2 V_\mathrm{A}P_\mathrm{w}^+ - 2(M_\mathrm{A}-1)^2 V_\mathrm{A}P_\mathrm{w}^-\,, \qquad \text{(A12)}$$

$$S = V_\mathrm{A}P_\mathrm{c} + 2(M_\mathrm{A}+1)V_\mathrm{A}P_\mathrm{w}^+ - 2(M_\mathrm{A}-1)V_\mathrm{A}P_\mathrm{w}^-\,. \qquad \text{(A13)}$$

where

$$M_\mathrm{A} = \frac{U}{V_\mathrm{A}}\,, \quad \eta = \frac{ac^2}{3\alpha V_\mathrm{A}P_\circ l}\,, \quad \beta = \frac{4ab}{3}\,. \qquad \text{(A14)}$$

Appendix B: Matching conditions for Alfvén wave pressure across a fast shock

The tangential momentum and electric field across a shock are continuous. Perturbating these matching conditions, McKenzie & Westphal (1969) showed that the transmission of Alfvén wave across a fast shock is given by (with $_1$ and $_2$ denote upstream and downstream, respectively):

$$\frac{\delta B_{\perp 2}^+}{\delta B_{\perp 1}^\pm} = \pm\sqrt{q}\frac{\delta U_{\perp 2}^+}{\delta U_{\perp 1}^\pm} = \frac{\sqrt{q}(1 \pm \sqrt{q})(M_\mathrm{A1} \pm 1)}{2(M_\mathrm{A1} + \sqrt{q})}\,, \qquad \text{(B1)}$$
$$\frac{\delta B_{\perp 2}^-}{\delta B_{\perp 1}^\pm} = \mp\sqrt{q}\frac{\delta U_{\perp 2}^-}{\delta U_{\perp 1}^\pm} = \frac{\mp\sqrt{q}(1 \mp \sqrt{q})(M_\mathrm{A1} \pm 1)}{2(M_\mathrm{A1} - \sqrt{q})}\,,$$

where $\delta B_\perp^\pm$ (and $\delta U_\perp^\pm$) are the amplitudes of the forward and backward propagating Alfvén waves; q is the compression ratio and M_A1 is the upstream Alfvén Mach number.

Wave pressure is defined as $P_\mathrm{w} = \langle(\delta B_\perp^+ + \delta B_\perp^-)^2/2\mu_o\rangle$ where the angular bracket represents an integration or average over the spectrum of the Alfvén wave (see e.g., Ko 1992). If the forward and backward propagating waves are incoherent, then $P_\mathrm{w} = \langle(\delta B_\perp^+)^2/2\mu_o\rangle + \langle(\delta B_\perp^-)^2/2\mu_o\rangle = P_\mathrm{w}^+ + P_\mathrm{w}^-$, because the cross term $\delta B_\perp^+ \delta B_\perp^-$ is averaged out. However, the downstream forward and backward waves generated by an upstream forward (or backward) wave are coherent. To keep the mathematics (and physics as well) simple enough for analysis, we assume that the length scale we are interested is much larger than the longest wavelength in the spectrum of Alfvén wave. The cross term $\delta B_{\perp 2}^+ \delta B_{\perp 2}^-$ from $\delta B_{\perp 1}^+$ (or $\delta B_{\perp 1}^-$) is averaged out again. (Note that the average is taken over a spatial scale smaller than the length scale we are interested in but larger than the largest wavelength in the spectrum. To obtain the wave pressure, an average over the spectrum has to be perforemed also.) With the vanishing of the cross terms, we obtain the matching conditions for the wave pressures as shown in Eq. (11).

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