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Analytic description of the r-mode instability in uniform density stars

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Abstract. We present an analytic description of the r-mode instability in newly-born neutron stars, using the approximation of uniform density. Our computation is consistently accurate to second order in the angular velocity of the star. We obtain formulae for the growth-time of the instability due to gravitational-wave emission, for both current and mass multipole radiation and for the damping timescale, due to viscosity. The $l = m = 2$ current-multipole radiation dominates the timescale of the instability. We estimate the deviation of the second order accurate results from the lowest order approximation and show that the uncertainty in the equation of state has only a small effect on the onset of the r-mode instability. The viscosity coefficients and the cooling process in newly-born neutron stars are, at present, uncertain and our analytic formulae enable a quick check of such effects on the development of the instability.

Key words: gravitational waves – instabilities – stars: neutron – stars: oscillations

1. Introduction

The recently discovered r-mode instability (Andersson 1998) in rotating neutron stars, has significant implications on the rotational evolution of a newly-born neutron star. The r-modes are unstable due to the Chandrasekhar-Friedman-Schutz (CFS) mechanism (Chandrasekhar 1970, Friedman & Schutz 1978) (see also Friedman & Morsink 1997). Two independent computations by Andersson, Kokkotas & Schutz (1998) and Lindblom, Owen and Morsink (1998) find that the r-mode instability is responsible for slowing down a rapidly rotating, newly-born neutron star to rotation rates comparable to that of the initial period of the Crab pulsar ($\sim$19 ms) or the recently discovered 16 ms X-ray pulsar in the supernova remnant N157B (Marshall et al. 1998) (with an estimated initial period of 6-9 ms). This is achieved by the emission of current-quadrupole gravitational waves, which reduce the angular momentum of the star. Additionally, as the initially rapidly rotating star spins down, an energy equivalent to roughly 1% of a solar mass is radiated in gravitational waves, which makes the process an interesting source of detectable gravitational waves (Owen et al. 1998).

In the present paper, we investigate the r-mode instability to 2nd order accuracy in the angular velocity of the star, in the approximation of uniform density (the actual density profile of realistic neutron stars is nearly uniform), where all results can be obtained analytically. [1] While our analytic results provide an independent check of the numerical results in Andersson et al. 1998 and Lindblom et al. 1998, our main objective is to present a simple set of equations, which enable one to obtain a qualitative insight into the mechanism of the r-mode instability and to quickly check the dependence of the instability on various important factors, such as the central density of the star, the different types of viscosity in neutron stars, the different possible cooling processes etc. Additionally, we expect that there is a number of issues related to the spinning mechanisms of pulsar such as accretion disc induced spin up, or the creation of millisecond pulsars due to accretion-induced collapse of a white dwarf (Andersson et al. 1998), for which one can use the simple relations provided here for a fast but still accurate evaluation of the various evolution scenarii.

2. The r-mode instability

Oscillations of stars are commonly described by the Lagrangian displacement vector $\boldsymbol{\xi}$, which describes the displacement of a given fluid element due to the oscillation. Since $\boldsymbol{\xi}$ is a vector on the (θ, ϕ) 2-sphere, it can be analyzed into a sum of spheroidal and toroidal components (or polar and axial components, in a different terminology). In a non-rotating star, the usual f, p and g modes of oscillation are purely spheroidal, characterized by the indices (l, m) of the spherical harmonic function Y_l^m. In a rotating star, modes that reduce to purely spheroidal modes in the non-rotating limit, also acquire toroidal components. Conversely, r-modes in a non-rotating star are purely toroidal modes with vanishing frequency. In a rotating star, the displacement vector acquires spheroidal components and the frequency in the rotating frame, to first order in the rotational frequency Ω of the star, becomes

$$\omega_r = \frac{2m\Omega}{l(l+1)} \,, \tag{1}$$

[1] After this paper was submitted, a preprint by Lindblom & Ipser (1998) appeared, where the study r-modes in Maclaurin spheroids and find the existence of more r-modes than the “classical” r-mode considered here.

for a given (l, m) mode. An inertial observer, measures a frequency of

$$\omega_i = \omega_r - m\Omega \; . \tag{2}$$

From (1) and (2) it can be deduced that a counter-rotating (with respect to the star, as defined in the co-rotating frame) r-mode, appears as co-rotating with the star to a distant inertial observer. Thus, to O(Ω), all r-modes with $l \geq 2$ are generically unstable to the emission of gravitational radiation, due to the Chandrasekhar-Friedman-Schutz (CFS) mechanism (note that the $l = 1$ r-mode is marginally unstable, to this order). The instability is active for as long as its growth-time is shorter than the damping-time due to the viscosity of neutron-star matter. Its effect is to slow-down, within a year, a rapidly rotating neutron star to slow rotation rates and this explains why only slowly-rotating pulsars are associated with supernova remnants (Andersson et al. 1998). Thus, the r-mode instability does not allow millisecond pulsars to be formed after an accretion-induced collapse of a white dwarf. It seems that millisecond pulsars can only be formed by the accretion-induced spin-up of old, cold, neutron stars.

3. The 2nd-order accurate slow rotation formalism

To O(Ω), the star is still spherical, and one can only determine the angular dependence of the r-mode eigenfunctions and their lowest order r-dependence (the latter is obtained by taking the curl of the perturbed equations of motion). For obtaining the second-order correction to the eigenfunctions and to the frequency, one must proceed to a consistent O(Ω^2) calculation. We follow the formalism for computing r-modes in Newtonian stars, due to Saio 1982, that was presented in more detail in Andersson et al. 1998. Here we will only summarize the equations needed for the uniform density case.

3.1. Assumptions

We make the following assumptions:

1. the perturbations are adiabatic,
2. the star is an incompressible barotrope of uniform density,
3. the rotation of the star is uniform, and
4. the perturbation of the gravitational potential can be neglected (Cowling approximation),

These assumptions are justified by the fact that, even for temperatures $T = 10^9 K$, the thermal energy of the star is much less than the Fermi energy of its interior ($>$ 60 MeV). Also, at such temperatures, the initially differentially rotating protoneutron star is rotating uniformly, due to the formation of a solid crust (see Stergioulas 1998, for a recent review on rotating neutron stars). The Cowling approximation has been shown to yield sufficiently accurate results for r-modes in slowly rotating, Newtonian stars (Saio 1982, Provost et al. 1981).

3.2. Definitions

In a slowly rotating star, the dominant correction to its structure is of O(Ω^2). The analysis of perturbations of the star is simplified by introducing a new radial coordinate a, defined through

$$r = a(1 + \epsilon) \; , \tag{3}$$

where $\epsilon = \epsilon(a, \theta)$ is a quantity of O(Ω^2), representing the deformation of the equilibrium structure from the non-rotating configuration. In the new coordinate system, all equilibrium quantities are functions of a only and the surfaces of constant a are equipotential surfaces.

Since equilibrium neutron stars are stationary and axisymmetric, a general oscillation can be analyzed into a sum of normal modes, with harmonic time-dependence $e^{i(m\phi+\omega_i t)}$.The displacement vector for a given r-mode can be written as (Saio 1982):

$$\begin{aligned}\boldsymbol{\xi}/a = \mathbf{T} + \mathbf{S} \;=\; & \left(0, K_{lm} \sin^{-1}\theta \partial_\phi, -K_{lm}\partial_\theta\right) Y_l^m \\ & + \sum_{\nu\mu} \left(S_{\nu\mu}, H_{\nu\mu}\partial_\theta, H_{\nu\mu}\sin^{-1}\theta\partial_\phi\right) Y_\nu^\mu,\end{aligned} \tag{4}$$

where $\mathbf{T}$ and $\mathbf{S}$ are the *toroidal* and *spheroidal* parts of the displacement, respectively. Note that the toroidal part has vanishing a-component and is described only by the function $K_{lm} = K_{lm}(a)$, which multiplies a toroidal angular vector. The spheroidal part has an non-vanishing a-component of O(Ω^2), described by the functions $S_{\nu\mu} = S_{\nu\mu}(a)$. The θ and ϕ-components of the spheroidal part are described by the functions $H_{\nu\mu} = H_{\nu\mu}(a)$ (also of O(Ω^2)), multiplying spheroidal angular vectors.

The perturbation in the pressure is expressed in terms of spheroidal radial functions $\zeta_{\nu\mu}$, as

$$\delta p = \rho g a \sum_{\nu\mu} \zeta_{\nu\mu} Y_\nu^\mu \; , \tag{5}$$

where δp is the Eulerian variation in the pressure (the variation of the pressure at a fixed point in space), ρ is the density and $g = -\rho^{-1} dP/da$ is the acceleration of gravity.

3.3. The propensity rule

For zero-temperature (barotropic) stars, it can easily be shown from the perturbation equations, that only modes with $l = m$ exist. Then, only the $\nu = l + 1$, $\mu = m$ terms contribute in the expansions for the displacement vector and the perturbation in the pressure (in the remainder of the text, we will drop the index m in these quantities). The absence of $l - 1$ terms (the spherical harmonics Y_{l-1}^m are zero) means that rotation excites only higher multipole spheroidal parts. This is in agreement with the "propensity" rule suggested by Chandrasekhar & Ferrari 1991, for the oscillations of slowly rotating relativistic stars, i.e. that the rotational coupling of a toroidal l-term with a spheroidal $l+1$-terms is strongly favored over the coupling with a spheroidal $l-1$-term. We find that, in uniform density stars, the "propensity" rule completely eliminates the coupling to lower-multipole terms.

3.4. The perturbation equations

A normal-mode solution to the perturbation equations satisfies the perturbed Euler equations, the perturbed continuity equation and the relation between the perturbations in density and pressure. We define dimensionless frequencies as

$$\bar{\omega}_r = \omega_r \left(\frac{R^3}{GM}\right)^{1/2}, \tag{6}$$

and

$$\varpi = \Omega \left(\frac{R^3}{GM}\right)^{1/2}, \tag{7}$$

and expand the frequency in the rotating frame as

$$\bar{\omega}_r = \sigma_0 \varpi + \sigma_2 \varpi^3. \tag{8}$$

Writing the distortion parameter ϵ as

$$\epsilon \equiv \left[\tilde{D}_1(a) + \tilde{D}_2(a) P_2(\cos\theta)\right] \varpi^2, \tag{9}$$

(where $P_2(\cos\theta)$ is the Legendre polynomial) and expanding the perturbation equations consistently to second order in the angular velocity of the star, we find that the eigenfunctions ζ_{l+1}, S_{l+1}, H_{l+1} and K_l are given by the following set of equations: [2]

$$a\frac{d\zeta_{l+1}}{da} = (l-1)\,\zeta_{l+1}\ , \tag{10}$$

$$a\frac{dS_{l+1}}{da} = -\,(4+l)\,S_{l+1} - h\zeta_{l+1}\ , \tag{11}$$

$$H_{l+1} = S_{l+1} + \frac{(2l+1)(l+1)}{8l\sigma_0}\left[(l+1)\sigma_2 + 6\tilde{D}_2\right]\zeta_{l+1}, \tag{12}$$

and

$$K_l = i\frac{(l+1)\sqrt{2l+3}}{2l\bar{\omega}_r\varpi}\zeta_{l+1}. \tag{13}$$

In (11), h is

$$h \;=\; \frac{1}{\sigma_0^2}\left\{\frac{(l+1)}{l}\left[(2l+3)\frac{\sigma_2}{\sigma_0} + 6(l-1)\tilde{D}_2\right] + 3\left(3\tilde{D}_2 + a\frac{d\tilde{D}_2}{da}\right)\right\}. \tag{14}$$

Note that the perturbation in the pressure is independent of the displacement vector and can be found by analytically integrating (10), while the toroidal function K_l is given algebraically in terms of the perturbation in the pressure. The spheroidal function S_{l+1} satisfies a differential equation that depends on the perturbation in the pressure and the structure of the star and can not be obtained analytically, but will not be needed for the remainder of this paper. The spheroidal function H_l is given algebraically in terms of S_{l+1} and ζ_{l+1}.

[2] These equations can be derived correspondingly from Saio's Eqs. (38), (36), (50) and (48).

3.5. Boundary conditions

From the leading terms of S_l and ζ_l near $a = 0$, one obtains the boundary condition at the center of the star:

$$(2l+3)S_{l+1} + h\zeta_{l+1} = 0\ . \tag{15}$$

At the surface of the star, the Lagrangian variation of the pressure vanishes (a fluid element on the surface of the unperturbed configuration must also be on the surface of the perturbed configuration):

$$\Delta p = \delta p + \boldsymbol{\xi}\boldsymbol{\nabla} p = 0, \tag{16}$$

or

$$\zeta_{l+1} = S_{l+1}. \tag{17}$$

To $O(\Omega)$, (16) is satisfied trivially, while to $O(\Omega^2)$ it yields the correction to the eigenfrequency to that order.

4. Eigenfunctions and eigenfrequencies

Eq. (10) for ζ_{l+1} implies a solution of the form

$$\zeta_{l+1} \sim a^{l-1}. \tag{18}$$

Since ζ_{l+1} is of order $O(\varpi^2)$, we normalize it to the dimensionless quantity

$$\zeta_{l+1} = \varpi^2 \left(\frac{a}{R}\right)^{l-1}. \tag{19}$$

Then, (13) yields

$$K_l = i\frac{(l+1)\sqrt{2l+3}}{2l}\left(\frac{\Omega}{\omega_r}\right)\left(\frac{a}{R}\right)^{l-1}, \tag{20}$$

where R is the radius of the star. These are the only two eigenfunctions needed for the remainder of the paper.

To O(Ω), the frequency of an $l = m$ r-mode is

$$\sigma_0 = \frac{2}{l+1}. \tag{21}$$

An expression for the second order correction to the eigenfrequency of a given mode can be obtained either directly from the boundary conditions (17), or by constructing an integral relation using the perturbed Euler equations. Applying the approximate integral relation given in the appendix of Saio (1982) to uniform density stars, we obtain:

$$\left(\frac{\bar{\omega}_r}{\varpi} - \sigma_0\right)\int_0^R a^{2l+2} da = -\frac{3l}{l+1}\sigma_0 \int_0^R a^{2l+2} D_2 da - \frac{3l}{(l+1)(2l+3)}\sigma_0 \int_0^R a^{2l+3}\frac{dD_2}{da} da \tag{22}$$

which leads, after integration by parts, to the approximate result

$$\sigma_2 \simeq \frac{5l}{(l+1)^2}, \tag{23}$$

and

$$\bar{\omega}_r \simeq \frac{2}{l+1}\varpi + \frac{5l}{(l+1)^2}\varpi^3. \tag{24}$$

For the derivation of Eq. (24), we have used the fact that $D_2(R) = -(5/6)\varpi^2$ for uniform density stars (cf. appendix II of Provost et al. 1981).

5. Dissipation time-scales

5.1. Energy of mode

The energy of the mode, measured in the rotating frame, is

$$E = \frac{1}{2}\int \rho|\dot{\xi}|^2 dV, \\ = \frac{l(l+1)}{2}\omega_r^2\rho\int_0^R a^4|K_l|^2 da, \tag{25}$$

which gives

$$E = \frac{(l+1)^3}{8l}\rho\Omega^2 R^5. \tag{26}$$

5.2. Dissipation due to gravitational waves

The dissipation of energy due to the emission of gravitational waves can be estimated from the standard multipole formula:

$$\left.\frac{dE}{dt}\right|_{\rm gw} = -\omega_r\sum_l N_l\omega_i^{2l+1}\left(|\delta D_l^m|^2 + |\delta J_l^m|^2\right), \tag{27}$$

where

$$N_l = 4\pi\frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \tag{28}$$

In (27)

$$\delta D_l^m = \int \delta\rho a^l Y_l^{m*} dV, \tag{29}$$

are the mass multipole moments and

$$\delta J_l^m = 2\sqrt{\frac{l}{l+1}}\int a^l(\rho\delta\boldsymbol{v} + \delta\rho\boldsymbol{v})\boldsymbol{Y}_l^{mB*}dV \tag{30}$$

are the current multipole moments, where $\boldsymbol{v}$ is the velocity of the fluid and $\boldsymbol{Y}_l^{mB*}$ are the "magnetic" vector harmonics (see Thorne 1980, Lindblom et al. 1998).

5.2.1. Mass multipoles

The dominant mass multipole moment is δD_{l+1}. For incompressible stars the Lagrangian variation of the density vanishes, $\Delta\rho=0$. From the relation between Lagrangian and Eulerian perturbations of a scalar quantity, it follows that

$$\delta\rho = -\boldsymbol{\xi}\boldsymbol{\nabla}\rho. \tag{31}$$

The derivative of the density across the surface is a Dirac delta-function at $a = R$, thus

$$\delta\rho = -aS_{l+1}Y_{l+1}^l\rho\delta(a-R) \tag{32}$$

The mass-multipole moment becomes

$$\delta D_{l+1} = -\varpi^2\rho R^{l+4}, \tag{33}$$

and, being of O(Ω^2), it contributes to $dE/dt|_{\rm gw}$ an O(Ω^{2l+8}) term.

5.2.2. Current multipoles

The dominant current multipole moment is

$$\delta J_l = 2l\omega_r\int_0^R \rho a^{l+3}K_l da, \tag{34}$$

which is

$$\delta J_l = i\frac{(l+1)}{\sqrt{2l+3}}\rho\Omega R^{l+4}. \tag{35}$$

The contribution of he dominant multipole moment to $dE/dt|_{\rm gw}$ is an O(Ω^{2l+4}) and an O(Ω^{2l+6}) term.

5.2.3. Growth-time

The growth time due to the emission of gravitational waves is

$$t_{\rm gw} = -\frac{2E}{dE/dt|_{gw}}. \tag{36}$$

Including both the mass and current multipole contributions and keeping the frequency to O(Ω^2), we obtain

$$t_{\rm gw} = -\frac{c^{2l+3}}{G}\frac{\pi(l+1)^3}{3l}\left\{(\sigma_0+\sigma_2\varpi^2)\left[l-\sigma_0-\sigma_2\varpi^2\right]^{2l+1} \times\left[\frac{(l+1)^2}{2l+3}N_l + \left[l-\sigma_0-\sigma_2\varpi^2\right]^2 N_{l+1}\varpi^4\right] \times MR^{2l}\Omega^{2l+2}\right\}^{-1}. \tag{37}$$

To lowest order in Ω, (37) reduces to

$$t_{\rm gw} = -\frac{c^{2l+3}}{24G}\frac{[(2l+3)!!]^2}{(2l+3)(l-1)^{2l}}\left(\frac{l+1}{l+2}\right)^{2l+2}\frac{\Omega^{-2l-2}}{MR^{2l}}. \tag{38}$$

5.3. Dissipation due to shear viscosity

The dissipation of energy because of the shear viscosity of neutron star matter is

$$\left.\frac{dE}{dt}\right|_{\rm sv} = -2\int \eta\delta\sigma^{ab}\delta\sigma_{ab}^* dV, \tag{39}$$

where

$$\delta\sigma_{ab} = \frac{i\omega_r}{2}\left(\nabla_a\xi_b + \nabla_b\xi_a - \frac{2}{3}g_{ab}\nabla_c\xi^c\right), \tag{40}$$

(see e.g. Ipser & Lindblom 1991) and η is the shear viscosity coefficient. We obtain

$$\left.\frac{dE}{dt}\right|_{\rm sv} = -l(l+1)\omega_r^2\eta\left[\int_0^R a^2|a\partial_a K_l|^2 da + (l-1)(l+2)\int_0^R a^2|K_l|^2 da\right], \tag{41}$$

which yields

$$\left.\frac{dE}{dt}\right|_{\rm sv} = -\frac{(l+1)^3(l-1)(2l+3)}{4l}\eta\Omega^2 R^3. \tag{42}$$

The damping time due to shear viscosity is

$$t_{\rm sv} = \frac{3}{4\pi(l-1)(2l+3)}\frac{M}{\eta R}. \quad (43)$$

5.4. Dissipation due to bulk viscosity

In a neutron star, bulk viscosity can arise, because of the departure from nuclear reaction equilibrium, such as beta-equilibrium, during the compression and expansion of matter caused by an oscillation. The energy is dissipated at a rate

$$\left.\frac{dE}{dt}\right|_{\rm bv} = -\int \zeta|\delta\sigma|^2 dV, \quad (44)$$

where ζ is the coefficient of bulk viscosity

$$\delta\sigma = -i\omega_r\frac{\Delta p}{\Gamma p}, \quad (45)$$

is the expansion of the fluid and Γ is the adiabatic index. The last relation follows from baryon conservation in an adiabatic oscillation. Strictly speaking, in a uniform density star, $\delta\sigma = 0$. But, we assumed the uniform density approximation only to make calculations easier. For the bulk viscosity we use an approximate timescale, that has been derived by Cutler & Lindblom 1987 for spheroidal oscillations in uniform-density stars. Since the bulk viscosity arises because of the change in density, for toroidal oscillations we use the spheroidal formula, but with l replaced by $l+1$:

$$\tau_{\rm bv} = \frac{3(2l+5)}{2\pi(l+1)^3}\frac{\Gamma^4 M}{\zeta R}. \quad (46)$$

For the purpose of estimating the bulk viscosity only, Γ is taken to be equal to 5, i.e. correspond to that of a stiff (nearly uniform density), $N = 0.25$ polytrope.

6. Critical angular velocities

Below the superfluid transition temperature, which is $T \sim 10^9$ K, the shear viscosity is dominated by electron-electron scattering and an approximate formula for the viscosity coefficient is

$$\eta = 6\times 10^{18}\frac{\rho_{15}^2}{T_9^2}\ {\rm g/cm\ s}, \quad (47)$$

(Cutler & Lindblom 1987), where the notation ρ_{15} means normalization of the density to 10^{15} gr/cm^3 and T_9 normalization of temperature to 10^9K. Above the superfluid transition temperature, the shear viscosity coefficient due to neutron-neutron interactions is

$$\eta = 2\times 10^{18}\frac{\rho_{15}^{9/4}}{T_9^2}\ {\rm g/cm\ s}, \quad (48)$$

(Flowers & Itoh 1979). The bulk viscosity will be important in hot, newly-born neutron stars, but its coefficient is not as certain as the coefficient for shear viscosity. Sawyer (1989) estimates the bulk viscosity in neutron star matter, assuming that the star cools through the modified URCA process and that it is transparent to neutrinos. The coefficient he obtains is

$$\zeta = 6\times 10^{25}\rho_{15}^2\omega_r^{-2}T_9^6 {\rm g/cms}. \quad (49)$$

It has been suggested (Lai & Shapiro 1995) that for temperatures larger than a few times $10^9 K$ the neutrino optical depth is still large and the bulk viscosity is thus inactive. If the star cools through the direct URCA reaction, the bulk viscosity will be much larger than in (49), but again only for temperatures for which the star is transparent to neutrinos. It becomes apparent that, depending on the cooling process and on the neutrino optical depth in a newly-born neutron star, the bulk viscosity can almost completely damp non-axisymmetric instabilities or have only a small effect on them. A more detailed study is needed and departure from equilibrium of other interactions (such as interactions between quarks at lower temperatures) should also be considered. For the time being we will use (49) as an conservative average of the large error bars associated with the bulk viscosity.

Another dissipation mechanism that can affect the instability is the superfluid mutual friction. Estimates by Mendell (1991) and Lindblom & Mendell (1995), suggests that mutual friction could suppress the gravitational - radiation - driven instability of f-modes, when the temperature of the star is between 10^7K$<T<10^9$K. It is not clear whether mutual friction will have the same effect for r-modes, and a new calculation of this effect is needed.

At each value of the temperature of the star, the critical angular velocity above which gravitations radiation has the shortest time-scale, compared to the viscosity time-scales, is obtained by solving the equation

$$\frac{1}{\tau_{\rm gw}} + \frac{1}{\tau_{\rm sv}} + \frac{1}{\tau_{\rm bv}} = 0. \quad (50)$$

We specialize to a specific neutron star model with radius $R = 12.47$km and mass $M = 1.5 M_\odot$ (same as in Andersson et al. 1998 and Lindblom et al. 1998). The density of the star is $\rho = 3.4\times 10^{14}{\rm gr/cm}^3$.

Fig. 1, shows the critical angular velocity as a function of temperature (in units of the angular velocity at the mass-shedding limit for Newtonian, uniform density and uniformly rotating stars, $\Omega_K \simeq 0.67\sqrt{\pi G\rho}$). The solid curve corresponds to the $O(\Omega^2)$ Eq. (37). A rapidly-rotating neutron star, born at temperatures 10^{11}K loses angular momentum because of the r-mode instability and slows-down. The minimum angular velocity it could reach is $\Omega_{\rm c} = 414{\rm s}^{-1}$ (or a period of 15ms) at $T \simeq 1\times 1.5\times 10^9$ K. The mass of the neutron star, or the adiabatic index, do not have a significant effect on the minimum critical angular velocity. The radius of the neutron star, however, (which can range from 10km to 15km), does have a considerable effect and the radius of our model represents a mean value of the expected radius of a typical $1.5M_\odot$ neutron star.

For low rotation rates, one can use (38) to construct approximate equations for the two parts of the curve in Fig. 1. The part of the critical curve where the shear viscosity dominates can be approximated by

$$\Omega_c^{(\rm sv)} = 581 \left(\frac{10{\rm km}}{R}\right)^{3/2} \left(\frac{10^9{\rm K}}{T}\right)^{1/3} \ {\rm s}^{-1}, \tag{51}$$

while the bulk viscosity dominated part is described by

$$\Omega_c^{(\rm bv)} = 362 \left(\frac{R}{10{\rm km}}\right)^{9/8} \left(\frac{T}{10^9{\rm K}}\right)^{3/4} \ {\rm s}^{-1}. \tag{52}$$

The two approximate expressions are shown as dotted curves in Fig. 1. For a period 1.56 ms (the period of the fastest known millisecond pulsar), the lowest order critical angular velocity differs from the $O(\Omega^2)$ result by $\sim 17\%$.

In a similar way, the lowest order approximations to the dissipation timescales are

$$t_{\rm gw} = -1.4 \times 10^6 \left(\frac{10^3 s^{-1}}{\Omega}\right)^6 \left(\frac{1.4 M_\odot}{M}\right) \left(\frac{10{\rm km}}{R}\right)^4 \ {\rm s}, \tag{53}$$

$$t_{\rm sv} = 3.6 \times 10^7 \left(\frac{R}{10{\rm km}}\right)^5 \left(\frac{T}{10^9{\rm K}}\right)^2 \left(\frac{1.4 M_\odot}{M}\right) \ {\rm s}, \tag{54}$$

and

$$t_{\rm bv} = 4.6 \times 10^9 \times \left(\frac{R}{10{\rm km}}\right)^5 \left(\frac{\Omega}{10^3 s^{-1}}\right)^2 \left(\frac{1.4 M_\odot}{M}\right) \left(\frac{10^9{\rm K}}{T}\right)^6 \ {\rm s}, \tag{55}$$

Our current results for the onset of the r-mode instability correspond to neutron stars with a very stiff equation of state. The results in Andersson et al. (1998), correspond to a much softer equation of state (an N=1.0 polytrope) and a comparison is shown in Fig. 2. The minimum critical temperature is roughly the same for both equations of state, although it occurs at a somewhat smaller temperature in the uniform density case. This shows that the uncertainty in the equation of state does not have a significant impact on the r-mode instability.

7. Discussion

Our analytical results for the onset of the r-mode instability in neutron stars agree well with numerically obtained results, for the same neutron star model. Using our analytic formulae, the uncertainty in the bulk viscosity can be easily explored for different present and future estimates of the bulk viscosity coefficient. The shear viscosity for lower temperatures is also uncertain, since a high shear viscosity due to the mutual friction between superfluid vortices below the superfluid transition temperature could suppress the instability. If future investigations provide a definite answer on the effect of mutual friction in a superfluid, one can easily study the implication on the r-mode instability using the analytic formulae presented in this paper.

We would also like to point out that, although an $l = 1$ dipole mode does not radiate in a non-rotating star, it does emit gravitational waves through the coupling to higher order terms, in rotating stars. According to (43), the shear viscosity for $l = 1$ vanishes and thus cannot affect the emission of gravitational waves, this is not true for realistic equations of state but still the damping times are extremely long. The $l = 1$ r-mode will radiate through the coupling to spheroidal $l = 2$ terms, i.e. it will generate mass quadrupole radiation. The frequency of this mode in the rotating frame is $\bar{\omega}_r = \varpi$, while in the inertial frame the frequency is $\bar{\omega}_i = (3/4)\varpi^3$. According to the criterion for the onset of the CFS-instability, the $l = 1$ r-modes are thus stable

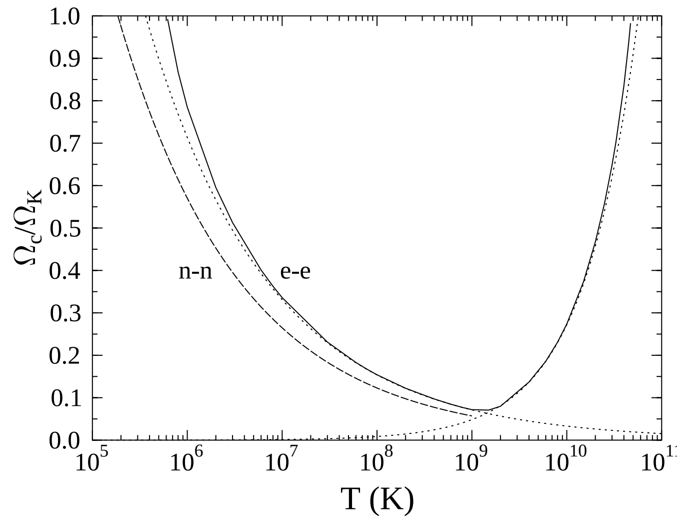

Fig. 1. Critical angular velocity for the onset of the r-mode instability as a function of temperature (for a 1.5 $M_\odot$ neutron star model). The solid line corresponds to the $O(\Omega^2)$ result using superfluid ($e^- - e^-$) shear viscosity, and Sawyer's (1989) estimate for the bulk viscosity. Dotted lines are lowest order approximations, while the dashed line corresponds to normal matter ($n - n$) shear viscosity.

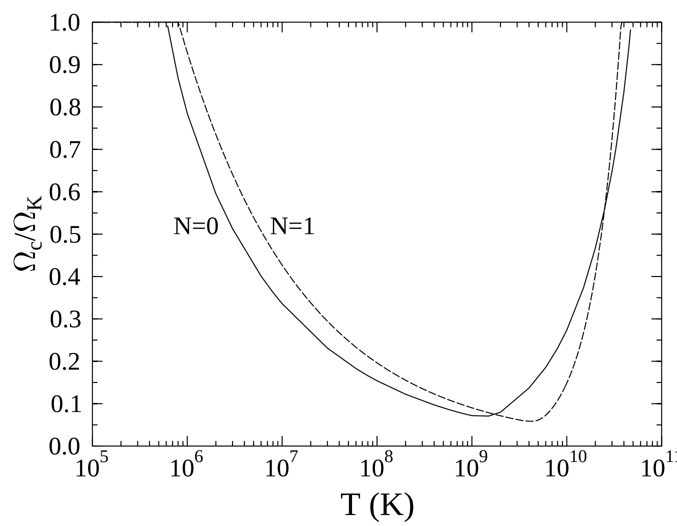

Fig. 2. Critical angular velocity for the onset of the r-mode instability as a function of temperature (for a 1.5 $M_\odot$ neutron star model). The present uniform density result ($N = 0$) is compared to the critical curve for the equation of state used in Andersson et al. (1998), ($N = 1.0$ polytrope). The minimum value of Ω_c/Ω_K is roughly the same, and the effect of the equation of state is mainly to shift the critical curve to different temperatures.

to the emission of gravitational waves, in contrast to the $l \geq 2$ modes. Such stable oscillations, unaffected by shear viscosity at low temperatures, could be excited during a neutron star glitch. In analogy to the $l = 1$ r-modes, an $l = 1$ spheroidal mode, like the f mode or the p-modes, will emit current quadrupole radiation and this case needs further study.

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