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On the post main sequence expansion of low mass stars

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Abstract. The post main sequence expansion of stars is investigated by means of a simple composite configuration: an isothermal He-core (allowing for non-relativistic electron degeneracy) is surrounded by a H-envelope of constant density (polytrope $n = 0$). Solving the equations of hydrostatic equilibrium for fixed values of total mass and temperature at the interface a one dimensional sequence of models is obtained with the mass of the core as parameter. As soon as the main part of the core becomes fully degenerate, the model stars expand rapidly. This behaviour is in good agreement with that of models obtained by numerical simulations.

The expansion is caused by an intermediate non-degenerate layer of large extension (but of very small mass content) just below the interface. It shifts the envelope to larger distances from the center and thus reduces the gravitational pull on it due to the highly contracted part of the core. Without this layer the thermal forces of the envelope – determined by the hydrogen burning temperatures at the interface – would be much too small to balance gravity. Such a loosely bound envelope extends to the large radii in question. Hence, the model suggests the fixed temperature required by hydrogen burning to be the ultimate reason for the post main sequence expansion.

Key words: stars: evolution – stars: interiors

1. Introduction

The immediate post main sequence state of stars is one of the best known phases of stellar evolution, observationally as well as theoretically. After using up their central hydrogen content by nuclear burning, stars produce their luminosity in a thin shell around a continually growing helium core. At this stage, they leave the main sequence of the Hertzsprung-Russell diagram and evolve towards higher luminosities and larger radii.

Which physical effects are responsible for this expansion? Simple physical reasoning based on the virial theorem would rather suggest decreasing radii as a consequence of increasing internal energy: due to the outward burning shell source a growing part of the star is at temperatures of 10^7K or higher. Results of numerical simulations show that this reasoning is wrong. What causes the stars to expand? There have been several suggestions in the literature:

One of the earliest explanations is due to Hoeppner and Weigert (1973). Experimenting with their stellar evolution code, they found large expansion after artificially placing a strong point mass in the center of a main sequence star. This result was further investigated by Weiss (1983). Thus, a strong gravitational field is needed for expansion.

Eggleton and Faulkner (1981) compared the structure of expanding stars with polytropes of index $n = 5$, which, at finite masses, have infinite radii. Indeed they found an increasing local polytropic index in those stars. Yahil and van der Horn (1985) investigated composite models and suggested a nearly isothermal structure just below the shell source as a cause for expanding envelopes – thus corroborating the reasoning of Eggleton and Faulkner (1981).

Applegate (1988) inspected the behaviour of the shell source: If the rate of energy generation exceeds a critical amount determined by the envelope the latter starts to expand. Thus the temperature of hydrogen burning is brought into play.

Finally, Whitworth (1989) investigated the overall problem: he considered series expansions around the main sequence state using parametrized properties of stellar matter. From his elegant but very involved formalism he concluded that it must be a complicated interaction between core and envelope which leads to expansion.

Is there really no simple physical answer? In what follows we study – very much in the line of Eggleton and Faulkner (1981) and Yahil and van der Horn (1985) – simple hydrostatic configurations with masses $M \leq 1.5 M_\odot$. The cores of these stars start to contract after using up their nuclear energy sources and soon reach a state of electron degeneracy. Then, the contraction stops and the cores become isothermal – due both to the lack of gravitational energy release and high thermal conductivity of degenerate electrons.

In constructing a composite configuration consisting of a fully degenerate isothermal core and a polytropic envelope with the hydrogen burning shell source at the interface, the following remarkable difficulty arises. The specific gravitational energy at

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the interface due to a polytropic core of $M_{core} = 0.25 M_\odot$ and $n = 1.5$ is

$$w = \frac{GM_{core}}{r_{core}} \sim 2 \cdot 10^{16} \mathrm{erg}; \tag{1}$$

the specific internal energy at the interface, as determined by the temperature of hydrogen burning $T_f = 2 \cdot 10^7$K is

$$u = \frac{3}{2} \frac{kT_f}{\mu_{env} H} \sim 10^{15} \mathrm{erg}. \tag{2}$$

The specific gravitational energy in excess of the specific internal energy strongly limits the mass of the envelope, which may be estimated as follows. The pressure at the interface exerted by an envelope of mass M_{env} is approximately

$$P_f \approx \frac{GM_{core} M_{env}}{4\pi r_{core}^4}.$$

According to the equation of state, this pressure is determined by the thermal conditions at the interface, T_f and $\rho_f < \overline{\rho_{core}}$, and hence the following inequality must be satisfied:

$$\frac{kT_f/\mu_{env} H}{GM_{core}/r_{core}} > 3 \frac{M_{env}}{M_{core}}.$$

Substituting the specific energies from Eqs. (1) and (2), we see that an envelope of only $M_{env}/M_{core} < 10^{-1}$ can be in hydrostatic balance with the compact core – a result well-known from the theory of white dwarfs.

This difficulty can be avoided either by increasing u or by decreasing w (and hence increasing r_{core}). Since $u \sim T_f$ is fixed by the conditions of hydrogen burning, the only possibility to increase M_{env}/M_{core} to the values needed for our composite model – $M_{env}/M_{core} \sim 5$ - 10, say – is to lift the interface to appreciably larger distances from the center. The envelope would then be considerably less bound to the core and it would expand.

How can a star manage to lift its core-envelope interface? To find an answer, the following composite hydrostatic model is considered: an isothermal core, in which the transition from non to complete degeneracy is taken exactly into account; and a hydrogen envelope of polytropic structure fitted to the core. Such a model is uniquely determined by prescribing the total mass, M, the (hydrogen burning) temperature at the interface, T_f, and the core mass, M_{core}. By continuously increasing M_{core} at fixed M, an "evolutionary sequence" of the model star is obtained. This model is presented in Sect. 2 and its solutions are discussed in Sect. 3. Here it is especially helpful to consider the specific energies w and u as functions of the fractional mass M_r/M for one of the expanding models; the functions w and u are interpreted in the light of the virial theorem and they are compared with those of an evolved solar model obtained by numerical simulation.

2. The model

As already outlined, we consider a polytropic envelope. Since the exact value of the polytropic index is of no great importance for the qualitative behaviour of the solutions, $n = 0$ was chosen for simplicity; an envelope of constant density can be treated analytically. Its structure is determined by

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho, \tag{3}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho. \tag{4}$$

The procedure to solve theses equations is straight-forward and will not be repeated here. Only the approximate solution for extendend envelopes should be considered more closely. In this case, M_r in Eq. (3) may be replaced by M_{core} because in extended envelopes the variation of the gravity acceleration GM_r/r^2 is by far more due to the variation of r by orders of magnitude than to the moderate variation of M_r. Thus

$$P = P_f - \frac{GM_{core}}{r_f}\rho_f + \frac{GM_{core}}{r}\rho_f \tag{5}$$

is obtained, where the constant of integration is determined at the interface. Applying Eq. (5) to the outer boundary $r = R$, $P = 0$ and expressing P_f by ρ_f and T_f, we obtain

$$\frac{kT_f}{\mu_{env} H} = \frac{GM_{core}}{r_f} - \frac{GM_{core}}{R}. \tag{6}$$

Eq. (6) relates the specific energies w and u at the interface – considered in the introduction – to the total radius R. Indeed, for $w \gg u$, only a small radius and hence a small mass of the envelope is possible. As w approaches u, the core-envelope interface is raised, the binding energy of the envelope goes to zero, and there are no restrictions on R and M_{env} anymore.

The structure of the isothermal core, allowing for non-relativistic electron degeneracy, is again determined by Eqs. (3) and (4) and the corresponding equation of state

$$P(\psi) = \frac{4\pi}{h^3} (2mkT)^{\frac{3}{2}} kT \left(\frac{3}{2} F_{\frac{3}{2}}(\psi) + \frac{\mu_E}{\mu_I} F_{\frac{1}{2}}(\psi) \right)$$

$$\rho(\psi) = \frac{4\pi}{h^3} (2mkT)^{\frac{3}{2}} \mu_E H F_{\frac{1}{2}}(\psi)$$

where μ_E^{-1},and μ_I^{-1} are the numbers of electrons, and nuclei per unit atomic weight respectively, and $F_{\frac{1}{2}}(\psi)$ and $F_{\frac{3}{2}}(\psi)$ are the Fermi-Dirac integrals. For $\psi \to \pm\infty$, the equations of state for complete and for non-degeneracy are recovered. The former is

$$P = K\rho^{\frac{5}{3}}; \qquad K = \frac{h^2}{20mH} \left(\frac{3}{\pi H}\right)^{\frac{5}{3}} \frac{1}{\mu_E^{\frac{5}{3}}} \tag{7}$$

with negligible contribution of the nuclei. This is a polytropic relation for $n = 1.5$ and a fixed polytropic constant K; with which w in Eq. (1) has been estimated.

The numerical solutions for $\psi_c \geq 20$ consist of two fairly distinct parts: a completely degenerate central part and a non-degenerate peripheral region on top of it with almost no transition in between. The core and envelope are fitted to each other at the interface $M_r = M_{core}$. There, the shell source is located,

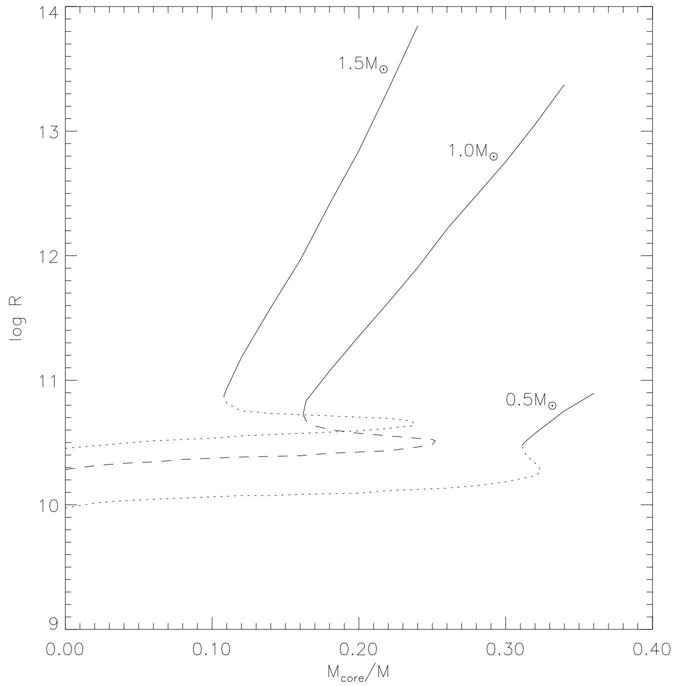

Fig. 1. Radius R as a function of M_{core}/M for models of three different masses $M = 0.5, 1.0, 1.5M_{\odot}$. Solid curves refer to expanding models; dotted and dashed parts belong to phases preceeding the formation of degenerate cores in Schoenberg–Chandrasekhar scenario.

assumed to be burning at $T_f = 2 \cdot 10^7$K. Hence, the chemical discontinuity and jump in density due to a He-core and a H-envelope

$$\frac{\rho_{int}}{\rho_{ext}} = \frac{\mu_{core}}{\mu_{env}} = \frac{8}{3}$$

has been properly taken into account.

Solutions were obtained for three different total masses $M = 0.5, 1.0, 1.5M_{\odot}$. All three sequences start at $M_{core} = 0$ ("main sequence state") and eventually reach a phase of rapidly increasing radii – see Fig. 1, where radius R is plotted against M_{core}/M.

Two different evolutionary phases are clearly discernible in Fig. 1. For small mass fractions of the core, there is only a moderate increase of the radius. This phase is terminated when the Schoenberg-Chandrasekhar limiting mass is reached. Evolution with increasing mass fraction only becomes possible after the formation of a degenerate core. The ensueing evolutionary phase is characterized by a dramatic increase of radius. It is this second phase which is relevant to our problem.

Obviously there is no quasistatic transition between the two phases. This is the consequence of an isothermal non-degenerate core (Schoenberg-Chandrasekhar scenario) during the first phase. According to numerical simulations, evolution towards shell source models proceeds along a somewhat different (strictly quasistatic) way. After some rearrangements on a thermal timescale (responsible for the Hertzsprung gap), the stars become structurally very similar to those of the above second phase (see Sect. 3 for comparison with a detailed numerical simulation).

Thus, sequences of expanding composite stellar models are obtained on an essentially hydrostatic basis for properly chosen equations of state. What they are teaching us physically will be discussed in the next section.

3. Discussion

The internal structure of the expanding models presented in the preceding section consists of:

1. a completely degenerate isothermal inner core of radius r_{core} (in fact its white dwarf radius) containing almost all of M_{core};
2. a non-degenerate isothermal outer core, located still below the interface, extending to radii r_f very much larger than r_{core}, but containing very little mass;
3. an interface between the He-core and the H-envelope, where the shell source is located; and
4. an extended envelope above the interface.

There is, most importantly, the intermediate layer 2 which reduces the strong gravitational pull of the compact core upon the envelope by elevating the core-envelope interface.

Formally, layer 2 is needed if the temperature at the interface, T_f, i.e. the temperature of the shell source, is prescribed. For a model consisting only of parts 1 and 4 only the quantities M and M_{core} may be prescribed, because of the universally fixed factor K in Eq. (7). Physically layer 2 is produced by the thermostatic action of the shell source. If it would not exist, the temperature T_f would have to be much higher than $2 \cdot 10^7 K$ to balance the pressure exerted by the envelope. Then the shell source would produce much more energy than could be transported by the envelope and the surplus energy would be used to lift M_{env} against gravity – until $T_f = 2 \cdot 10^7 K$ is restored. We are not able to follow these processes within the present framework. However, it is very suggestive that exact numerical simulations yield models of the same structure as obtained here (see Sect. 3).

This intermediate layer can only serve its purpose if, for decreasing energy w in the outward direction, the thermal energy u does not decrease too much as well. This requirement is well satisfied, because no energy is released from the core and hence the layer is isothermal. This layer can even extend into the shell source region, where the temperature is kept constant by hydrogen burning. This behaviour is not realized in the present model, where the shell source region is nearly a "sheet", but is found in more realistic simulations. The formation of the intermediate isothermal layer may be the reason for an increasing polytropic index in stars evolving off the main sequence, as pointed out by Eggleton and Faulkner (1981).

Since the intermediate layer is below the shell source and since it must be non-degenerate, the shell source itself must always be located in the non-degenerate regime. Furthermore, the intermediate layer is an extension of the degenerate central part of the core; hence this part must be an almost complete

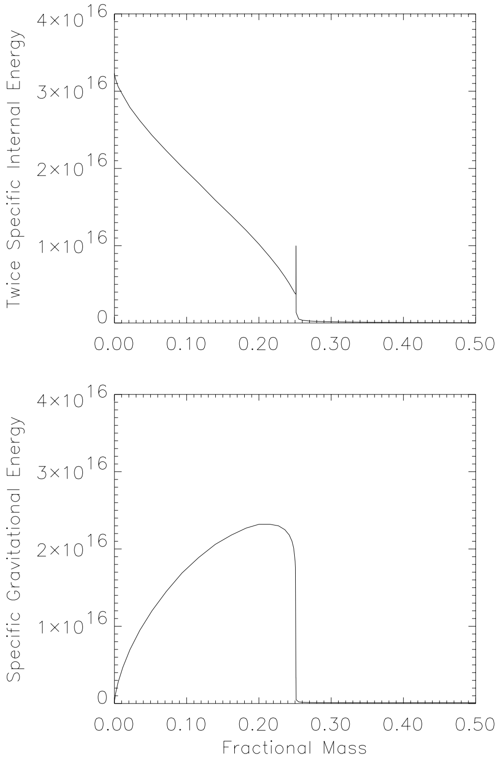

Fig. 2. Specific internal and gravitational energy u and w as functions of fractional mass M_r/M for the $1M_{\odot}$-model with $M_{core}/M = 0.25$.

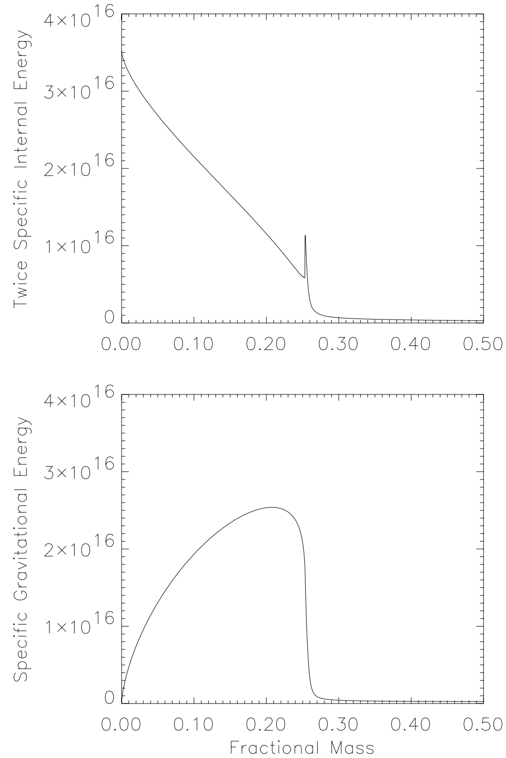

Fig. 3. Specific internal and gravitational energy u and w as functions of fractional mass M_r/M for the sun with $M_{core}/M = 0.25$ as obtained by numerical simulation (Stix, 1997).

polytrope $n = 1.5$ (i.e. a white dwarf inside the star). Both properties are well known from numerical simulations.

The hydrostatic arguments outlined so far may be visualized by the specific energies w and u as functions of the fractional mass M_r/M – see Fig. 2 – and for $M = 1M_{\odot}$ and $M_{core} = 0.25M$. According to the virial theorem, the areas below these curves must be equal. The run of u is fixed by $T_f = 2 \cdot 10^7 K$ at $M_r = M_{core}$, which is very small compared to the contribution of the degenerate core. Hence, almost all of the internal energy of the whole star is concentrated in the core. Accordingly, the specific gravitational energy must also attain very small values at M_{core}, otherwise the virial theorem would not be satisfied. Thus, a drop of w at $M_r \approx M_{core}$ by more than a factor 10 becomes necessary. This drop is due to the intermediate layer, which, in this representation, appears almost like a discontinuity in radius. Fig. 2 also shows why the naive argument for the behaviour of R given in the introduction must fail. The main contribution to the energies are made by the core – and in fact it does show decreasing r_{core} for increasing M_{core}, whereas R is completely decoupled from the energy balance. Because of the weakly bound envelope, R must be very large – regardless of the detailed structure of the envelope, which only fixes the exact value of the large radii.

These specific energies w and u as functions of M_r/M may now be compared with those obtained from numerical simulations. For this purpose Stix (1997) ran a solar model from the main sequence to a state where, after $1.415 \cdot 10^{10}$y, a He-core of $0.25M$ has developed – as in the model discussed in the preceding paragraph. The functions w and u taken from the numerical results are shown in Fig. 3. There is good agreement between Figs. 2 and 3, especially the behaviour of the specific gravitational energy. Detailed inspection reveals that the drop of w at $M_r \approx M_{core}$ does not exclusively occur below the shell source but extends into it; the drop is also slightly smaller than in our model. This behaviour shows that the shell source itself is also contributing to the separation between core and envelope – a result possible only if the shell source is resolved (see Fig. 3). Again, the envelope is unimportant as far as energies are concerned. Taking its detailed structure into account, a total radius $R = 6.24 \cdot 10^{11}$cm is obtained, compared to $1.12 \cdot 10^{12}$cm for an envelope with constant density in our model.

4. Concluding remarks

The expansion phase of our model stars is due to the formation of a non-degenerate isothermal zone between the degenerate central part of the core and the polytropic envelope. This intermediate zone is formed to bridge the mismatch between the specific gravitational energy right on top of the fully degenerate part of the core and the specific internal energy fixed by the temperature for hydrogen burning. Otherwise, an envelope of a substantial amount of mass would not be in hydrostatic equilibrium with the core. Due to this intermediate zone, w approaches u, the envelope becomes less strongly bound and – according to Eq. (6) – it expands. This model includes the suggestions mentioned in the introduction as different aspects: There is a strong gravitational field of the core, there is a layer with $n \geq 5$, and there is a shell source with limited luminosity – all of them fitting into simple hydrostatic requirements.

This argument depends on degenerate cores forming in low mass stars. What about high mass stars with continually contracting cores? Contraction yields values of the specific gravitational energy on top of the cores which are again in excess of the value of the specific internal energy as determined by the hydrogen burning temperature. As will be shown in a forthcoming paper, these cores – under the assumtion of homologous contraction – consist of a nearly polytropic central part (with $n = 3$) surrounded by a zone with larger n. This situation is similar to the present one and again leads to expanded envelopes.

Finally, for higher burning temperatures, the discrepancy between w and u could be avoided from the outset and post main sequence contraction should be expected. Indeed, evolved helium stars with T_f now higher by a factor 10 evolve towards the left of the main sequence position (Deinzer and Salpeter, 1964). This once more points to the low hydrogen burning temperatures as the cause for expansion.

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