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A proposal for a sufficient test of luminosity functions

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Abstract. A new statistic o/o_{max} is introduced in this paper. If the proposed luminosity function correctly describes the distribution of the adopted sample, then the values of $o/o_{\max}$ would be uniformly distributed in the interval $[0, 1]$. We find that the $o/o_{\max}$ test of the luminosity function is a necessary one, and the $o/o_{\max}$ test is independent of the n/n_{max} test which is a revision of the famous V/V_{max} test, presented in our previous work (Qin & Xie 1997). It is proved that two given distributions of $o/o_{\max}$ and $n/n_{\max}$ correspond to a unique normalized luminosity function. The distributions of $o/o_{\max}$ and $n/n_{\max}$ therefore provide a sufficient test for any adopted forms of the luminosity function. As a demonstration, we present a simple application of the method to a luminosity function and a sample found in the literature.

Key words: methods: statistical – galaxies: luminosity function – galaxies: quasars: general – cosmology: observations

1. Introduction

The motivation for studying the luminosity function of quasars is that the luminosity function is closely related to the structure of the Universe. A correct luminosity function of quasars can not only describe the space and luminosity distributions of the objects, but also provide a test for cosmological models (e.g., the q_0 test described by Turner 1979a and 1979b). As the redshifts of quasars are high, the luminosity function of the objects might provide fateful constraints on cosmological models, especially when concerning the early state of the Universe. The luminosity function of the objects can be estimated with flux-limited samples by applying some statistical methods, such as the $1/V_a$ method (Schmidt 1968; Hartwick & Schade 1990) and the C method (Lynden-Bell 1971; Choloniewski 1987) as well as the Turner method (Turner 1979a, 1979b). However, many astronomers are not satisfied with only getting an empirical luminosity function. Because it can plainly exhibit the evolution of density or luminosity of quasars and then reveals quantitatively the relation of the distribution of the objects with the nature of the Universe, an analytical form of the luminosity function which correctly describes the distribution of the objects is strongly desired. So, many analytical forms of the luminosity function have been proposed and discussed. They are classified into several models. Among the many models of the luminosity function, the pure luminosity evolution model is the most successful one and has been frequently used (Mathez 1976, 1978; Marshall et al. 1984; Boyle et al. 1988, 1990; Pei 1995; Van Waerbeke et al. 1996).

To check if a proposed luminosity function is correct, one needs a powerful statistical method. There have been many statistical tests applicable to the study of distribution of sources (for example, the test of the constant functional form assumption for the luminosity function proposed by Turner 1979b). Among them, the famous V/V_{max} test, first introduced by Schmidt (1968), has been widely used. It can be used to check if the space distribution of objects is uniform, as well as to check if the distributions of different kinds of objects (e.g., the more luminous objects and the less luminous objects) or different samples are the same (see, e.g., Schmidt & Green 1983; Osmer 1980). The test was first applied to check the pure luminosity evolution model by Mathez (1976), and it was still used recently for the cosmological test under the assumption of the pure luminosity evolution model recently (see Van Waerbeke et al. 1996). A revision of the test, the V'/V'_{max} test, first introduced by Schmidt (1968), can be used to check if a proposed luminosity function belonging to the pure density evolution model is correct, where $dV' = \rho(z)dV$ is the "density-weighted volume element" with $\rho(z)$ being the density law assumed (see also, Turner 1979b; Avni & Bahcall 1980; Hartwick & Schade 1990). Applying both the V/V_{max} and V'/V'_{max} tests to radio sources, Schmidt (1968) showed that the space distribution of the sources is not at all uniform, and the pure density evolution model may account for the distribution of the sources. However, many analyses show that, while the luminosity-dependent density evolution and the pure luminosity evolution models are likely to meet the true distribution of quasars, the pure density evolution model is unlikely to meet the distribution (e.g., Schmidt & Green 1983; Boyle et al. 1987, 1988). Therefore, the application of the V'/V'_{max} test to the study of the luminosity function of quasars is limited. In our previous work, we generalized the V/V_{max} test to a new form, the n/n_{max} test, so that it can be applicable to any kinds of luminosity function models (see Qin & Xie 1997).

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A common character of the various statistical tests used so far to check the correctness of a proposed form of the luminosity function is that they were all necessary. For a quantitative study of analytical forms of the luminosity function, it is desirable to have a sufficient test. In this paper, we make an investigation on this issue.

In Sect. 2, we introduce a new statistic and discuss some of its properties. In Sect. 3, a proposal for the sufficient test of any adopted forms of the luminosity function is presented and a simple application of the method is demonstrated. A summary of this work is given in Sect. 4.

2. A new statistic and some of its properties

The V/V_{max} statistic and its various revisions, e.g., the V_e/V_a statistic (Avni & Bahcall 1980), the V'/V'_{max} statistic (Schmidt 1968), and the n/n_{max} statistic (Qin & Xie 1997), are all concerned in the space distribution of sources. The luminosity distribution of sources is not reflected by them. For example, from the definition of the statistic n/n_{max} (Qin & Xie 1997) we find that both luminosity functions $\Phi(M,z)$ and $f(M)\Phi(M,z)$ give the same values of n/n_{max}. A new statistic independent of n/n_{max} should be that concerned in the luminosity distribution of sources. Therefore, we define

$$o(M,z) \equiv \int_{M_{\min}}^{M} \Phi(M,z)dM \tag{1}$$

and

$$o_{\max}(z) \equiv \int_{M_{\min}}^{M_{\max}(z)} \Phi(M,z)dM, \tag{2}$$

where $\Phi(M,z)$ is the luminosity function of sources, $M_{\min}$ is the absolute magnitude of the most luminous object in a sample, and $M_{\max}(z)$ is the absolute magnitude of the faintest object detectable by the adopted flux limit $m_{\max}$ at the distance of redshift z, which is

$$M_{\max}(z) = M(m_{\max}, z). \tag{3}$$

With the definition of $o(M,z)$ and $o_{\max}(z)$ we have the following statistic

$$\frac{o}{o_{\max}} \equiv \frac{\int_{M_{\min}}^{M} \Phi(M,z)dM}{\int_{M_{\min}}^{M_{\max}(z)} \Phi(M,z)dM}. \tag{4}$$

It is obvious that the values of $o/o_{\max}$ satisfy $1 \geq o/o_{\max} \geq 0$.

The statistic $o/o_{\max}$ reflects the luminosity distribution of the sources obeying $\Phi(M,z)$, and is expected to follow a certain distribution (see the following). Thus, it may provide a test for checking if $\Phi(M,z)$ is correct.

In the following, we study some properties of the statistic $o/o_{\max}$ based on the assumption that $\Phi(M,z)$ correctly describe the distribution of the adopted sample.

Property 1: The values of the statistic $o/o_{\max}$ are uniformly distributed in the interval $[0,1]$.

For a distribution of one variable, the values of the cumulative function so defined are well known to be uniformly distributed in the interval $[0,1]$. Here, $\Phi(M,z)$ is a distribution of two variables. One can verify that, for such a distribution, the cumulative function so defined is also a statistic uniformly distributed in the interval $[0,1]$ (see, e.g., the statistic $n/n_{\max}$ in Qin & Xie 1997).

We notice that Property 1 is free of particular forms of $\Phi(M,z)$ so long as $\Phi(M,z)$ is correct. For the uniform distribution of luminosity, $\Phi(M,z)$ must be independent of luminosity, i.e., it is only a function of z. In this case, the statistic can be used to test the luminosity uniformity of a sample.

Property 2: Property 1 can be held in any redshift range.

This will be well understood when the adopted sample is defined in the given redshift range.

Property 3: The mean value of $o/o_{\max}$ is $1/2$.

As a symmetric center of the uniform distribution, $1/2$ is obviously the mean value of the statistic. The variance of this mean value should be the same as that of any statistics uniformly distributed in the interval $[0,1]$, which is $1/12N$ (see, e.g., Avni & Bahcall 1980; Schmidt & Green 1983).

In addition to the above plain properties, we also find some useful conclusions.

Statement 1: The luminosity function corresponding to a given distribution of $o/o_{\max}$ is not unique.

Proof. Let $\Phi(M,z)$ correspond to a given distribution of $o/o_{\max}$ [that is, $o/o_{\max}$ is valued by Eq. (4) with $\Phi(M,z)$], $\{\frac{o}{o_{\max}}(M,z)\}$, and $g(z)\Phi(M,z)$ correspond to other distribution of $o/o_{\max}$, $\{\frac{o'}{o'_{\max}}(M,z)\}$. According to the definition, we have

$$\begin{aligned}\frac{o'}{o'_{\max}}(M,z) &= \frac{\int_{M_{\min}}^{M} g(z)\Phi(M,z)dM}{\int_{M_{\min}}^{M_{\max}(z)} g(z)\Phi(M,z)dM} \\ &= \frac{\int_{M_{\min}}^{M} \Phi(M,z)dM}{\int_{M_{\min}}^{M_{\max}(z)} \Phi(M,z)dM} = \frac{o}{o_{\max}}(M,z).\end{aligned} \tag{5}$$

This completes the proof.

Therefore, the $o/o_{\max}$ test of the luminosity function can only be a necessary one.

Statement 2: The $o/o_{\max}$ test is independent of the $n/n_{\max}$ test.

Proof. The proof of Statement 1 shows that different luminosity functions $\Phi(M,z)$ and $g(z)\Phi(M,z)$ correspond to a same distribution of $o/o_{\max}$. Let $\Phi(M,z)$ correspond to $\{\frac{n}{n_{\max}}(M,z)\}$ and $g(z)\Phi(M,z)$ to $\{\frac{n'}{n'_{\max}}(M,z)\}$. According to the definition of $n/n_{\max}$, we find that

$$\begin{aligned}\frac{n'}{n'_{\max}}(M,z) &= \frac{\int_{0}^{z} g(z)\Phi(M,z)dV(z)}{\int_{0}^{z_{\max}(M)} g(z)\Phi(M,z)dV(z)} \\ &\neq \frac{\int_{0}^{z} \Phi(M,z)dV(z)}{\int_{0}^{z_{\max}(M)} \Phi(M,z)dV(z)} = \frac{n}{n_{\max}}(M,z)\end{aligned} \tag{6}$$

when $g(z)$ is an arbitrary function of z. In the same way we can also find that, while $\Phi(M,z)$ and $f(M)\Phi(M,z)$ correspond to a same distribution of $n/n_{\max}$, they may correspond to different distributions of $o/o_{\max}$. This implies that a given distribution of $o/o_{\max}$ does not necessarily correspond to a unique distribution of $n/n_{\max}$ or vice versa. Therefore, the $o/o_{\max}$ test is independent of the $n/n_{\max}$ test. The proof is then completed.

3. A proposal for the sufficient test

Because the $o/o_{\max}$ test is independent of the $n/n_{\max}$ test, even though both of them are necessary tests of luminosity functions, the combination of them may provide a sufficient test. Such a test will require that the correspondence between the luminosity function and two given distributions of $o/o_{\max}$ and $n/n_{\max}$ be one to one. In the following we will see that this requirement can be satisfied with a weak constraint.

Statement 3: Two given distributions of $o/o_{\max}$ and $n/n_{\max}$ correspond to a unique normalized luminosity function.

Proof. Let the normalized luminosity function $\Phi(M,z)$ correspond to two given distributions of $o/o_{\max}$ and $n/n_{\max}$, $\{\frac{o}{o_{\max}}(M,z)\}$ and $\{\frac{n}{n_{\max}}(M,z)\}$, and $\Phi'(M,z)$ correspond to $\{\frac{o'}{o'_{\max}}(M,z)\}$ and $\{\frac{n'}{n'_{\max}}(M,z)\}$. Let $\frac{o}{o_{\max}}(M,z) = \frac{o'}{o'_{\max}}(M,z)$ and $\frac{n}{n_{\max}}(M,z) = \frac{n'}{n'_{\max}}(M,z)$ for any (M,z). According to definitions, we have

$$\int_{M_{\min}}^{M} \Phi(M,z)dM = g_o(z)\int_{M_{\min}}^{M} \Phi'(M,z)dM \tag{7}$$

and

$$\int_0^z \Phi(M,z)dV(z) = f_n(M)\int_0^z \Phi'(M,z)dV(z), \tag{8}$$

where

$$g_o(z) \equiv \frac{\int_{M_{\min}}^{M_{\max}(z)} \Phi(M,z)dM}{\int_{M_{\min}}^{M_{\max}(z)} \Phi'(M,z)dM} \tag{9}$$

and

$$f_n(M) \equiv \frac{\int_0^{z_{\max}(M)} \Phi(M,z)dV(z)}{\int_0^{z_{\max}(M)} \Phi'(M,z)dV(z)}. \tag{10}$$

Eq. (7) should be held for any given M and Eq. (8) be held for any given z. This will be true when and only when

$$\Phi(M,z) = g_o(z)\Phi'(M,z) \tag{11}$$

and

$$\Phi(M,z) = f_n(M)\Phi'(M,z). \tag{12}$$

Comparing Eqs. (11) and (12) yields

$$g_o(z) = f_n(M). \tag{13}$$

Eq. (13) leads to

$$g_o(z) = f_n(M) = k, \tag{14}$$

where k is a constant. Thus

$$\Phi(M,z) = k\Phi'(M,z). \tag{15}$$

Since both $\Phi(M,z)$ and $\Phi'(M,z)$ are normalized luminosity functions, constant k must be 1. Therefore,

$$\Phi(M,z) = \Phi'(M,z). \tag{16}$$

This shows that two given distributions of $o/o_{\max}$ and $n/n_{\max}$ correspond to a unique normalized luminosity function. The proof is then completed.

Since two given distributions of $o/o_{\max}$ and $n/n_{\max}$, $\{\frac{o}{o_{\max}}(M,z)\}$ and $\{\frac{n}{n_{\max}}(M,z)\}$, correspond to a unique normalized luminosity function $\Phi(M,z)$, when $\{\frac{o}{o_{\max}}(M,z)\}$ and $\{\frac{n}{n_{\max}}(M,z)\}$ are true distributions, $\Phi(M,z)$ must be the true luminosity function. Therefore, a statistical analysis depending on both distributions of $o/o_{\max}$ and $n/n_{\max}$ provides a sufficient test for normalized luminosity functions. The only constraint on the test is that the proposed luminosity function must be normalized. This constraint is weak because it is only a simple process of data calculation.

As a demonstration of application, we apply both the $o/o_{\max}$ and $n/n_{\max}$ tests to a luminosity function and a sample found in literature. The sample employed is the AAT sample (Boyle et al. 1990). It comes from a UVX survey and contains 420 quasars. The objects detected in the survey could be as faint as $B \sim 21mag$. In converting b magnitudes in the original data into B magnitudes, we follow Boyle et al. (1990) to adopt $B = b - 0.14mag$. For K-corrections, we follow Boyle et al. (1990) to take $\alpha = 0.5 (f_\nu \propto \nu^{-\alpha})$. The luminosity function adopted is the B model in Boyle et al. (1990), which is a two power law, pure luminosity evolution model of $q_0 = 0.5$ and the luminosity evolution form $(1+z)^{k_L}$, where, the parameters of the model were determined by the best fit of the model to the data of the AAT sample. For the pure luminosity evolution model with the luminosity evolution form $(1+z)^{k_L}$, the luminosity function obeys $\Phi(M,z) = \Phi[M + 2.5k_L\log(1+z), 0]$ (see, e.g., Boyle et al. 1988).

To check if a model correctly describes the distribution of a sample, one can apply the powerful Kolmogorov-Smirnov test or the χ^2 test. For simplicity, one can also apply the mean value test, and, if this test fails to be passed, then there will be less hope for other powerful tests to be passed. For the B luminosity function and the AAT sample, our calculation gives $\langle o/o_{\max}\rangle = 0.401$ and $\langle n/n_{\max}\rangle = 0.303$. For $N = 420$, the allowed 1σ deviation is $1/\sqrt{12N} = 0.014$. The deviations of the mean values from the expected value 0.5 are as big as several σs. Both distributions of $o/o_{\max}$ and $n/n_{\max}$ obviously deviate from the uniform distribution in the interval $[0,1]$.

Supposing the luminosity function is correct, then the above deviations must be due to the incompleteness of the sample. It is known that the UVX technique is free of substantial bias or incompleteness in identifying QSOs with redshifts less than 2.2 (Véron 1983). For the AAT sample, the incompleteness at any redshift is generally less than 10 per cent, with the only significant exception occurring for redshifts $0.5 < z < 0.9$, where the incompleteness is as much as $15-20$ per cent (Boyle et al. 1987). In order to make the analysis with a more complete sample, we require $z \in [0.3, 2.2]$ and get a subsample of 391 sources (called S1 sample) from the AAT sample. For this subsample, we have

$\langle o/o_{\max}\rangle = 0.400$ and $\langle n/n_{\max}\rangle = 0.438$. The allowed 1σ deviation is $1/\sqrt{12N} = 0.015$, where $N = 391$. Compared with the above results we find that, while the luminosity distribution remains unchanged, the space distribution has been significantly improved, showing that S1 sample is indeed more complete than the AAT sample so long as the luminosity function is correct. However, the discrepancies with 0.5 found in both $\langle o/o_{\max}\rangle$ and $< n/n_{\max} >$ are sill very large (the allowed 1σ deviation is only 0.015). Should these discrepancies be due to the incompleteness of the sample? For a random variable (say x) uniformly distributed over the range $[0, 1]$, the incompleteness of 10 per cent can cause as much as 0.05 deviation, that is $\sigma_{in} \equiv |\langle x\rangle - 0.5| \leq 0.05$. This shows that, at 1σ level, the incompleteness does not account for the above discrepancies, but at 2σ level it does. When taking off the sources in the redshift range $(0.5, 0.9)$, we have a subsample (called S2 sample) of 320 sources from S1 sample. For S2 sample, we have $\langle o/o_{\max}\rangle = 0.405$ and $\langle n/n_{\max}\rangle = 0.441$. The allowed 1σ deviation is $1/\sqrt{12N} = 0.016$, where $N = 320$. The results have been slightly improved, showing that S2 sample might probably be more complete than S1 sample so long as the adopted luminosity function is correct, and the interpretation of the worse incompleteness (as much as $15 - 20$ per cent) inside the redshift range $(0.5, 0.9)$ might be true.

On the contrary, if the sample is complete, then the luminosity function might be incorrect. As shown by the above discussion, the large discrepancies with 0.5 found in both $\langle o/o_{\max}\rangle$ and $\langle n/n_{\max}\rangle$ can only be accounted for the incompleteness of 10 per cent of the AAT sample at 2σ level. From a theoretical point of view, it has been shown that: (1) given the observed counts and any arbitrary cosmology, one can derive an evolution law; (2) or, given the observed counts and any evolution model, one can derive cosmological constraints (see, e.g., Mustapha et al. 1997). Once cosmology and evolution are derived, the empirical luminosity function is easily computed from the data. The B model of Boyle et al. (1990) consists of both evolution and cosmology conjointly (two power law of the pure luminosity evolution model; $q_0 = 0.5$ for the $\Lambda = 0$ cosmological model). Boyle et al. (1990) fixed the corresponding evolution parameter and those of the adopted functional form of the luminosity function through a global fitting procedure. This model is obviously overconstrained [according to the points (1) and (2) above] and only marginally compatible with their data. Van Waerbeke et al. (1996) showed that this model-plus-data is rejected at the 3σ level by the $V/V_{\max}$ test. If this model-plus-data is also rejected at the 3σ level by the $n/n_{\max}$ and $o/o_{\max}$ tests (which may provide as much as 0.048 deviation for $N = 320$), then, together with the 1σ deviation caused by the incompleteness of 10 per cent (which may be as much as 0.05), one can explain the large discrepancies with 0.5 found in both $\langle o/o_{\max}\rangle$ and $\langle n/n_{\max}\rangle$ for the S2 sample. For the S1 sample, the discrepancies can also be understood.

In this section, we provide only a demonstration of a simple application of the $o/o_{\max}$ and $n/n_{\max}$ tests. Further discussion of the correctness of the B luminosity function and the completeness of the AAT sample is beyond the scope of this paper and then is not presented here.

4. Conclusions

A new statistic o/o_{max} is introduced and some of its properties are studied in Sect. 2. Based on the assumption that the proposed luminosity function correctly describes the distribution of the adopted sample, the values of $o/o_{\max}$ are expected to be uniformly distributed in the interval $[0, 1]$. We find that the $o/o_{\max}$ test of the luminosity function is a necessary one and the $o/o_{\max}$ test is independent of the n/n_{max} test. In Sect. 3 we show that, even though both the $o/o_{\max}$ and $n/n_{\max}$ tests are necessary for checking the luminosity function, the distributions of the two statistics provide a sufficient test due to that two given distributions of $o/o_{\max}$ and $n/n_{\max}$ correspond to a unique normalized luminosity function. As a demonstration of application, we apply both the $o/o_{\max}$ and $n/n_{\max}$ tests to the B luminosity function and the AAT sample presented in Boyle et al. (1990). The results are briefly discussed.

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