Astron. Astrophys. 341, 318–327 (1999)

ASTRONOMY
AND
ASTROPHYSICS

Solar system planetary motion to third order of the masses

X. Moisson

Bureau des Longitudes, URA 707 du CNRS, 77, avenue Denfert-Rochereau, F-75014 Paris, France

Received 10 March 1998 / Accepted 27 August 1998

Abstract. We present in this paper a new solution for planetary motion. It consists in developing the perturbations up to the third order of the planetary masses and uses the IERS92 masses set. The relativistic perturbations expressed in isotropic coordinates and the perturbations of the Moon on the inner planets are also included. Effects of asteroids on Earth-Moon barycenter, Mars and Jupiter have been taken into account. We used adjustments to numerical DE403 and DE405 integrations of the Jet Propulsion Laboratory (JPL) to obtain the theory's integration constants.

Key words: celestial mechanics, stellar dynamics – ephemerides – solar system: general

1. Introduction

This work has been motivated by the current need of very accurate analytical theories of the motion of the solar system's planets. The theory of Earth rotation as well as the reference systems theory are two fields among many which use analytical theories as the one we derived. These theories contain much physical information because of their expressions in terms of some physically relevant arguments as the mean mean longitudes of the planets.

In this work, the great accuracy we try to reach in our solution leads us to build a secular variation theory in which the long period's arguments (that is arguments with periods between few tens of thousand years and few million years) have been expanded with respect to time. Our analytical ephemeris has been compared with the numerical integrations of the Jet Propulsion Laboratory (JPL): DE403 and DE405.

In the first part of this paper, we describe the previous analytical VSOP theory which is a solution for all the planets except Pluto. We develop in the second part the method used for the construction of third order perturbations with respect to the masses. In the third part, we compare our solution to the ephemeris DE403 of the JPL. Results show great improvement compared to the former VSOP82 (Bretagnon, 1982) theory for the inner planets. In doing so, we determine the theory's integration constants and the rotation matrix between the equator and the ecliptic in J2000 at a precision level of 0.″00014. In a fourth part, we describe the results obtained in comparing the theory to the numerical DE405 integration of the JPL. Attempts to explain the differences found in terms of the effect of mass modifications, integration constants modifications and asteroids handling are investigated. The results are not relevant so that the intrinsic differences between DE403 and DE405 canot be explained in this way.

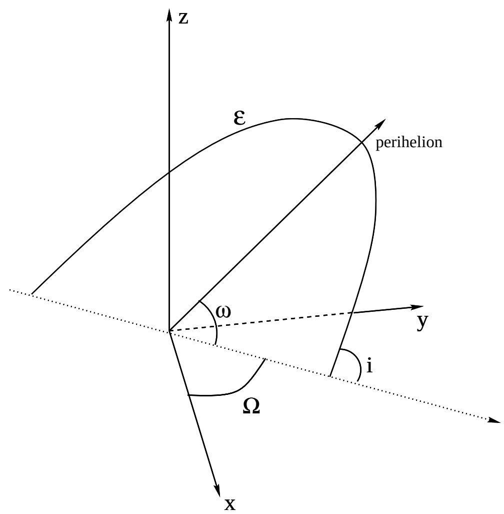

Fig. 1. Elliptic elements i, ω and Ω for an osculating ellipse $\mathcal{E}$

2. The VSOP82 theory

Analytical theories for the motion of the solar system's planets were built in the early 1980's by (Bretagnon, 1982) at the Bureau des Longitudes and are called VSOP solutions for "Variation séculaire des orbites planétaires". These solutions are based on the integration of Lagrange differential equations for the elliptic elements a, λ, k, h, q and p. a and λ are the semi-major axis and the mean longitude respectively. k, h, q and p are deduced from the variables e, i, ω and Ω described in the Fig. 1 by the following formulae:

$$\begin{aligned} \varpi &= \Omega + \omega \\ k &= e\cos\varpi \quad ; \quad q = \sin\frac{i}{2}\cos\Omega \\ h &= e\sin\varpi \quad ; \quad p = \sin\frac{i}{2}\sin\Omega \end{aligned} \tag{1}$$

Table 1. Frequency ν in $''$/year

Planet	**Merc.**	**Venus**	**Earth**	**Mars**
Frequency ($''$/year)	2000	1000	300	300
Period (years)	648	1296	4320	4320
Planet	**Jup.**	**Sat.**	**Uran.**	**Nept.**
Frequency ($''$/year)	200	200	200	200
Period (years)	6480	6480	6480	6480

Table 2. Biggest differences between VSOP82 and DE200 over the time span 1891–2000. Units are 10^{-10} for k, h, q and p, 10^{-10} rad for the mean longitude λ and 10^{-10} astronomical unit (au) for the semi major axis. An equivalent in arcseconds is given for λ.

Planet	**a**	λ	k	h	q	p
Mercury	3	29 ($0.''0006$)	22	24	7	9
Venus	39	205 ($0.''0042$)	130	100	64	54
Earth	61	168 ($0.''0034$)	161	141	83	60
Mars	270	534 ($0.''0110$)	326	339	96	112
Jupiter	3017	1874 ($0.''0387$)	1300	1403	152	104
Saturn	12836	4468 ($0.''0922$)	3345	2647	145	203
Uranus	39081	2172 ($0.''0448$)	1864	2101	177	200
Neptune	253770	8143 ($0.''1680$)	7210	3670	132	212

where e, i, ω and Ω are respectively the eccentricity, the inclination, the argument of the perihelion and the longitude of the ascending node of the orbit.

Results of integration are expressed under the form of Poisson's series of time in which the long-period arguments (that is to say arguments with periods between few tens of thousand years and few million years) have been expanded with respect to time.

The precision of VSOP solutions is determined by two main factors: the level of truncation of the series and the precision of the computation of each term. Analytical series of VSOP have been truncated at the level of 10^{-1} mas (1 mas = $0.''001$) for the inner planets and 2×10^{-1} mas for the outer planets. In terms of the accuracy of the computations, the following rule applies: for each planet, a minimum frequency is imposed, say ν, so that each term with a frequency $\nu' \geq \nu$ has been computed at the precision of 1 mas at least and the other ones are computed at the accuracy $\nu/\nu' \times 1 \text{ mas} > 1 \text{ mas} \quad ((\nu/\nu')^2 \times 1 \text{ mas}$ in the case of the mean longitude). Table 1 gives, for each planet, the frequency ν and the period associated to ν.

Both factors described above lead to an intrinsic accuracy of the VSOP theory which has to be confirmed by a comparison to a numerical integration such as the ephemeris DE200 of the JPL (Standish, 1982). This procedure permits the computation of the solution's integration constants as well as the discrepancies between the two ephemerides which roughly approximate the accuracy of VSOP analytical solution.

Table 2 summarizes these differences for a time span of more than a century [25 september 1891, 0 h - 26 february 2000, 0 h]. Units are 10^{-10} for the dimensionless variables k, h, q and p, 10^{-10} rad for the mean longitude λ and 10^{-10} astronomical units (au) for the semi major axis. We also give the amount in arcseconds for λ.

The goal of this work is to improve the VSOP theory. To achieve this, we have used values for the masses which were recently adopted (IERS92) and we have performed computations with an accuracy one hundred times better than the one used for the construction of the solution VSOP. Moreover, we have computed the perturbations of the following asteroids: Vesta, Iris, Bamberga, Ceres and Pallas on the Earth-Moon barycenter, Mars and Jupiter.

3. Perturbations up to the third order

In this section, we briefly describe the method used to integrate Lagrange differential equations with the help of a Taylor expansion of their right-hand members with respect to masses. This method has been used to derive the VSOP solution as well as our solution.

3.1. Notations

The Lagrange differential equations for a general n-body problem (Sun + (n-1) planets) consist in a set of 6(n-1) differential equations of the first order. If we suppose that the right hand members of these equations are known, the Lagrange differential equations can be seen as a set of linear differential equations and the principle of superposition may then be applied. Under this hypothesis, it is sufficient to model the following problem: the Sun, a disturbing planet (P') with mass m' and a perturbed one (P) with mass m.

We denote by σ any twelve elliptic elements a, λ, k, h, q and p of (P) or (P'); $\mathbf{V}$ and $\mathbf{V}'$ are the heliocentric position vectors of (P) and (P') respectively. We write $r = |\mathbf{V}|$, $|.|$ being the euclidian norm, $r' = |\mathbf{V}'|$ and $\Delta = |\mathbf{V} - \mathbf{V}'|$

3.2. Method

Let us write the Lagrange differential equation at hand with the simplified form:

$$\frac{d\sigma}{dt} = f_\sigma\left(x_i\right) \tag{2}$$

where $x_i = x_i^{(0)} + \Delta^{(1)} x_i + \Delta^{(2)} x_i$ is one of the twelve elements of the perturbed planet and the perturbating one. $x_i^{(0)}$ represents the solution of the Keplerian problem, $\Delta^{(1)} x_i$ the perturbations of the first order of the element x_i with respect to the masses and $\Delta^{(2)} x_i$ the perturbations of the second order. We perform a Taylor expansion of f_σ around the elliptic motion $x_i^{(0)}$ limited to the third order terms:

$$\frac{d\sigma}{dt} = f_\sigma\left(x_i^{(0)}\right) + \sum_{i=1}^{12}\left(\frac{\partial f_\sigma}{\partial x_i}\right)\Delta^{(1)} x_i + \sum_{i=1}^{12}\left(\frac{\partial f_\sigma}{\partial x_i}\right)\Delta^{(2)} x_i$$

$$+\frac{1}{2}\sum_{i=1}^{12}\sum_{j=1}^{12}\left(\frac{\partial^2 f_\sigma}{\partial x_i \partial x_j}\right)\Delta^{(1)}x_i\Delta^{(1)}x_j \quad (3)$$

To compute the derivatives found in (3), the function f_σ is expressed in the following form:

$$f_\sigma(x_i) = D_\sigma \mathbf{V}.(R_1\mathbf{V} + R_2\mathbf{V}') \quad (4)$$

with:

$$R_1 = -\frac{n\ a\ m'}{1+m}\frac{1}{\Delta^3}$$
$$R_2 = \frac{n\ a\ m'}{1+m}\left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) \quad (5)$$

and where each $D_\sigma \mathbf{V}$ is a vectorial quantity.

In this way, computations of the first and second order derivatives of f_σ are obtained in a closed form in Chapront et al. (1975) and in Bretagnon (1980). The main advantage in computing this closed form is the possibility to deduce from the results the precision required for each step of the computer calculation. Indeed, it appears that each derivative can be written as an expansion:

$$T_0 + T_1\frac{1}{\Delta^3} + T_2\frac{1}{\Delta^5} + T_3\frac{1}{\Delta^7} \quad (6)$$

where T_k are series depending on the solution of the Keplerian problem for each planet.

The method consists then in computing the perturbations order by order. The perturbations $\Delta^{(1)}x_i$ are obtained by integrating $f_\sigma\left(x_i^{(0)}\right)$, the integration of $\sum_{i=1}^{12}\left(\frac{\partial f_\sigma}{\partial x_i}\right)\Delta^{(1)}x_i$ gives the second-order perturbations $\Delta^{(2)}x_i$, and finally, we compute the third-order perturbations in integrating the last term of the expression (3).

3.3. Integration of the equations

We have integrated the Lagrange differential equations as described above using the IERS 92 masses set given in Table 3. For the integration, we used the J2000 constants for the elliptic elements a, λ, k, h, q and p determined in Bretagnon (1982). We used the frequency ν given in Table 1 but the corresponding accuracy is 10^{-2} mas for the inner planets and 10^{-1} mas for the outer ones. The series have been truncated at the level of 10^{-4} mas for the inner planets and 2.10^{-3} mas for the outer one.

Results of the integration are expressed with Poisson's series of time T limited to the degree 3 because of the lack of higher order perturbations in our theory:

$$\sigma = \sum_{m=0}^{3} T^m P_m \quad (7)$$

where P_m is a periodic series:

$$P_m = \sum_{j\in J} A_{m,j}\sin(\phi_{m,j}) + B_{m,j}\cos(\phi_{m,j}) \quad (8)$$

Table 3. IERS 1992 masses and mean mean motions of the theory. Unit is rad/year for N. For the planetary masses, we give the ratio $\mathrm{M_{SUN}/M_{Planet}}$

Planet	$\mathrm{M_{SUN}/M_{Planet}}$	N
Mercury	6 023 600.0000	26.087 903 141 574 2
Venus	408 523.7100	10.213 285 546 211 0
Earth	328 900.5600	6.283 075 849 991 4
Mars	3 098 708.0000	3.340 612 426 699 8
Jupiter	1 047.3486	0.529 690 965 094 6
Saturn	3 497.9000	0.213 299 095 438 0
Uranus	22 902.9400	0.074 781 598 567 3
Neptune	19 412.2400	0.038 133 035 637 8

with $(A_{m,j}, B_{m,j}) \in I\!R^2$, J being a finite set. Arguments $\phi_{m,j}$ in (8) are linear combinations of the mean mean longitudes $\lambda_l = \lambda_l^0 + N_l\,T$, that is:

$$\phi_{m,j} = \sum_{l=1}^{8} k_{m,j,l}\lambda_l \quad (9)$$

for $k_{m,j,l} \in \mathbb{Z}$.

The mean mean motions $N_l,\ l = 1..8$ found in the expression of the mean mean longitudes λ_l are given in Table 3 and correspond to the adjustment of the VSOP82 theory to DE200. Corrections to these mean mean motions will be deduced from the adjustment of our solution to DE403 in the next part.

3.4. Relativity, lunar complements, asteroids

The Schwarzschild relativistic contributions in isotropic coordinates have been added to the results of the analytic integration for all the planets. Moreover, perturbations of the Moon on the Earth-Moon barycenter and on the inner planets have also been included in the analytical solution. Both contributions are expressed with Poisson's series of time. Although relativistic and lunar contributions are deduced from the VSOP solution and should be computed with the help of the new theory, the effect of such an approximation is of second order and does not significantly affect the results of the comparisons to numerical DE403 ephemeris presented hereafter.

We computed first-order perturbations on Earth-Moon barycenter, Mars and Jupiter of the minor planets: Vesta, Iris, Bamberga, Ceres and Pallas, the J2000 elliptic elements of which are listed in Table 4.

The contributions of this first-order perturbations have been added to our solution too.

4. Comparison to DE403

The analytical solution we have built (perturbations up to the third-order + relativistic complements + Moon complements on the inner planets + asteroids contributions on Earth-Moon barycenter, Mars and Jupiter) is physically relevant if and only if it i s adjusted in some sense to observations, this sense be-

Table 4. Elliptic elements used for the minor planets. For the planetary masses, we give the ratio $\mathrm{M_{MP}/M_{SUN}}$ where MP stands for minor planet. Units is $rad/year$ for N, rad for λ_0 and 10^{-10} for the ratio $\mathrm{M_{MP}/M_{SUN}}$.

Minor planet	N	λ_0	k	h	q	p	$\mathrm{M_{MP}/M_{SUN}}$
Vesta	1.7311765	4.09149775810	−.024226855	−.086147151	−.015012622	.060412702	1.340
Iris	1.7044508	1.71780146320	.162676547	.162154931	−.008434956	−.047370906	0.054
Bamberga	1.4289030	5.59335131690	.331352426	.071208447	.082299375	−.051004062	0.088
Ceres	1.3647634	2.80627678560	−.069102469	.035248334	.015146103	.091100211	4.640
Pallas	1.3619495	2.03486053350	−.127958153	.194688580	−.297351380	.035411460	1.050

Table 5. Integration constants of our solution by evaluation of DE403. Units are $rad/year$ for N and rad for λ^0

Planet	N	λ^0	k	h	q	p
Mercury	26.0879031406104	4.40260863104	0.04466063600	0.20072330403	0.04061565406	0.04563549265
Venus	10.2132855474372	3.17613445055	−0.00449281984	0.00506684795	0.00682411337	0.02882282378
Earth	6.2830758508994	1.75346989883	−0.00374081565	0.01628447938	0.00000000000	0.00000000000
Mars	3.3406124316303	6.20349999629	0.08536560315	−0.03789973949	0.01047043002	0.01228448716
Jupiter	0.5296909553481	0.59955957616	0.04698613142	0.01200421559	−0.00206543955	0.01118381015
Saturn	0.2132991252582	0.87398398117	−0.00296090913	0.05542790595	−0.00871729544	0.01989130734
Uranus	0.0747816536222	5.48122292038	−0.04595424617	0.00564826786	0.00185927847	0.00648605481
Neptune	0.0381329126393	5.31189740459	0.00599840984	0.00669074462	−0.01029144772	0.01151685425

ing defined as a fit to the numerical integrations of JPL which themselves were adjusted to fit observations.

Practically, this means that we have to determine the theory's integration constants. To achieve this, we chose to adjust our solution to the ephemeris DE403 (Standish and al., 1995), which includes a set of observations (optical, radar, spacecraft measurements...) covering more or less the period [1914–1995]. In the same time, we can co mpare our analytical solution and DE403 and estimate the divergences between the two ephemerides.

We decided to use the time span [January 1st 1900, 0h - January 1st 2000, 0h] for the determination of the theory's integration constants and the comparison. This choice has been made for two reasons. Firstly, as I already mentioned it, the observationa l data to which DE403 ephemeris was fitted covers almost the same period. Secondly, comparison of the VSOP82 theory to DE200 (see part 1.) has been performed on a comparable time span. Thus, the differences found will be comparable to the ones listed in Table 2.

4.1. Link between the reference frames of both solution

The first step of this comparison is to determine the two angles ε and φ occuring in the rotation matrix between the equator plane of the DE403 solution and the dynamical inertial ecliptic in which we built our solution. Let us write:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{AS} = \mathcal{M}\left(\varepsilon,\ \varphi\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{DE403} \tag{10}$$

where AS stands for analytical solution, $(x, y, z)_{\mathcal{R}}$ are the cartesian coordinates of a planet in the frame $\mathcal{R}$ and:

$$\mathcal{M}\left(\varepsilon,\ \varphi\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

To determine ε and φ, we proceeded by successive iterations from the values: $\varepsilon_0 = 23^o26'21.''409$ and $\varphi_0 = -2604.605\ 10^{-10}$ rad. A linear regression is fitted to the residuals between the two solutions for the variables p and q of the Earth. The intercept constants at J2000 of these linear regressions give the two quantities Δq and Δp from which are deduced the contributions $\Delta\varepsilon = \varepsilon - \varepsilon_0$ and $\Delta\varphi = \varphi - \varphi_0$ by the formulae:

$$\left\{ \begin{array}{lcl} \Delta\varepsilon & = & -2\Delta q \\ \Delta\varphi & = & -\frac{2\Delta p}{\sin\varepsilon_0} \end{array} \right. \tag{12}$$

We obtained:

$$\left\{ \begin{array}{lcl} \varepsilon_{DE403} & = & 23^o26'21.''40872 \\ \varphi_{DE403} & = & -0.''05340 \end{array} \right. \tag{13}$$

With these values of ε and φ, the rotation matrix (11) annuls the inclination variables q and p of the Earth in J2000.

4.2. Integration constants

The procedure indicated above is also used to determine the new integration constants of the theory. For each element σ, we have computed in the same way the quantities $\Delta\sigma$ which have to be added to the J2000 constants used for the integration. The new constants are reported in Table 5. The comparison of our solution to DE403 for the integration constants determination implies that the mean mean motions N are expressed in the same time scale as the one used in DE403.

Because of the limitation to third-order perturbations in the theory, there is no reason to consider these constants as defini-

Table 6. Biggest differences between our solution and DE403 over the time span 1900–2000. Units are 10^{-10} for k, h, q and p, $10^{-10} rad$ for the mean longitude λ and 10^{-10} astronomical unit (au) for the semi major axis. An equivalent in arcseconds is given for λ

Planet	a	λ	k	h	q	p
Mercury	0.8	17 (0.″00035)	16	16	2.6	2.7
Venus	8.2	26 (0.″00054)	28	24	7.4	3.6
Earth	33	64 (0.″00132)	71	71	1.6	2.9
Mars	176	365 (0.″00753)	281	225	6.6	5.7

tive. Nevertheless, for the inner planets, their amplitude should not be affected much by the influence of higher order perturbations. It is no longer the case for the outer ones, for which the results of the iterative method will be of most importance.

4.3. Differences between DE403 and our solution

After determining both rotation matrix between frames of our solution and DE403 and integration constants, we have computed systematically the differences between these two solutions. Table 6 gives the biggest differences found on the time span already mentioned before [January 1st 1900, 0h - January 1st 2000, 0h]. The results are not shown for the outer planets because the theory of these bodies is not accurate enough when limited to the third-order perturbations. As in Table 2, the units are 10^{-10}, $10^{-10} rad$ for λ and 10^{-10} astronomical unit (au) for the semi major axis and we give an equivalent in arcseconds for λ.

A first comparison with Table 2 indicates that our solution greatly improves the VSOP one especially for the inclination variables of the Earth q and p. This point is of great importance because it allows us to determine the dynamical inertial ecliptic with a theoretical precision of 0.14 mas.

To illustrate the differences of Table 6, we plot the residuals for the elliptic variables a, λ, k, h, q and p of Mercury on Fig. 2. We also ploted the differences for the mean longitude λ of Venus, the Earth and Mars on the Fig. 3.

5. Comparison to DE405. Secular drifts between DE403 and DE405

We have proceeded with the same analysis as described for DE403 with the numerical DE405 solution of the JPL (Standish et al. 1997). In this case, the rotation matrix (11) is defined by the values of ε and φ:

$$\begin{cases} \varepsilon_{DE405} &= 23^o 26' 21.'' 40893 \\ \varphi_{DE405} &= \quad -0.'' 05101 \end{cases} \tag{14}$$

We then determined the discrepencies between DE405 and our solution over the time span [January 1st 1900, 0h - January 1st 2000, 0h] with these values of ε and φ. The results are given for the inner planets in Table 7.

In comparing Table 7 to Table 6, we see that our analytical solution is slightly closer to DE405 than to DE403 for the semi-major axis and the mean longitudes of the inner planets. However, the results for Mercury's k, h variables are worse, and the same phenomenon holds for the variables q and p of all planets. A direct comparison between DE403 and DE405 shows that they are secular discrepancies between the two solutions (see Fig. 4).

Table 7. Biggest differences between our solution and DE405 over the time span 1900–2000. Units are 10^{-10} for k, h, q and p, 10^{-10} rad for the mean longitude λ and 10^{-10} astronomical unit (au) for the semi-major axis. An equivalent in arcseconds is given for λ

Planet	a	λ	k	h	q	p
Mercury	0.7	17 (0.″00035)	<u>251</u>	<u>67</u>	<u>21</u>	<u>34</u>
Venus	7.5	26 (0.″00054)	27	24	<u>13</u>	<u>6.0</u>
Earth	31	63 (0.″00130)	69	69	<u>6.4</u>	<u>3.7</u>
Mars	172	359 (0.″00741)	284	227	<u>13</u>	<u>9.8</u>

Table 8. Slope of the linear regression fitted to the differences between our solution and DE405 for the k, h, q and p variables. Units are 10^{-10}/century.

Planet	A_k	A_h	A_q	A_p
Mercury	245.374	−56.418	21.283	33.685
Venus	−9.731	−2.885	13.400	5.473
Earth	−23.640	−21.918	6.109	3.086
Mars	−102.747	−33.778	11.734	8.338

These secular discrepancies are responsible for all the worse results of the comparison of our solution to DE405. Indeed, we have written out on Table 8 the slopes A of the linear regressions $A * T + B$ fitted to the differences between our complete third-order solution and DE405.

As we can see, the important differences given in the Table 7 (underlined quantities in this table) are mainly due to the contributions of the secular terms $A * T$ over one century.

5.1. How to explain the secular drifts between DE403 and DE405

Unlike the mean mean motions N, the secular terms of the k, h, q and p variables are perturbations computed analytically in our solution. Only the J2000 value of these variables are adjusted to the observations. Therefore, it is impossible to adjust the secular terms of the k, h, q and p variables of our analytical solution in order to cancel the secular discrepancies between DE405 and our solution.

The force model for DE405 is nearly the same as that for DE403. The most significant is in the handling of the asteroids. In DE403, keplerian formulae with secular rates for the elements are used. For DE405, the orbits of the asteroids are integrated using DE404 (Standish, private communication).

Then, we investigated three possibilities to explain the intrinsic secular differences between the two ephemerides:

– the mass modification

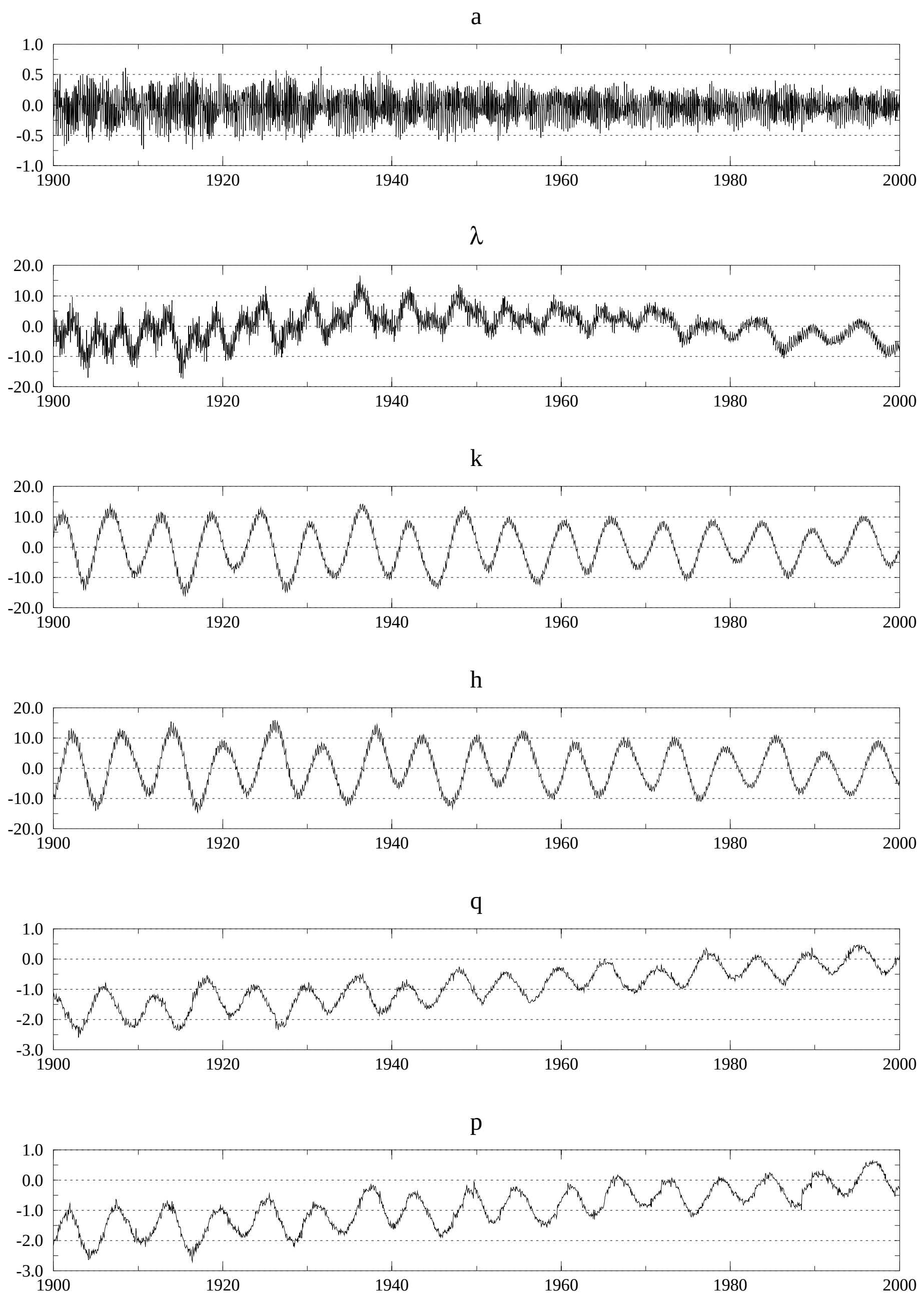

Fig. 2. Differences between our solution and DE403 for Mercury. Units are 10^{-10} astronomical units (au) for the semi major axis a, 10^{-10} rad for the mean longitude λ and 10^{-10} for the k, h, q and p variables.

– the influence of theory's integration constants
– the handling of the asteroids.

To make explicit the influence of each modification, we will exhibit the case of the variable k of Mercury as a comprehensive example. For this planet, we have obtained from our third order theory:

$$\begin{aligned} k \;=\; & 0.044\,660\,636\,00 \qquad \text{(integration constant)} \\ & -\; 5102887.47\; 10^{-10}\; T \text{ (1st order secular perturbation)} \\ & -\; 1012.25\; 10^{-10}\; T \quad \text{(2nd order secular perturbation)} \\ & -\; 4.85\; 10^{-10}\; T \quad \text{(3rd order secular perturbation)} \\ & +\; \ldots \end{aligned}$$

where T is measured in century from J2000.

5.1.1. Mass modification

The only mass modification between DE403 and DE405 affects the Earth-Moon barycenter. The Earth-Moon barycenter's GM values are:

$$GM_{DE403} \;=\; 0.8997011374291877 \times 10^{-9}$$
$$GM_{DE405} \;=\; 0.8997011346712499 \times 10^{-9}$$

We have computed that the maximum induced contributions to the secular terms of the k, h, q and p variables of Mercury, Venus the Earth and Mars are a few 10^{-15} per year that is a few 10^{-13} after one century. Thus, the contribution of this mass modification is not sufficient.

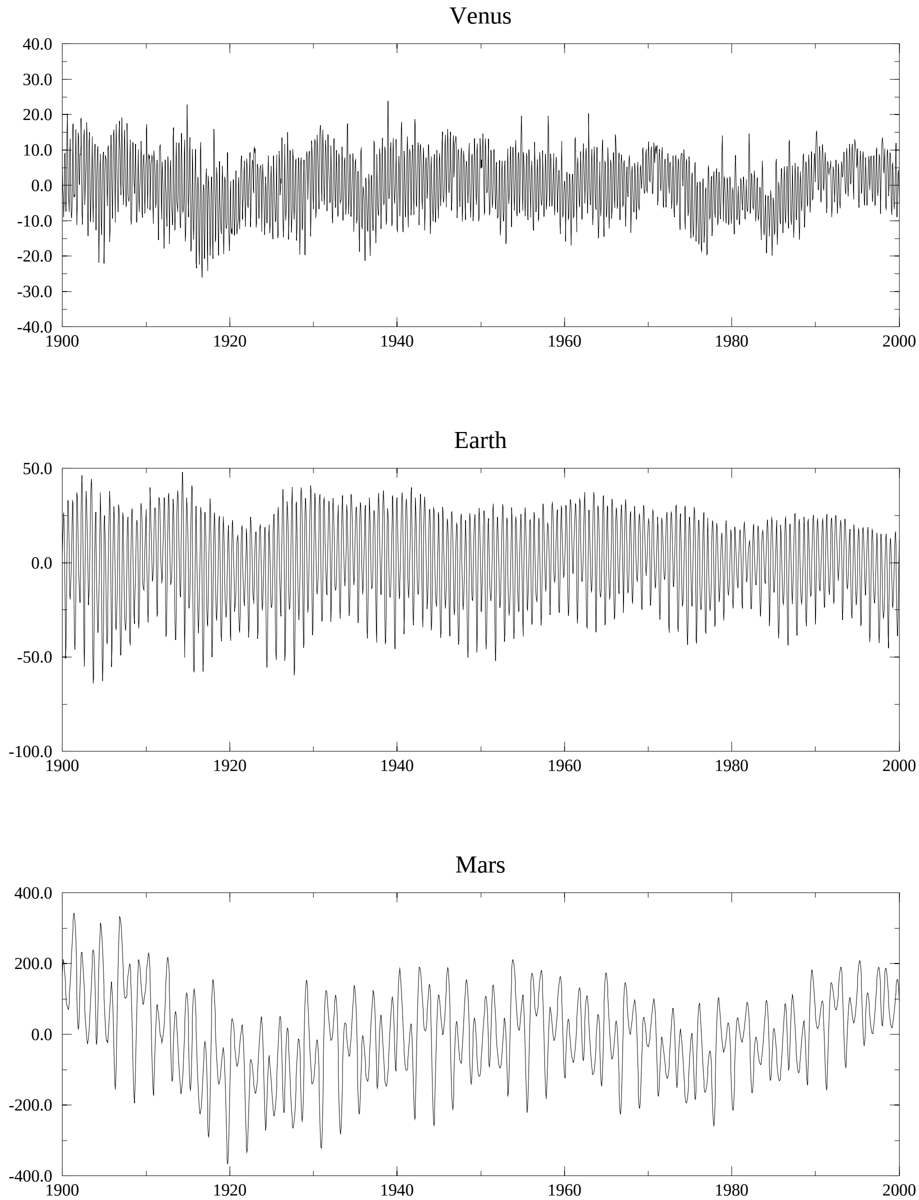

Fig. 3. Differences between our solution and DE403 for the mean longitude λ of Venus, the Earth and Mars. Unit is 10^{-10} rad.

Table 9. Integration constants of our solution by comparison to DE405. Units are $rad/year$ for N and rad for λ^0

Planet	N	λ^0	k	h	q	p
Mercury	26.08790314072	4.4026086402872	0.04466063156	0.20072330594	0.04061564895	0.04563550715
Venus	10.21328554733	3.1761344529928	−0.00449281986	0.00506684784	0.00682411642	0.02882282244
Earth	6.28307585085	1.7534699031363	−0.00374081570	0.01628447938	0.00000000000	0.00000000000
Mars	3.34061243173	6.2035000031353	0.08536560340	−0.03789973899	0.01047042972	0.01228448722
Jupiter	0.52969095278	0.5995595026467	0.04698613723	0.01200422322	−0.00206561747	0.01118389565
Saturn	0.21329912411	0.8739839547220	−0.00296081707	0.05542779719	−0.00871736917	0.01989137396
Uranus	0.07478165727	5.4812230869150	−0.04595407683	0.00564815620	0.00185921733	0.00648608038
Neptune	0.03813291938	5.3118978682528	0.00599871879	0.00669072445	−0.01029152731	0.01151689940

In the case of our example, we can roughly estimate:

$$|\Delta k|_{Mass}| \leq 0.01\ 10^{-10}\ T$$

where T is measured in century.

5.1.2. Theory's integration constants modification

The theory's integration constants determined in Sect. 4 and the ones obtained in comparing our solution to DE405 are not the

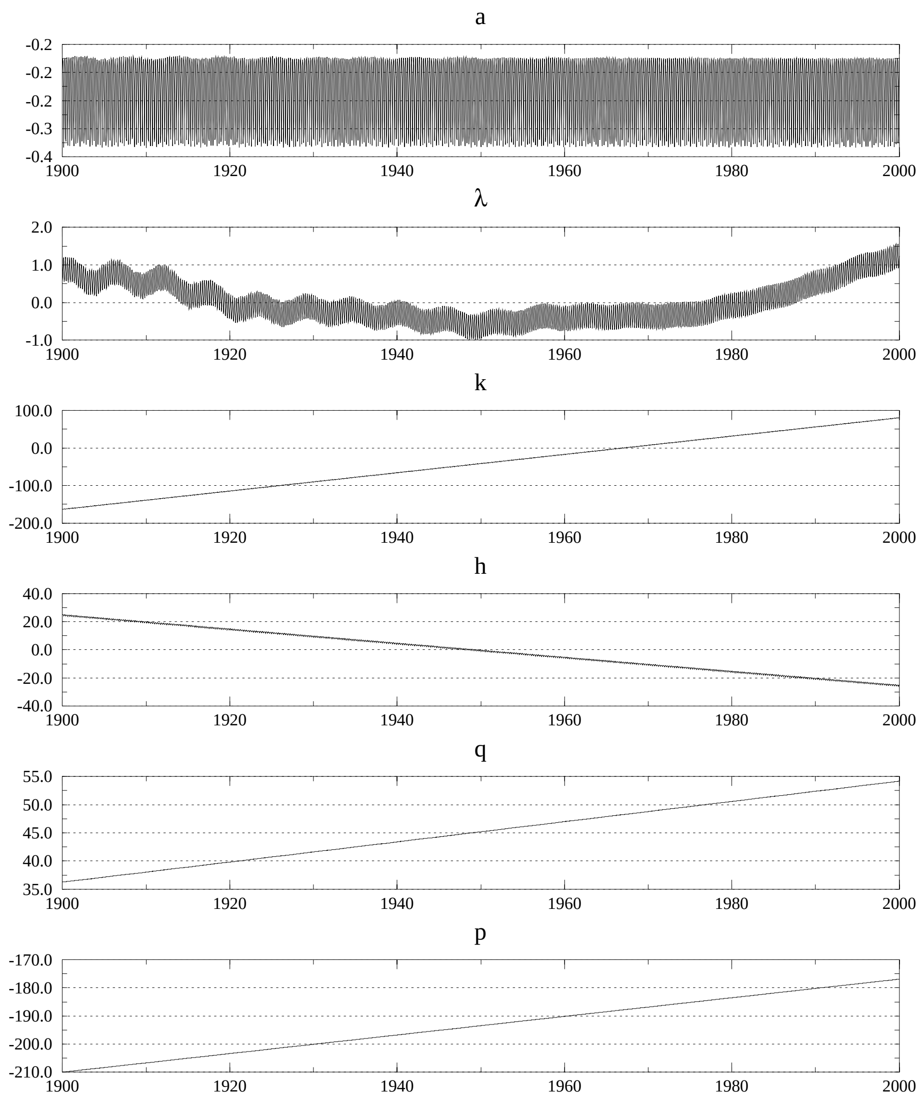

Fig. 4. Differences between DE403 and DE405 for Mercury variables a, λ, k, h, q, and p. Units are 10^{-10} astronomical units (au) for the semi major axis a, 10^{-10} rad for the mean longitude λ and 10^{-10} for the k, h, q and p variables.

same. These theory's integration constant modifications could explain the secular differences.

To confirm or invalidate this hypothesis, we have computed first-order solutions, which represents 90% of the global one, with the two integration constant set given in Table 5 (DE403) and in Table 9 (DE405). The results of the integration are noted as σ_{DE403} and σ_{DE405} respectively. We then computed the $\sigma_{DE403} - \sigma_{DE405}$ differences between these two first-order solutions for the k, h, q and p variables of the inner planets.

For our example, we give hereafter the difference found for the variable k of Mercury:

$$\begin{aligned}
k_{DE403} - k_{DE405} \;=\;& 0.000\,000\,004\,44 \\
&+\; 0.416\ 10^{-12} \sin 2\lambda_5 \\
&+\; 0.703\ 10^{-12} \cos 2\lambda_5 \\
&-\; 0.070\ 10^{-12} \sin \lambda_5 \\
&-\; 0.403\ 10^{-12} \cos \lambda_5 \\
&-\; 0.130\ 10^{-12} \sin 3\lambda_6 \\
&-\; 0.252\ 10^{-12} \cos 3\lambda_6 \\
&-\; 0.131\ 10^{-12} \sin (2\lambda_1 - 5\lambda_2) \\
&+\; 0.185\ 10^{-12} \cos (2\lambda_1 - 5\lambda_2) \\
&+\; \ldots \\
&-\; 0.38600\ 10^{-10}\ T \\
&+\; \ldots
\end{aligned}$$

where T is measured in century, λ_1 stands for $\lambda_{\mathrm{Mercury}}$, λ_2 for λ_{Venus}, λ_5 for $\lambda_{\mathrm{Jupiter}}$ and λ_6 for $\lambda_{\mathrm{Saturn}}$.

Table 10 gives the secular coefficients $\Delta\sigma|_s$ ($\sigma \in \{k,\ h,\ q,\ p\}$) of these differences (i.e. the coefficient of T in the serie of the difference).

Obviously, there is no chance to explain the secular differences between DE403 and DE405 with only the help of theory's integration constant modifications, except for the q and p variables of the Earth and Mars. Indeed, in comparing Tables 8 and 10, we see that these changes of integration constants modify the secular rates of the k, h, q and p variables by only a tenth to a thousandth of the expected contribution necessary to explain the secular differences between DE403 and DE405.

Table 10. Secular coefficients of the differences, for the k, h, q and p variables, between a first order solution computed with the DE403 integration constants set and a first order solution computed with the DE405 integration constants set. Units are 10^{-10}/century.

Planet	$\Delta k\|_s$	$\Delta h\|_s$	$\Delta q\|_s$	$\Delta p\|_s$
Mercury	-0.386	0.476	0.320	1.421
Venus	-0.330	0.272	1.906	3.890
Earth	-0.383	0.460	3.009	6.171
Mars	0.186	0.211	6.235	12.796

Table 11. Secular terms of the first order perturbations of the k, h, q and p variables of Mercury induced by the two groups of asteroids. Units are 10^{-10}/century.

Minor Planet	$\Delta k\|_s$	$\Delta h\|_s$	$\Delta q\|_s$	$\Delta p\|_s$
Vesta	-2.50	0.56	-0.22	-0.85
Iris	-0.08	0.01	0.04	-0.03
Bamberga	-0.12	-0.02	0.06	0.03
Ceres	-4.92	1.29	-1.20	-0.83
Pallas	0.70	0.44	-0.06	-1.73
Second group	-4.04	1.06	-0.98	-0.68
Total	-10.96	3.34	-2.36	-4.09

5.1.3. Handling of the asteroids

Our analytical solution has been completed with the first order perturbations of Vesta, Iris, Bamberga, Ceres and Pallas on the Earth-Moon barycenter, Mars and Jupiter. Let us complete our analysis of secular discrepancies by estimating the contributions of the asteroids to the secular terms of the k, h, q and p variables of Mercury and Venus.

For this purpose, we have used the asteroids considered in creating DE245 (Standish, 1993) which have been splited in two groups: the first group includes Vesta, Iris, Bamberga, Ceres and Pallas for a total mass of $7.17\ 10^{-10}$ solar mass and the second group is made up of 295 other asteroids for a total mass of $3.807\ 10^{-10}$ solar mass.

For the second group, we have made the assumption that all the 295 asteroids were glued into a single one following the same orbit than Ceres. This is of course a very poor modelisation of the asteroid belt but our purpose is only to get a rough estimate of the influence of these 295 asteroids on Mercury and Venus.

The secular contributions to the k, h, q and p variables of Mercury and Venus due to the two groups of asteroids are listed in Table 11 for Mercury and 12 for Venus.

Thus, for the case of Mercury, we have:

$$\begin{aligned}\Delta k|_{Ast.} \;=\; & -6.92\ 10^{-10}\ T \\ & \text{(1st group secular perturbation)} \\ & -4.04\ 10^{-10}\ T \\ & \text{(2nd group secular perturbation)}\end{aligned}$$

As one can see in comparing Tables 11 and 12 to Table 8, the contributions of the asteroids considered are not able to reduce the secular discrepencies between DE403 and DE405 except for the q and p variables of Venus.

Table 12. Secular terms of the first order perturbations of the k, h, q and p variables of Venus induced by the two groups of asteroids. Units are 10^{-10}/century.

Minor Planet	$\Delta k\|_s$	$\Delta h\|_s$	$\Delta q\|_s$	$\Delta p\|_s$
Vesta	-1.35	0.15	-1.14	-0.78
Iris	0.09	-0.09	0.12	-0.02
Bamberga	0.01	-0.20	0.15	0.14
Ceres	0.44	1.21	-4.51	0.60
Pallas	-0.29	0.32	-0.13	-3.89
Second group	0.36	0.99	-3.70	0.49
Total	-0.74	2.38	-9.21	-3.46

This result also shows that the different handling of the asteroids in DE403 and DE405 is not responsible for the secular differences between the two solutions. Indeed, the secular perturbations due to the asteroids are at least ten times smaller than the slopes listed in Table 8 for Mercury. Thus, a different model for the motion of the asteroids, which would only modify by a few percents these secular perturbations, cannot remove the secular discrepancies between DE403 and DE405.

5.1.4. Conclusion

To conclude this analysis, let us summarize the example of the variable k of Mercury. For this variable, we found a secular drift between DE403 and DE405 with a slope of about $245\ 10^{-10}$ per century. The mass modification can explain a slope of only a few 10^{-13} per century. The integration constants modification implies a secular perturbation of $0.386\ 10^{-10}$ per century. The asteroids contribute to $-6.92\ 10^{-10}$ per century for the first group and the 295 others cannot explain more than $4.04\ 10^{-10}$ per century. Then, the different handling of the asteroids can only contribute to a few percents of these two last perturbations. Thus, the slope between DE403 and DE405 for the case of the variable k of Mercury is not explainable by this way and the analysis we have made shows that the same phenomenon holds for the k and h variables of the inner planets and for the q and p variables of Mercury.

6. Conclusion

In this paper, we have presented a new solution for the solar system bodies motion. This solution has been built using the IERS92 masses set at a very high precision. We have included to the solution the Schwarzschild relativistic complements and the perturbations of the Moon on the inner planets. The effects of minor planets on Earth-Moon barycenter, Mars and Jupiter have been taken into account.

The results of the comparisons of our solution to the numerical DE403 ephemeris, necessary to determine the theory's integration constants, have shown that our solution improves the

VSOP theory for the inner planets in that sense that it is closer to observational data. The inclination variables q and p of the Earth-Moon barycenter do not diverge with respect to DE403 for more than a few 10^{-10} which enables ones to determine the dynamical inertial ecliptic with an accuracy of 0.14 mas.

The same comparison has been carried out with the numerical DE405 solution. The results have permitted to exhibit secular discrepancies between DE403 and DE405 ephemerides for the inner planets which are neither explainable by the mass modification influence nor by the theory's integration constants modifications nor by the handling of the asteroids.

For what concerns the future prospects of this solution, it should serve as an excellent basis for an iterative method. The iterative method consists in computing the perturbative function (the right hand members of (2)) with the help of the iteration $N-1$ and to construct the iteration N by integrating (2). This method takes advantage of the fact that there is no need to make explicit the succesive derivatives of f_σ to obtain the perturbations of order greater than three.

The usefulness of this method for the theory of the outer planets has been pointed out in Bretagnon (1980). Its utilisation for the whole solar system has never been undertaken until now and we think that it should be very efficient to manage the construction of a very accurate analytical theory of the motion of all the planets.

Acknowledgements. The author thanks Dr. E.M. Standish for providing useful information about the DE403 and DE405 numerical integrations. The author also thanks Dr. P. Bretagnon and Dr. J.L. Simon for fruitful discussions and suggestions.

References

Bretagnon P., 1980, A&A 84, 329
Bretagnon P., 1982, A&A 114, 378
Bretagnon P., Francou G., 1988, A&A 202, 309
Bretagnon P., Rocher P., Simon J.-L., 1997, A&A 319, 305
Chapront J., Bretagnon P., Mehl M., 1975, C&M 11, 379
Chapront-Touzé M., Chapront J., 1983, A&A 124, 50
Simon J.L., Bretagnon P., Chapront J., et al., 1994, A&A 282, 663
IERS, 1992, In: McCarthy D.D. (ed.) IERS standards (1992). IERS Technical Note 13, Observatoire de Paris
Standish E.M., 1982, A&A 114, 297
Standish E.M., Newhall X.X., Williams J.G., Folkner W.F., 1995, JPL IOM 314.10-127
Standish E.M., 1997, Magnetic tapes.