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The statistics of the gravitational field arising from an inhomogenous system of particles

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Abstract. In this paper we extend Chandrasekhar and von Neumann's analysis of the statistics of the gravitational field to systems in which particles (e.g. stars, galaxies) are not homogeneously distributed. We derive a distribution function $W(\mathbf{F}, d\mathbf{F}/dt)$ giving the joint probability that a test particle is subject to a force **F** and an associated rate of change of **F** given by d**F**/dt. We calculate the first moment of d**F**/dt to study the effects of inhomogenity on dynamical friction.

Key words: cosmology: large-scale structure of Universe – galaxies: interactions – galaxies: clusters: general – galaxies: star clusters – stars: statistics – methods: analytical

1. Introduction

The study of the statistics of the fluctuating gravitional force in infinite homogeneous systems was pionered by Chandrasekhar & von Neumann in two classical papers (Chandrasekhar & von Neumann 1942, 1943 hereafter CN43) and in several other papers by Chandrasekhar (1941, 1943a,b,c,d,e, 1944a, 1944b). The analysis of the fluctuating gravitional field, developed by the quoted authors, was formulated by means of a statistical treatment. In their papers Chandrasekhar & von Neumann considered a system in which the stars are distributed according to a uniform probability density, no correlation among the positions of the stars is present and where the number of stars constituting the system tends to infinity while keeping the density constant. The force **F**, for unit mass, acting on a star of a generical star system is given by the well known equation:

$$\mathbf{F} = -G \sum_{i=1}^{N} \frac{M_i}{|\mathbf{r}_i|^3} \mathbf{r}_i \tag{1}$$

where M is the mass of the i-th field star, N is the total number of the stars in the system and $\mathbf{r}_i$ is the position vector of the i-th test star relative to the field one. The summation includes all the neighboring stars. The motion of the stars in the neighborhood of the test star produces a time variation of **F**. The exact dependence of **F** on the position and time cannot be exactly predicted, while it is possible to study the fluctuation of **F** from a statistical point of view.

Two distributions are fundamental for the description of the fluctuating gravitional field:

1) $W(\mathbf{F})$ which gives the probability that a test star is subject to a force **F** in the range **F**, **F**+ d**F**;
2) $W(\mathbf{F}, \mathbf{f})$ which gives the joint probability that the star experiences a force **F** and a rate of change **f**, where $\mathbf{f} = d\mathbf{F}/dt$.

The first distribution, known as Holtsmark's law (Holtsmark 1919), in the case of homogeneous distribution of the stars, gives information only on the number of stars experiencing a given force but it does not describe some fundamental features of the fluctuations in the gravitional field such as the *speed of the fluctuations* and the dynamical friction. These features can be described using the second distribution $W(\mathbf{F}, \mathbf{f})$. As shown by CN43, the *speed of the fluctuations* can be adequately expressed in terms of the mean life of a state **F**:

$$T = \frac{|\mathbf{F}|}{\sqrt{\langle|\mathbf{f}_{\mathbf{F}}^2|\rangle}} \tag{2}$$

where $\langle|\mathbf{f}^2|\rangle$ is the second moment of **f**. Hence, for the definition of the speed of fluctuations and of the dynamical friction one must determine the distribution $W(\mathbf{F}, \mathbf{f})$. For a test star moving whith velocity **v** in a sea of field stars characterized by a random probability distribution of the velocities, $\Phi(\mathbf{u})$, we may write:

$$\langle\mathbf{V}\rangle = \langle\mathbf{u}\rangle - \mathbf{v} = -\mathbf{v} \tag{3}$$

where $\mathbf{V}$ represents the velocity of a typical field star relative to the one under consideration, $\mathbf{u}$ denotes the velocity of a field star. This asymmetry of the distribution of the relative velocities produces, as shown by CN43, a deceleration of the test star in the direction of motion. This effect is known, after Chandrasekhar papers, as "*dynamical friction*". Some information on dynamical friction can be obtained by means of the first moment of **f**. As shown by CN43:

$$\langle\frac{d\mathbf{F}}{dt}\rangle_{\mathbf{F},\mathbf{v}} = \frac{-2\pi}{3} GmnB(\beta) \left[\mathbf{v} - \frac{3\mathbf{F}\cdot\mathbf{v}}{|\mathbf{F}|^2} \cdot \mathbf{F}\right] \tag{4}$$

$$\langle\frac{d|\mathbf{F}|}{dt}\rangle_{\mathbf{F},\mathbf{v}} = \frac{4\pi}{3} GmnB(\beta) \frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{F}|} \tag{5}$$

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where m is the mass of a field star, n_l is the local density, and $B(\beta)$ is definited in CN43 Eq. (98). These equations show that the amount of acceleration in the direction of $-\mathbf{v}$, when $\mathbf{v}\cdot\mathbf{F} \leq 0$, is greater than that in the direction $+\mathbf{v}$, when $\mathbf{v}\cdot\mathbf{F} \geq 0$. The star suffers a deceleration being the a priori probability that $\mathbf{v}\cdot\mathbf{F} \geq 0$ equal to the probability that $\mathbf{v}\cdot\mathbf{F} \leq 0$.

Several authors have stressed the importance of stochastic forces and in particular dynamical friction in determining the observed properties of clusters of galaxies (White 1976; Kashlinsky 1986, 1987) while others studied the role of dynamical friction in the orbit decaying of a satellite moving around a galaxy or in the merging scenario (Bontekoe & van Albada 1987; Seguin & Dupraz 1996; Dominguez-Tenreiro & Gomez-Flechoso 1998) which is not only the framework for galaxy formation picture in hierarchical cosmological models, but also important for the study of particular aspects of the evolution of a number of astronomical systems, such as galactic nuclei, cD galaxies in rich galaxy clusters. Finally, here, we want to remember as other the statistical description of dynamical friction other works have been made on it based on different approaches for example: Fokker Planck equation based polarization cloud (Rosenblunth et al. 1957; Binney & Tremaine 1987); resonant particle interactions (Tremaine & Weinberg 1984; Weinberg 1986); fluctation dissipation (Berkenstein & Maoz 1992; Maoz 1993).

Chandrasekhar's theory (and in particular his classical formula - see Chandrasekhar 1943b) is widely employed to quantify dynamical friction in a variety of situations, even if all the theory developed by the quoted authors is based on the hypothesis that the stars are distributed uniformly and it is well known that in stellar systems, the stars are not uniformly distributed, (Elson et al. 1987; Wybo & Dejonghe 1996; Zwart et al. 1997) as well as in galactic systems, the galaxies are not uniformly distributed (Peebles 1980; Bahcall & Soneira 1983; Sarazin 1988; Liddle, & Lyth 1993; White et al. 1993; Strauss & Willick 1995). It is evident that an analysis of dynamical friction taking account of the inhomogeneity of astronomical systems can provide a more realistic representation of the evolution of these systems itself. Moreover from a pure theorethical ground we expect that inhomogeneity affects all the aspects of the fluctuating gravitational field (Antonuccio & Colafrancesco 1994; Del Popolo 1994; Del Popolo et al. 1996; Del Popolo & Gambera 1996, 1997; Gambera 1997). Firstly the Holtsmark distribution is no more correct for inhomogeneous systems. For these systems, as shown by Kandrup (1980a,b, 1983), the Holtsmark distribution must be substituted with a generalized form of the Holtsmark distribution characterized by a shift of $W(\mathbf{F})$ towards larger forces when inhomogeneity increases. This result was already suggested by the numerical simulations of Ahmad & Cohen (1973, 1974). Hence when the inhomogeneity increases the probability that a test particle experiences a large force increases, secondly $W(\mathbf{F},\mathbf{f})$ is changed by inhomogeneity. Consequently, the values of the mean life of a state, the first moment of $\mathbf{f}$ and the dynamical friction force are changed by inhomogeneity with respect to those of homogeneous systems.

This paper must be intended as the first part of a work pointed to:

a) study the effects of inhomogeneity on the distribution functions of the stochastic forces and on dynamical friction (present paper);
b) test the result against N-body simulations;
c) find a formula that describes dynamical friction in homogeneous and inhomogeneous systems only on the basis of the statistical theory.

Before continuing we want to stress that when we speak of inhomogeneity we refer to inhomogeneity in position distribution and not to that of velocity distribution. Our work follows the spirit of Kandrup's (1980a) in the sense that we are interested in the effect of a non-uniform distribution in position of stars on the distributions of the stochastic force.

The plan of the paper is the following: in Sect. 2 we sketch the calculations needed to obtain the distribution function $W(\mathbf{F},\mathbf{f})$ after having released the hypothesis of homogeneity. The complete calculations are developed in the appendix. Then in Sect. 3 we calculate the first moment of **f** and in Sect. 4 we show how dynamical friction is influenced by inhomogeneity. Finally, in Sect. 5 we draw our conclusions.

2. The distribution function $W(\mathbf{F},\mathbf{f})$ in inhomogeneous systems

To calculate $W(\mathbf{F},\mathbf{f})$ in an inhomogeneous system we consider a particle moving with a velocity $\mathbf{v}$, subject to a force, per unit mass, given by Eq. (1) and to a rate of change given by

$$\mathbf{f} = \frac{d\mathbf{F}}{dt} = G\sum_{i=1}^{N} M_i \left[\frac{\mathbf{V}_i}{|\mathbf{r}_i|^3} - \frac{3\mathbf{r}_i(\mathbf{r}_i\mathbf{V}_i)}{|\mathbf{r}_i|^5}\right] \qquad (6)$$

where $\mathbf{V}_i$ is the velocity of the field particle relative to the test one.

The expression of $W(\mathbf{F},\mathbf{f})$ is given following Markoff's method by (CN43):

$$W(\mathbf{F},\mathbf{f}) = \frac{1}{64\pi^6}\int_0^\infty\int_0^\infty A(\mathbf{k},\mathbf{\Sigma})\cdot \{\exp\left[-i(\mathbf{k}\mathbf{\Phi}+\mathbf{\Sigma}\mathbf{\Psi})\right]\} d\mathbf{k}d\mathbf{\Sigma} \qquad (7)$$

with $A(\mathbf{k},\mathbf{\Sigma})$ given by

$$A(\mathbf{k},\mathbf{\Sigma}) = e^{-nC(\mathbf{k},\mathbf{\Sigma})} \qquad (8)$$

being

$$C(\mathbf{k},\mathbf{\Sigma}) = \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \times\tau\left[1-\exp i(\mathbf{k}\mathbf{\Phi}+\mathbf{\Sigma}\mathbf{\Psi})\right] d\mathbf{r}d\mathbf{V}dM \qquad (9)$$

where n is the average number of stars per unit volume while $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are given by the following relations:

$$\mathbf{\Phi} = G\frac{M}{|\mathbf{r}|^3}\mathbf{r} \qquad (10)$$

$$\mathbf{\Psi} = \frac{d\mathbf{F}}{dt} = M\left[\frac{\mathbf{V}}{|\mathbf{r}|^3} - \frac{3\mathbf{r}(\mathbf{r}\mathbf{V})}{|\mathbf{r}|^5}\right] \tag{11}$$

and $\tau(\mathbf{V},\mathbf{r},M)d\mathbf{V}d\mathbf{r}dM$ is the probability that a star has velocity in the range $\mathbf{V},\mathbf{V}+d\mathbf{V}$, positions in $\mathbf{r},\mathbf{r}+d\mathbf{r}$ and mass in $M, M+dM$.

Now we suppose that τ is given by:

$$\tau = \frac{a}{r^p}\psi(j^2(M)|\mathbf{u}|^2) \tag{12}$$

where a is a constant that can be obtained from the normalization condition for τ, j a parameter (of dimensions of velocity^{-1}), ψ an arbitrary function, $\mathbf{u}$ the velocity of a field star. In other words we assume, according to CN43 and Chandrasekhar & von Newmann (1942), that the distribution of velocities is spherical, i.e. the distribution function is $\psi(\mathbf{u}) \equiv \psi(j^2(M)|\mathbf{u}|^2)$, but differently from the quoted papers we suppose that the positions are not equally likely for stars, that is the stars are inhomogeneously distributed in space. A lenghty calculation leads us (see Appendix for a derivation and the meaning of simbols) to find the function $A(\mathbf{k},\mathbf{\Sigma})$:

$$\begin{aligned} A(\mathbf{k},\mathbf{\Sigma}) \; = \; & e^{-\tilde{a}k^{\frac{3-p}{2}}}\{1 - \; igp(\mathbf{k},\mathbf{\Sigma}) \\ & + \tilde{b}k^{\frac{-(3+p)}{2}}] \cdot [Q(\mathbf{\Sigma}) + kR(\mathbf{\Sigma})]\} \end{aligned} \tag{13}$$

This last equation introduced into Eq. (7) solves the problem of finding the distribution $W(\mathbf{F},\mathbf{f})$ and makes it possible to find the moments of $\mathbf{f}$ that give information regarding the dynamical friction.

3. Evalutation of $\overline{f}$

As we stressed in the introduction, the study of the dynamical friction is possible when we know the first moment of $\mathbf{f}$. This calculation can be done using the components of $\mathbf{f}$ (f_i, f_j, f_k) in the system of coordinates previously introduced. We have that:

$$f_i = \frac{\int_{-\infty}^{\infty} W(\mathbf{F},\mathbf{f})f_i d\mathbf{f}}{W(\mathbf{F})} \tag{14}$$

and similar equations for the other components of the force. The distribution function $W(\mathbf{F})$, giving the number of stars subject to a force $\mathbf{F}$, can be calculated as follows:

$$\begin{aligned} W(\mathbf{F}) \; = \; & \frac{1}{64\pi^6}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\{e^{[-i(\mathbf{kF}+\mathbf{\Sigma f})]}\} \\ & \cdot A(\mathbf{k},\mathbf{\Sigma}) \; d\mathbf{k}d\mathbf{\Sigma}d\mathbf{f} \end{aligned} \tag{15}$$

integrating we find:

$$W(\mathbf{F}) = \frac{1}{2\pi^2 F}\int_0^{\infty}\{e^{[\;-a^2k^{(3-p)/2}]}\}\cdot k\sin(kF)dk \tag{16}$$

This equation gives the generalized Holtsmark distribution obtained by Kandrup (1980a - Eq. 4.17) and provides the probability that a star is subject to a force $\mathbf{F}$ in a inhomogeneous system.

As previously stressed, to calculate the first moment of $\mathbf{f}$ we need only an approximated form for $A(\mathbf{k},\mathbf{\Sigma})$:

$$A(\mathbf{k},\mathbf{\Sigma}) = \{e^{[\;-a^2k^{(3-p)/2}]}\}\cdot[1 - igp(\mathbf{k},\mathbf{\Sigma})] \tag{17}$$

Using this last expression for $A(\mathbf{k},\mathbf{\Sigma})$ and Eq. (A36), Eq. (A37), Eq. (14), Eq. (16), Eq. (17) and performing a calculation similar to that by CN43 the first moment of $\mathbf{f}$ is given by:

$$\begin{aligned} \overline{\mathbf{f}} \; = \; & -\left(\frac{1}{2}\right)^{\frac{3}{3-p}}\cdot A(p)\cdot B(p)^{\frac{p}{3-p}} \\ & \cdot\frac{\alpha^{\frac{3}{3-p}}GML(\beta)}{\pi H(\beta)\beta^{\frac{2-p}{2}}}\cdot\left[\mathbf{v} - \frac{3\mathbf{F}\cdot\mathbf{v}}{|\mathbf{F}|^2}\cdot\mathbf{F}\right] \end{aligned} \tag{18}$$

where

$$\begin{aligned} L(\beta) \; = \; & 6\int_0^{\infty}\left[e^{(x/\beta)^{\frac{(3-p)}{2}}}\right]\left[\frac{\sin x}{x^{(2-p)/2}} - \frac{\cos x}{x^{p/2}}\right]dx \\ & - 2\int_0^{\infty}\left[e^{(x/\beta)^{\frac{(3-p)}{2}}}\right]\cdot\frac{\sin x}{x^{(p-2)/2}}dx \end{aligned} \tag{19}$$

and for $p = 0$ Eq. (18) reduces to:

$$\overline{\mathbf{f}} = -\frac{\alpha}{6}Gm\left[\frac{L(\beta)}{\pi\beta H(\beta)}\right]_{p=0}\cdot\left[\mathbf{v} - \frac{3\mathbf{F}\cdot\mathbf{v}}{|\mathbf{F}|^2}\cdot\mathbf{F}\right] \tag{20}$$

and consequently

$$\begin{aligned} [L(\beta)]_{p=0} \; = \; & \int_0^{\infty} e^{[-\;(x/\beta)^{3/2}]}\,\cdot \\ & \left[\frac{6\sin x}{x} - 6\cos x + 2x\sin x\right]dx \end{aligned} \tag{21}$$

this last expression can also be written as:

$$[L(\beta)]_{p=0} = 3\pi\int_0^{\beta}H(\beta)d\beta - \pi\beta H(\beta) \tag{22}$$

being

$$H(\beta) = \frac{2}{\pi\beta}\int_0^{\infty}\left\{e^{\left[(\frac{-x}{\beta})^{3/2}\right]}\right\}\cdot x\sin x dx \tag{23}$$

In this way we can written Eq. (20) as:

$$\begin{aligned} \overline{\mathbf{f}} \; = \; & -\frac{2\pi}{3}Gmn\left[\frac{3\cdot\int_0^{\beta}H(\beta)d\beta}{\beta\cdot H(\beta)} \; - \; 1\right] \\ & \cdot\left[\mathbf{v} - \frac{3\mathbf{F}\cdot\mathbf{v}}{|\mathbf{F}|^2}\cdot\mathbf{F}\right] \end{aligned} \tag{24}$$

this last equation coincides with Eq. (105) by CN43.

The results obtained by us for an inhomogeneus system are different [see Eq. (18)], as expected, from that obtained by CN43 for a homogeneous system (CN43 - Eq. 105). At the same time it is very interesting to note that for $p = 0$ (homogeneous system) our result coincides, as obvious, with the results obtained by CN43. In a inhomogeneous system, in a similar way to what happens in a homogeneus system, $\mathbf{f}$ depends on $\mathbf{v}$, $\mathbf{F}$ and θ (the angle between $\mathbf{v}$ and $\mathbf{F}$) while differently from homogeneous systems, $\mathbf{f}$ is a function of the inhomogeneity parameter p. The dependence of $\mathbf{f}$ on p is not only due to the functions $A(p)$, $B(p)$ and to the density parameter α but also to the parameter $\beta = |\mathbf{F}|/\mathbf{Q_H}$. In fact in inhomogeneous systems the *normal* field Q_H is given by $Q_H = GM(\alpha B(p)/2)^{2/(3-p)}$, clearly dependent on p.

4. Dynamical friction in inhomogeneous systems

The introduction of the notion of dynamical friction is due to CN43. In the stochastic formalism developed by CN43 the dynamical friction is discussed in terms of $\mathbf{f}$:

$$\overline{\mathbf{f}} = \frac{-2\pi}{3} GmnB(\beta)\left[\mathbf{v} - \frac{3\mathbf{F}\cdot\mathbf{v}}{|\mathbf{F}|^2}\cdot\mathbf{F}\right] \quad (25)$$

where

$$B(\beta) = \frac{3\cdot\int_0^\beta W(\beta)d\beta}{\beta\cdot W(\beta)} \ - \ 1 \quad (26)$$

and $\beta = |\mathbf{F}|/Q_H = |\mathbf{F}|/2.603GMn^{2/3}$. As shown by CN43, the origin of dynamical friction is due to the asymmetry in the distribution of relative velocities. As previously told, if a test star moves with velocity $\mathbf{v}$ in a spherical distribution of field stars, namely $\phi(\mathbf{u})$ then we have that:

$$\overline{\mathbf{V}} = \overline{\mathbf{u}-\mathbf{v}} = -\mathbf{v} \quad (27)$$

The asymmetry in the distribution of relative velocities is conserved in the final Eq. (25). In fact from Eq. (25) we have:

$$\frac{d|\mathbf{F}|}{dt} = \frac{4\pi}{3}GMnB(\beta)\cdot\frac{\mathbf{vF}}{\mathbf{F}} \quad (28)$$

(CN43). This means that when $\mathbf{v}\cdot\mathbf{F}\geq 0$ then $\frac{d|\mathbf{F}|}{dt}\geq 0$; while when $\mathbf{v}\cdot\mathbf{F}\leq 0$ then $\frac{d|\mathbf{F}|}{dt}\leq 0$. As a consequence, when $\mathbf{F}$ has a positive component in the direction of $\mathbf{v}$, $|\mathbf{F}|$ increases on average; while if $\mathbf{F}$ has a negative component in the direction of $\mathbf{v}$, $|\mathbf{F}|$ decreases on average. Moreover, the star suffers a greater amount of acceleration in the direction $-\mathbf{v}$ when $\mathbf{v}\cdot\mathbf{F}\leq 0$ than in the direction $+\mathbf{v}$ when $\mathbf{v}\cdot\mathbf{F}\geq 0$.

In other words the test star suffers, statistically, an equal number of accelerating and decelerating impulses. Being the modulus of deceleration larger than that of acceleration the star slows down.

At this point we may show how dynamical friction changes due to inhomogeneity. From Eq. (18) we see that $\frac{d\mathbf{F}}{dt}$ differs from that obtained in homogeneus system only for the presence of a dependence on the inhomogeneity parameter p. If we divide Eq. (18) for Eq. (25) we obtain:

$$\frac{\left(\overline{\frac{d\mathbf{F}}{dt}}\right)_{Inh.}}{\left(\overline{\frac{d\mathbf{F}}{dt}}\right)_{Hom.}} = -\left(\frac{1}{2}\right)^{\frac{3}{3-p}} \ B(p)^{\frac{p}{3-p}} \ \frac{\alpha^{\frac{3}{3-p}}L(\beta)}{\pi H(\beta)\beta^{\frac{2-p}{2}}}\cdot\frac{3\cdot A(p)}{2\pi nB(\beta)} = -\left(\frac{1}{2}\right)^{\frac{6-p}{3-p}}\cdot\frac{3\alpha^{\frac{3}{3-p}}L(\beta)B(p)^{\frac{p}{3-p}}\cdot A(p)}{n\cdot\pi^2 H(\beta)B(\beta)\beta^{\frac{2-p}{2}}} \quad (29)$$

If we consider a homogeneous system, $p=0$, the previous equation reduces to:

$$\frac{\left(\overline{\frac{d\mathbf{F}}{dt}}\right)_{Inh.}}{\left(\overline{\frac{d\mathbf{F}}{dt}}\right)_{Hom.}} = 1 \quad (30)$$

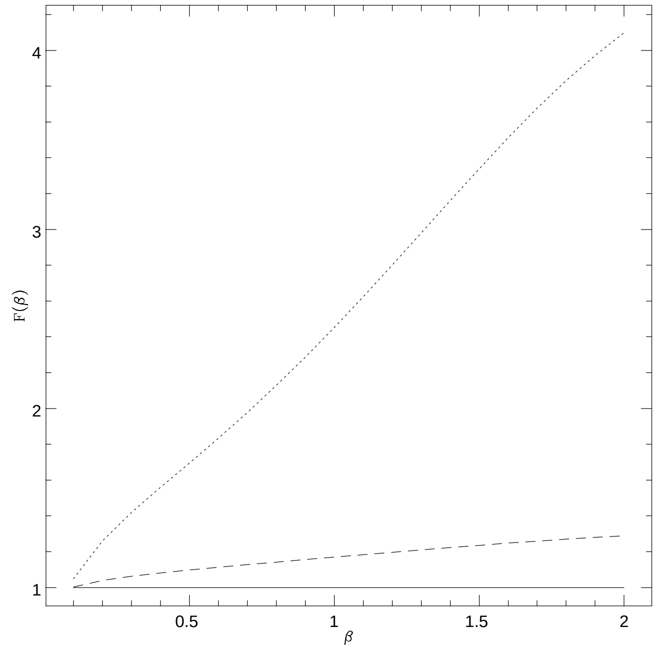

Fig. 1. The function $F(\beta)$ for several values of the inhomogeneity parameter p; solid line $p=0$, dashed line $p=0.1$, dotted line $p=0.5$.

In the case of an inhomogeneous system, $p\neq 0$, we see that:

$$\frac{\left(\overline{\frac{d\mathbf{F}}{dt}}\right)_{Inh.}}{\left(\overline{\frac{d\mathbf{F}}{dt}}\right)_{Hom.}} = n^{p/(3-p)}F(\beta(n,p)) \quad (31)$$

where

$$F[\beta(n,p)] \ = \ -\left(\frac{1}{2}\right)^{\frac{6-p}{3-p}}\cdot B(p)^{\frac{p}{3-p}} \cdot\frac{3L(\beta)\cdot A(p)}{n\cdot\pi^2 H(\beta)B(\beta)\beta^{\frac{2-p}{2}}} \quad (32)$$

As we show in Fig. 1, this last equation is an increasing function of p. This means that for increasing values of p the star suffers an even greater amount of acceleration in the direction $-\mathbf{v}$ when $\mathbf{v}\cdot\mathbf{F}\leq 0$ than in the direction $+\mathbf{v}$ when $\mathbf{v}\cdot\mathbf{F}\geq 0$, with respect to the homogeneous case. This is due to the fact that the difference between the amplitude of the decelerating impulses and the accelerating ones is, as in homogeneous systems, statistically negative, but now larger, being the scale factor greater. This finally means that, for a given value of n, the dynamical friction increases with increasing inhomogeneity in the space distribution of stars (it is interesting to note that this effect is fundamentally due to the inhomogeneity of the distribution of the stars and not to the density n). In other words two systems having the same n will have their stars slowed down differently according to the value of p. This is strictly connected to the asymmetric origin of the dynamical friction.

In addition, by increasing n the dynamical friction increases, just like in the homogeneous systems, but the increase is larger than the linear increase observed in homegeneous systems.

5. Conclusion

In this paper we calculated the distribution function and the first moment of $\mathbf{f} = d\mathbf{F}/dt$ for an inhomogeneous system. We obtained an expression relating $\mathbf{f} = d\mathbf{F}/dt$ to the degree of inhomogeneity in a gravitational system. In the last part of the paper we showed the implications of this result on the dynamical friction in a inhomogeneous system and in particular how inhomogeneity acts as an amplifier of the asimmetry effect giving rise to dynamical friction. Here we want to stress that the distribution function that we have obtained is valid for every inhomogeneous system and consequently it is more general than the distribution function obtained by CN43, that is reobtained when we assume $p = 0$ in our model. Moreover Kandrup's (1980a) theory of the stationary distribution $W(\mathbf{F})$ for inhomogeneous systems is reobtained in the limit $t \to \infty$.

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Appendix A: derivation of $A(\mathrm{k}, \mathbf{\Sigma})$

In order to find $A(\mathbf{k}, \mathbf{\Sigma})$ we have to calculate the integral in Eq. (9) (see Sect. 2). We change the integration variable in Eq. (9) from $\mathbf{r}$ to $\mathbf{\Phi}$ using the following relations:

$$d\mathbf{r} = -\frac{1}{2}(GM)^{1.5}\Phi^{-4.5}d\mathbf{\Phi} \tag{A1}$$

$$r = \sqrt{\frac{GM}{\Phi}} \tag{A2}$$

we have that Eq. (9) can be rewritten as:

$$\begin{aligned} C(\mathbf{k},\mathbf{\Sigma}) =& \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty (GM)^{1.5} d\mathbf{V} dM \\ &\times \int_{-\infty}^\infty \tau\left[1 - e^{i(\mathbf{k\Phi}+\mathbf{\Sigma\Psi})}\right] \\ &\times a\left(\frac{GM}{\Phi}\right)^{-\frac{p}{2}} \Phi^{-\frac{9}{2}} d\mathbf{\Phi} \end{aligned} \tag{A3}$$

and if we express the product $\Sigma \cdot \Psi$ as a function of Φ (CN43 - Eq. 18) we can write $C(\mathbf{k}, \mathbf{\Sigma})$ in the following form:

$$C(\mathbf{k},\mathbf{\Sigma}) = \frac{1}{2}G^{1.5}\int_0^\infty \int_{-\infty}^\infty M^{1.5} D(\mathbf{k},\mathbf{\Sigma}) d\mathbf{V} dM \tag{A4}$$

where $D(\mathbf{k}, \mathbf{\Sigma})$ is given by:

$$D(\mathbf{k},\mathbf{\Sigma}) = \int_{-\infty}^\infty \tau\left[1 - e^{i(\mathbf{k\Phi}+\mathbf{\Sigma\Psi})}\right]\Phi^{-4.5}d\mathbf{\Phi} \tag{A5}$$

or equivalently:

$$\begin{aligned} D(\mathbf{k},\mathbf{\Sigma}) =& \int_{-\infty}^\infty \tau\left[1 - e^{(i\mathbf{k\Phi})}\right]\Phi^{-4.5}d\mathbf{\Phi} \\ &+ \int_{-\infty}^\infty \tau[e^{i(\mathbf{k\Phi})}]\cdot[1 - e^{(i\mathbf{\Sigma\Psi})}]\cdot\Phi^{-4.5}d\mathbf{\Phi} \end{aligned} \tag{A6}$$

The first integral in Eq. (A6) can be easily calculated:

$$\begin{aligned} I_1 =& \int_{-\infty}^\infty \tau\left[1 - e^{(i\mathbf{k\Phi})}\right]\Phi^{-4.5}d\mathbf{\Phi} \\ =& \frac{4\pi a}{(GM)^{p/2}}\cdot B(p)\cdot k^{\frac{(3-p)}{2}} \end{aligned} \tag{A7}$$

with

$$B(p) = \int_0^\infty \frac{z - \sin z}{z^{(7-p)/2}}\cdot dz \tag{A8}$$

For $p = 0$ Eq. (A7) gives:

$$I_1 = \frac{8}{15}\cdot(2\pi k)^{1.5} \tag{A9}$$

that coincides with the first term in the right hand side of Eq. (22) in CN43.

The second integral is more difficult to evaluate and is given by:

$$\begin{aligned} I_2 =& \int_{-\infty}^\infty \tau[e^{(i\mathbf{k\Phi})}]\cdot[1 - e^{(i\mathbf{\Sigma\Psi})}]\cdot\Phi^{-4.5}d\mathbf{\Phi} \\ =& a\int_{-\infty}^\infty \left(\frac{\Phi}{GM}\right)^{p/2}[e^{(i\mathbf{k\Phi})}]\,[1 - e^{(i\mathbf{\Sigma\Psi})}]\,\Phi^{-\frac{9}{2}}d\mathbf{\Phi} \end{aligned} \tag{A10}$$

We have obtained this last result using Eq. (10) and Eq. (12) (see Sect. 2). Since we are interested in the moments of $\frac{d\mathbf{F}}{dt}$ for a given $\mathbf{F}$ we need only the behaviour of the function $A(\mathbf{k}, \mathbf{\Sigma})$ for $|\mathbf{\Sigma}| \Rightarrow 0$. To obtain this we expand the term $[1 - e^{(i\mathbf{\Sigma\Psi})}]$ in powers of $\mathbf{\Sigma}$ and $\mathbf{\Psi}$ in Eq. (A10) and then we have:

$$\begin{aligned} I_2 =& \frac{a}{(GM)^{p/2}}\int_{-\infty}^\infty \left[-i\mathbf{\Sigma\Psi} + \frac{(\mathbf{\Sigma\Psi})^2}{2}\right] \\ &\times\Phi^{\frac{p-9}{2}}\,[e^{(i\mathbf{k\Phi})}]d\mathbf{\Phi} \end{aligned} \tag{A11}$$

or

$$\begin{aligned} I_2 =& \frac{-ia}{(GM)^{p/2}}\cdot\int_{-\infty}^\infty \mathbf{\Sigma\Psi}\cdot\Phi^{(p-9)/2}\cdot[e^{(i\mathbf{k\Phi})}]d\mathbf{\Phi} \\ &+ \frac{a}{2(GM)^{p/2}}\cdot\int_{-\infty}^\infty (\mathbf{\Sigma\Psi})^2\cdot\Phi^{(p-9)/2}\cdot[e^{(i\mathbf{k\Phi})}]d\mathbf{\Phi} \\ =& -D_1 + D_2 \end{aligned} \tag{A12}$$

where $\mathbf{\Sigma}\cdot\mathbf{\Psi}$ is given by

$$\mathbf{\Sigma}\cdot\mathbf{\Psi} = \Phi^{1.5}(\mathbf{\Sigma_1 V}) - 3\Phi^{-0.5}(\mathbf{\Phi V})\cdot(\mathbf{\Phi\Sigma_1}) \tag{A13}$$

being $\mathbf{\Sigma_1} = \mathbf{\Sigma}/(GM)^{0.5}$. Substituting these last equations in Eq. (A12) we have:

$$\begin{aligned} D_1(\mathbf{k},\mathbf{\Sigma}) =& \frac{ia}{(GM)^{p/2}}\int_{-\infty}^\infty \left[\mathbf{\Sigma_1 V} - 3\cdot\frac{(\mathbf{\Phi V})\cdot(\mathbf{\Phi\Sigma_1})}{\Phi^2}\right] \\ &\cdot\Phi^{(p-6)/2}\cdot[e^{(i\mathbf{k\Phi})}]d\mathbf{\Phi} \end{aligned} \tag{A14}$$

Now, we evalutate this integral following CN43. We introduce a system of coordinates with the z-axis in the direction of $\mathbf{k}$ and letting $\mathbf{\Sigma}_1 = (s_1, s_2, s_3)$, $\mathbf{V} = (V_1, V_2, V_3)$ and $\mathbf{l}_\Phi =$

$(l, m, n) = (\sin\theta\cos w, \sin\theta\sin w, \cos\theta)$. The result of this integration is given by:

$$D_1(\mathbf{k}, \mathbf{\Sigma}) = \frac{4\pi i a}{(GMk)^{p/2}} \cdot (s_1 V_1 + s_2 V_2 - 2 s_3 V_3) \cdot A(p) \quad \text{(A15)}$$

where

$$A(p) = \int_0^\infty \left[\frac{\sin x}{x^{(4-p)/2}} - \frac{3\sin x}{x^{(8-p)/2}} + \frac{3\cos x}{x^{(6-p)/2}} \right] \cdot dx \quad \text{(A16)}$$

For $p = 0$ Eq. (A15) becomes

$$D_1(\mathbf{k}, \mathbf{\Sigma}) = -\frac{4\pi i}{3} \cdot (s_1 V_1 + s_2 V_2 - 2 s_3 V_3) \quad \text{(A17)}$$

which coincides with Eq. (37) in CN43.

The second integral in Eq. (A12) is:

$$D_2(\mathbf{k}, \mathbf{\Sigma}) = \frac{a}{2(GM)^{p/2}} \cdot \int_{-\infty}^{\infty} (\mathbf{\Sigma\Psi})^2 \, \Phi^{\frac{p-9}{2}} \cdot \left[e^{(i\mathbf{k\Phi})} \right] d\mathbf{\Phi} \quad \text{(A18)}$$

If we define the variable

$$z = |\mathbf{k}| \cdot |\mathbf{\Phi}| \quad \text{(A19)}$$

and we introduce Eq. (A13) in Eq. (A18) we obtain:

$$D_2(\mathbf{k}, \mathbf{\Sigma}) = \frac{a}{2(GM)^{p/2}} \, k^{-\frac{3+p}{2}} \int_0^\infty \int_{-1}^1 \int_0^{2\pi} [e^{(izt)}] \cdot [\mathbf{\Sigma}_1 \mathbf{V} - 3(\mathbf{l}_\Phi \cdot \mathbf{V})(\mathbf{l}_\Phi \cdot \mathbf{\Sigma}_1)]^2 \, z^{\frac{p+1}{2}} dw dt dz \quad \text{(A20)}$$

where $t = cos\theta$ and w is the azimuthal angle. The integral can be calculated regarding z and t as complex variables and using an appropriate chosen contour (see Chandrasekhar & von Newmann 1942). The result of the integration in terms of the original variable $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ is:

$$D_2(\mathbf{k}, \mathbf{\Sigma}) = \frac{-\pi a k^{-(3+p)/2}}{2(GM)^{p/2}} \Gamma\left(\frac{p}{2} + \frac{3}{2}\right) \left\{ \cos\left[-\frac{\pi}{4}\left(\frac{p}{3}+1\right)\right] - \sin\left[-\frac{\pi}{4}\left(\frac{p}{3}+1\right)\right] \right\} \{\Sigma_1^2[a_* V_1^2 + bV_2^2 + cV_3^2] + \times\Sigma_2^2[bV_1^2 + a_* V_2^2 + cV_3^2] + \Sigma_3^2[cV_1^2 + eV_2^2 + dV_3^2] + +f(\Sigma_2\Sigma_3 V_2 V_3 + \Sigma_1\Sigma_3 V_1 V_3) + e\Sigma_2\Sigma_1 V_2 V_1\} \quad \text{(A21)}$$

where

$$a_* = \frac{-2}{p+1} - \frac{24}{(p+1)(p-3)} - \frac{216}{(p+1)(p-3)(p-7)} \quad \text{(A22)}$$

$$b = \frac{-72}{(p+1)(p-3)(p-7)} \quad \text{(A23)}$$

$$c = \frac{-36}{(p-3)(p-7)} \quad \text{(A24)}$$

$$d = \frac{-2}{p+1} - \frac{12}{3-p} + \frac{18}{7-p} \quad \text{(A25)}$$

$$e = \frac{-4}{p+1} - \frac{48}{(p+1)(p-3)} - \frac{288}{(p+1)(p-3)(p-7)} \quad \text{(A26)}$$

$$f = \frac{-4}{p+1} - \frac{24}{(p+1)(p-3)} - \frac{12}{3-p} + \frac{144}{(p-3)(p-7)} \quad \text{(A27)}$$

then we have obtained:

$$D(\mathbf{k}, \mathbf{\Sigma}) = I_1 + I_2 = \frac{4\pi a k^{-\frac{3+p}{2}}}{(GM)^{p/2}} B(p) - \frac{-4\pi i a k^{-p/2}}{(GM)^{p/2}} \cdot (\Sigma_1 V_1 + \Sigma_2 V_2 - 2\Sigma_3 V_3) \cdot A(p) + D_2 \quad \text{(A28)}$$

Now substituting Eq. (A28) in Eq. (A4) we have:

$$C(\mathbf{k}, \mathbf{\Sigma}) = 2\pi a (G\overline{M})^{(3-p)/2} k^{(3-p)/2} B(p) - 2\pi i a k^{-p/2} \cdot (G\overline{M})^{(2-p)/2} (\Sigma_1 V_1 + \Sigma_2 V_2 - 2\Sigma_3 V_3) \cdot A(p) + -\frac{\pi a}{2} (G\overline{M})^{(1-p)/2} k^{-(3+p)/2} \Gamma\left(\frac{p}{2} + \frac{3}{2}\right) \cdot \left\{ \cos\left[-\frac{\pi}{4}\left(\frac{p}{3}+1\right)\right] - \sin\left[-\frac{\pi}{4} \cdot \left(\frac{p}{3}+1\right)\right] \right\} \cdot \{\Sigma_1^2[a\overline{V}_1^2 + b\overline{V}_2^2 + c\overline{V}_3^2] + \Sigma_2^2[b\overline{V}_1^2 + a\overline{V}_2^2 + c\overline{V}_3^2] + +\Sigma_3^2[c\overline{V}_1^2 + e\overline{V}_2^2 + d\overline{V}_3^2] + f(\Sigma_2\Sigma_3\overline{V_2 V_3} + +\Sigma_1\Sigma_3\overline{V_1 V_3}) + e\Sigma_2\Sigma_1\overline{V_2 V_1}]\} \quad \text{(A29)}$$

where we have used bars to indicate that the corresponding quantities have been averaged with the weight function $\tau(\mathbf{V}, M)$.

If the distribution of the velocities of the field stars, $\mathbf{u}$, is spherical and the test star moves with velocity $\mathbf{v}$ we have:

$$\overline{\mathbf{V}} = \overline{\mathbf{u}} - \overline{\mathbf{v}} = -\mathbf{v} \quad \text{(A30)}$$

and also:

$$\overline{V}_u = \overline{v}_u \quad \text{(A31)}$$

$$\overline{V}_u^2 = \frac{1}{3}|\mathbf{u}|^2 + v_u^2 \quad \text{(A32)}$$

$$\overline{V}_u V_\nu = v_u v_\nu \quad \text{(A33)}$$

and if we use the system of coordinates introduced by CN43 that is $v_1 = |\mathbf{v}| sen\gamma$, $v_2 = 0$ and $v_3 = |\mathbf{v}| cos\gamma$ where γ is the angle between $\mathbf{k}$ and $\mathbf{v}$ we can simplify Eq. (A29) and then the Eq. (8) (see Sect. 2) becomes:

$$A(\mathbf{k}, \mathbf{\Sigma}) = e^{-nC(\mathbf{k},\mathbf{\Sigma})} = e^{[-\ \frac{\alpha}{2}(GMk)^{\frac{3-p}{2}} \cdot B(p)\ +\ \frac{i\alpha}{2} k^{\frac{-p}{2}}} \cdot (GM)^{\frac{2-p}{2}} \, |\mathbf{v}| \cdot [\Sigma_1 \sin\gamma \ - \ 2\Sigma_3 \cos\gamma] \cdot A(p) + - \frac{\alpha}{8}(GM)^{\frac{1-p}{2}} k^{\frac{-(3+p)}{2}} \cdot \Gamma\left(\frac{p}{2} + \frac{3}{2}\right) \cdot \left\{ \cos\left[-\frac{\pi}{4}\left(\frac{p}{3}+1\right)\right] \ - \ \sin\left[-\frac{\pi}{4} \cdot \left(\frac{p}{3}+1\right)\right] \right\} \cdot$$

$$\left\{|\mathbf{u}|^2 \cdot \left[\frac{1}{3}\ (a+b+c)(\Sigma_1^2+\Sigma_2^2)\ +\ \frac{\Sigma_3^2}{3}(2c+d)\right]\right.$$
$$+\ |\mathbf{v}|^2 \cdot \left[(a\sin^2\gamma + c\cos^2\gamma)\Sigma_1^2\ +\ (b\sin^2\gamma + \right.$$
$$+\ c\cos^2\gamma)\Sigma_2^2 +\ (c\sin^2\gamma + d\cos^2\gamma)\Sigma_3^2\ +$$
$$+\ f\Sigma_1\Sigma_3\sin\gamma\cos\gamma\,]\}\ ] \quad \text{(A34)}$$

where $\alpha = 4\pi na$. We define, now, the following constants and functions:

$$\tilde{a} = \frac{\alpha}{2}(GM)^{\frac{3-p}{2}} \cdot B(p) \quad \text{(A35)}$$

$$g = \frac{\alpha}{2}(GM)^{\frac{2-p}{2}} \cdot |\mathbf{v}| \cdot A(p) \quad \text{(A36)}$$

$$p(\mathbf{k},\mathbf{\Sigma}) = k^{\frac{-p}{2}} \cdot [\Sigma_1\sin\gamma\ -\ 2\Sigma_3\cos\gamma] \quad \text{(A37)}$$

$$\tilde{b} = \frac{\alpha}{8}(GM)^{\frac{1-p}{2}} \cdot \Gamma\left(\frac{p}{2}+\frac{3}{2}\right) \cdot$$
$$\left\{\cos\left[-\frac{\pi}{4}\left(\frac{p}{3}+1\right)\right] - \sin\left[-\frac{\pi}{4}\cdot\left(\frac{p}{3}+1\right)\right]\right\} \cdot |\mathbf{u}|^2 \quad \text{(A38)}$$

$$Q(\mathbf{\Sigma}) = \left[\frac{1}{3}\ (a_* + b + c)(\Sigma_1^2+\Sigma_2^2) + \frac{\Sigma_3^2}{3}(2c+d)\right] \quad \text{(A39)}$$

$$R(\mathbf{\Sigma}) = (a_*\sin^2\gamma + c\cos^2\gamma)\Sigma_1^2\ +\ (b\sin^2\gamma + c\cos^2\gamma)\Sigma_2^2$$
$$+\ (c\sin^2\gamma + d\cos^2\gamma)\Sigma_3^2\ +\ f\Sigma_1\Sigma_3\sin\gamma\cos\gamma$$

$$k = \frac{|\mathbf{v}|^2}{|\mathbf{u}|^2} \quad \text{(A40)}$$

so we can re-write Eq. (A34) as:

$$A(\mathbf{k},\mathbf{\Sigma}) = e^{-\tilde{a}k^{\frac{3-p}{2}}}\{1-\ igp(\mathbf{k},\mathbf{\Sigma})$$
$$+\ \tilde{b}k^{\frac{-(3+p)}{2}}\,] \cdot [Q(\mathbf{\Sigma}) + kR(\mathbf{\Sigma})]\} \quad \text{(A41)}$$

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