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Statistical characteristics of radiation formed in atmosphere with randomly distributed inhomogeneities

A.G. Nikoghossian[1,3], **S. Pojoga**[2,3], **and Z. Mouradian**[3]

[1] Byurakan Astrophysical Observatory, 378433 Byurakan, Armenia
[2] Astronomical Institute of the Romanian Academy, RO-75212 Bucharest, Romania
[3] Observatoire de Paris-Meudon, DASOP, F-92195 Meudon Cedex, France

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Abstract. In this paper we continue studying the spatial brightness variations of solar quiescent prominences. The theory is developed to determine the mean intensity and the relative mean square deviation (RelMSD) for the line radiation emerging from an one-dimensional atmosphere with randomly distributed inhomogeneities. Both the local thermodynamic equilibrium (LTE) and Non-LTE cases are considered. The results previously obtained for the LTE atmosphere are extended to encompass the more realistic situation when the random parameters describing the physical properties of structural elements may take an arbitrary number of possible values. The profiles of spectral lines constructed by solving the stochastic problem of radiative transfer in the LTE atmosphere allow one to check the accuracy of the commonly used approximations.

The problem of determining the statistical characteristics of the line radiation formed in a multicomponent Non-LTE atmosphere is treated for the case of conservative scattering. We derive a closed-form analytical expression for the RelMSD of the outgoing intensity for an arbitrarily large number of structural elements in the medium.

The theoretical results are employed for interpreting some of the features specific to the spatial brightness fluctuations of prominences in extreme ultraviolet (EUV) lines. The estimates obtained for the mean value of the line-of-sight number of threads are in agreement with those inferred by other authors. Some important conclusions on the multithread structure of prominences are drawn.

Key words: radiative transfer – Sun: prominences – Sun: transition region

1. Introduction

In astronomy we are faced frequently with problems in which the physical and geometrical properties of a radiating medium undergo random variations. In general, two possible approaches to such problems can be represented. For simplicity's sake one may deal with averaged characteristics of the radiating atmosphere and address a deterministic problem of the radiative transfer to give an approximate, in a certain sense, description of the real physical situation. The alternative approach is to tackle a much more complicated stochastic problem and use the statistical features of the observed radiation as additional information for the determination of the structural pattern of the atmosphere and its variations. Thus, we are led to a kind of inverse problem, whose solution may not generally be unique.

In this latter approach, we need in a suitable theory of radiative transfer through an atmosphere with randomly distributed inhomogeneities. In this connection, it is of particular interest to consider a multicomponent atmosphere, i.e., an atmosphere composed of a single type of structure, randomly distributed in space with or without an intervening ambient medium. The physical parameters characterizing each structural element are also assumed to undergo random variations, taking one or another value with some probability. The statistical properties of the radiation emerging from a multicomponent LTE atmosphere were studied by Jefferies and Lindsey (1988) in connection with the interpretation of total solar eclipse observations in the far-infrared. A model problem considered by these authors assumes two possibilities for the values taken by the random physical parameters, characteristics of the structural elements. Nevertheless, as we shall see below, the results remain to be valid for an arbitrary number of possibilities. The procedure developed in the mentioned work supposes that variations in the physical properties of two adjacent elements are non-correlated. More recently Lindsey and Jefferies (1990) extended their theory to handle transfer for structure scale lengths smaller than the scale size of the homogeneity, i.e., for the microscopic domain of inhomogeneities, when random variations of different elements are essentially correlated.

In this paper we shall limit ourselves by working in the macroscopic domain of inhomogeneities, so that no correlations exist between variations of different structural elements. Our treatment of the stochastic transfer problems is motivated by the study of EUV spectra of the solar quiescent prominences, whose fine structure may now be considered as wellestablished. Digital raster images of prominences obtained with the Harvard College Observatory (HCO) aboard ATM-Skylab represent a rich observational resource for investigating the statistical properties of the line formation in multicomponent media.

Send offprint requests to: Z. Mouradian

In general, the spatial brightness variations of prominence are due to several factors (see Pojoga et al. 1998; hereafter Paper I). These include the random nature of the filling factor, the statistical variations in the line-of-sight number of structural elements, and instrumental errors. Perhaps most significant factor is physical inhomogeneities characterizing the structures, which is the primary subject of this paper. It is natural to expect that the study of prominence radiation in optically thin lines, such as C II λ1336 Å, C III λ977 Å and O VI λ1032 Å, as well as in the extremely opaque lines of the Lyman series provides necessary information on the physical and geometrical characteristics of the fine structure of prominences. The procedure explored by Jefferies and Lindsey for LTE-atmospheres is obviously applicable only to optically thin lines with high temperatures of formation, while the formation of the Lyman series lines is controlled by the multiple scattering effects and needs the development of a suitable Non-LTE theory. This theory is essentially involved, as far as the physical conditions at a given point of a medium are now determined not only by the local values of thermodynamic parameters but also by the radiation field throughout the atmosphere. The Non-LTE atmosphere for conservative scattering was discussed by Nikoghossian et al. (1997; hereafter Paper II) by using Ambartsumian's (1960) procedure of addition of layers. The present paper is a further extension of results previously obtained for both the LTE and Non-LTE atmospheres. Particularly in the latter case, a new approach based on the concept of escape probability is explored. This enables one to advance as against Paper II and derive a closed-form analytical expression for the relative mean square deviation (RelMSD) for any number of structural elements.

The outline of this paper is as follows: we begin, in the opening Sect. 2, by reconsidering the LTE atmosphere. A somewhat more detailed derivation of basic equations and their physical consequences than in Paper II, are given. It is shown that the equations for the mean intensity and RelMSD remain valid for sufficiently general assumptions on the random character of the inhomogeneities. The results of numerical calculations are discussed. In Sect. 3 we construct the profiles of spectral lines formed in the LTE atmosphere with randomly varying characteristics. It is shown that the discrepancy between real line profiles and those obtained for an atmosphere with previously averaged values of physical parameters, may become substantial. The Non-LTE atmosphere with conservative scattering is considered in Sect. 4. The analytical expression for the RelMSD is derived for an arbitrary number of structural elements. The results obtained and their applications to prominences are discussed in Sect. 5.

2. The LTE atmosphere

Let us treat the number N of structural elements each of which is described by the power of the energy release, B, and optical thickness, τ. The value of B is assumed to be constant within each individual element. As earlier in Paper II, we begin by considering the simplest situation, when the pair of quantities (B, τ) takes randomly only two possible sets of values (B_1, τ_1) and (B_2, τ_2), each occurring with probability p_1 and $p_2 = 1 - p_1$. One may think of a more general problem when various elements take different values of τ for the same value of B. Referring the interested reader to the paper by Jefferies and Lindsey (1988) for this problem, we note that all the results to be obtained can be easily extended to comprise this case as well.

Thus, suppose that we have a set of N radiating elements with randomly varying properties (B_i, τ_i; $i = 1, 2$). We are interested in the averaged characteristics of the emerging radiation such as the mean intensity $\langle I_N \rangle$ and the RelMSD $\delta_N = \left(\langle I_N^2 \rangle / \langle I_N \rangle^2\right) - 1$. As was emphasized in Paper II, the LTE atmosphere differs from the Non-LTE atmosphere essentially by the absence of reflected radiation from the structural components. This fact greatly simplifies the problem since the averaged characteristics of radiation emerging from a given medium are not affected as a result of the addition of new elements to the medium. This implies that the averaging process may be performed in parts, which allows to write directly

$$\langle I_N \rangle = \alpha \langle I_{N-1} \rangle + \langle I_1 \rangle, \tag{1}$$

where $\alpha = \langle \exp(-\tau) \rangle$ and $\langle I_1 \rangle = \langle B\,(1 - \exp(-\tau)) \rangle$, or the more general result

$$\langle I_N \rangle = \alpha^k \langle I_{N-k} \rangle + \langle I_k \rangle. \tag{2}$$

A more rigorous derivation of Eq. (1) might proceed along the following lines. Let a certain configuration of $N-1$ elements occur with probability P. Adding a new element to the front part of such a medium we shall observe a radiation of intensity $I'_N = I_{N-1} \exp(-\tau_1) + J_1$ or $I''_N = I_{N-1} \exp(-\tau_2) + J_2$ with probability $p_1 P$ and $p_2 P$, respectively. Here we introduced the quantities $J_i = B_i\,(1 - \exp(-\tau_i))$ $(i = 1, 2)$ characterizing the intensity of radiation emitted by an individual structural element of each type. Multiplying each value of intensity by the proper probability, and adding up the results we are led to Eq. (1). The same reasonings applied to I_N^2 yields

$$\langle I_N^2 \rangle = \beta \langle I_{N-1}^2 \rangle + 2\,K \langle I_{N-1} \rangle + \langle I_1^2 \rangle, \tag{3}$$

where $\beta = \langle \exp(-2\tau) \rangle$, and $K = \langle J \exp(-\tau) \rangle$.

An important comment to be made at this stage is that the physical considerations and ratiocinations underlying Eqs. (1) and (3), do not depend on the number n of the possible values taken by the pair of parameters (B, τ). Therefore, the mentioned equations along with those to be obtained are valid for an arbitrary number of realizations of the components' physical properties. Moreover, Eqs. (1) and (3) remain true also for the continual analogue of the problem at hand, i.e., when B and τ (hence J and $J \exp(-\tau)$) are continuum-valued random quantities. Knowing the probability distributions of these quantities, one may easily find the proper averaged parameters (see below Eq. 12) being necessary for evaluation of the mean intensity and the RelMSD. For expository reasons, however, in what follows we continue studying the discrete problem, keeping in mind that the physical conclusions at which we arrive can be easily reformulated for the continuous distributions of B and τ.

Let us now establish several important equations for the mean intensity and the RelMSD that will be in use in further discussion. By applying successively the recurrence relation (1) we may write

$$\langle I_N\rangle = L_N\langle I_1\rangle, \tag{4}$$

$$\langle I_N^2\rangle = \beta\langle I_{N-1}^2\rangle + C_{N-1}, \tag{5}$$

where $L_N = \left(1-\alpha^N\right)/(1-\alpha)$ and $C_N = 2KL_N\langle I_1\rangle + \langle I_1^2\rangle$.

Eq. (5) allows us to find $\langle I_N^2\rangle$ in terms of $\langle I_1^2\rangle$ and write

$$\langle I_N^2\rangle = \beta^{N-1}\langle I_1^2\rangle + \sum_{k=2}^{N-1} C_k\beta^{N-k-1}. \tag{6}$$

The summation in the right-hand side of Eq. (6) can be easily performed to yield

$$\begin{aligned}\sum_{k=2}^{N-1} C_k\beta^{N-k-1} &= \langle I_1^2\rangle \sum_{k=2}^{N-1}\beta^{N-k-1} \\ &\quad + 2K\langle I_1\rangle \sum_{k=2}^{N-1} L_k\beta^{N-k-1} \\ &= M_{N-1}\langle I_1^2\rangle + 2KA_N\langle I_1\rangle,\end{aligned} \tag{7}$$

where $M_N = \left(1-\beta^N\right)/(1-\beta)$, and $A_N = (L_N - M_N)/(\alpha-\beta)$.

Thus, we finally have

$$\langle I_N^2\rangle = M_N\langle I_1^2\rangle + 2KA_N\langle I_1\rangle, \tag{8}$$

which by virtue of Eqs. (4) leads to the requisite equation for the RelMSD:

$$\delta_N = \frac{M_N}{L_N^2}(1+\delta_1) + 2\frac{KA_N}{\langle I_1\rangle L_N^2} - 1, \tag{9}$$

where $\delta_1 = \left(\langle I_1^2\rangle/\langle I_1\rangle^2\right) - 1$. Note also that Eqs. (3) and (8) allow us to write two equivalent recurrence relations for determining δ_N, which we present here in a more visualizable form

$$\begin{aligned}L_{N+1}^2(1+\delta_{N+1}) &= \beta L_N^2(1+\delta_N) \\ &\quad + (2KL_N/\langle I_1\rangle) + 1 + \delta_1,\end{aligned} \tag{10}$$

$$\begin{aligned}L_{N+1}^2(1+\delta_{N+1}) &= L_N^2(1+\delta_N) \\ &\quad + (2KD_N/\langle I_1\rangle) + \beta^N(1+\delta_1),\end{aligned} \tag{11}$$

where $D_N = A_{N+1} - A_N = \left(\alpha^N - \beta^N\right)/(\alpha-\beta)$.

We now have at our disposal all of the equations needed to discuss the statistical properties of radiation emerging from the LTE atmosphere. An important salient trait inherent in Eqs. (4) and (9) (see also Eqs. 10 and 11) is that the values of the mean intensity and the RelMSD for the multicomponent atmosphere with an arbitrary number of elements are determined by only a few parameters. These parameters for the general case of n realizations of (B,τ) are

$$\alpha = \sum_{i=1}^n p_i\exp(-\tau_i), \qquad \beta = \sum_{i=1}^n p_i\exp(-2\tau_i),$$

$$\langle I_1\rangle = \sum_{i=1}^n p_iJ_i, \qquad K = \sum_{i=1}^n p_iJ_i\exp(-\tau_i), \tag{12}$$

$$\delta_1 = \frac{1}{2\langle I_1\rangle^2}\sum_{i=1}^n p_i\sum_{k=1}^n p_k(J_i-J_k)^2.$$

It is seen that all the above parameters are the averaged characteristics of a single structural element. Thus we arrive at the conclusion that, in general, there must exist a great number of configurations, having a fixed number of components, that are characterized by the same values of the mean intensity and the RelMSD. Furthermore, the values of $\langle I_N\rangle$ and δ_N for such media exhibit the same behaviour with varying N.

It is of particular interest to study the asymptotic behaviour of fluctuations when $N\to\infty$. Eqs. (4) and (9) show that while $\langle I_N\rangle \to \langle I_1\rangle/(1-\alpha)$, the RelMSD goes generally to the nonzero limit, viz.,

$$\delta_\infty = \frac{(1-\alpha)^2}{1-\beta}(1+\delta_1) + \frac{2K}{\langle I_1\rangle}\frac{1-\alpha}{1-\beta} - 1, \tag{13}$$

in contrast (as will be seen later) to the Non-LTE atmosphere. Let us now consider several special cases when one of the physical parameters is the same for all components of an atmosphere:

(i) Let τ_i be τ, common for all structural elements. Then $K/\langle I_1\rangle = \alpha = \exp(-\tau)$, $\beta = \alpha^2$ and we obtain from Eq. (9)

$$\delta_N = \left(\frac{1-\exp(-\tau)}{1+\exp(-\tau)}\right)\left(\frac{1+\exp(-N\tau)}{1-\exp(-N\tau)}\right)\delta(B), \tag{14}$$

where $\delta(B) = \left(\langle B^2\rangle/\langle B\rangle^2\right) - 1$. As might be expected, the fluctuations stem from the differences in values of B, and fall off with $N\to\infty$, tending to the nonzero limit $\delta_\infty = [(1-\exp(-\tau))/(1+\exp(-\tau))]\delta(B)$. The larger the optical thickness of the components, the greater δ_∞, and the faster the passing of δ_N to its asymptotic plateau, $\delta(B)$. For τ sufficiently small such that $N\tau \ll 1$, Eq. (14) simplifies to

$$\delta_N = \delta(B)/N. \tag{15}$$

(ii) Now we suppose that all components of a medium are characterized by the same value of B, i.e., the atmosphere is homogeneous, so that fluctuations in the observed radiation are due to variations in the total optical thickness. It is obvious that only in this case the arrangement of the elements for a given proportion of various species is not essential. Taking into account that now $\delta_1 = (1-2\alpha+\beta)/(1-\alpha)^2$ and $K/\langle I_1\rangle = (\alpha-\beta)/(1-\alpha)$, we obtain

$$\delta_N = \frac{\beta^N - \alpha^{2N}}{\left(1-\alpha^N\right)^2}. \tag{16}$$

It follows from Eq. (16) that for a homogeneous atmosphere the RelMSD tends exponentially to 0, when $N\to\infty$. One may show that this is the only case in which fluctuations vanish with increasing N.

(iii) Also of interest is the situation in which all structural elements are emitting equal amounts of energy J, so that the

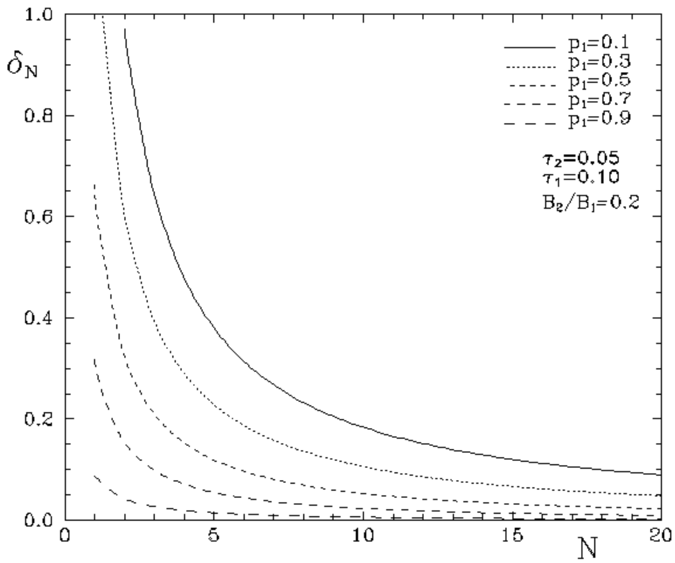

Fig. 1. The function δ_N of N for various p_1 and indicated values of other parameters in the case of $B_1 > B_2$, $\tau_1 > \tau_2$. The less probable the appearance of the brightest component, the larger values of δ_N. The function δ_N for $p_1 = p_2 = 0.5$ occupies some intermediate position.

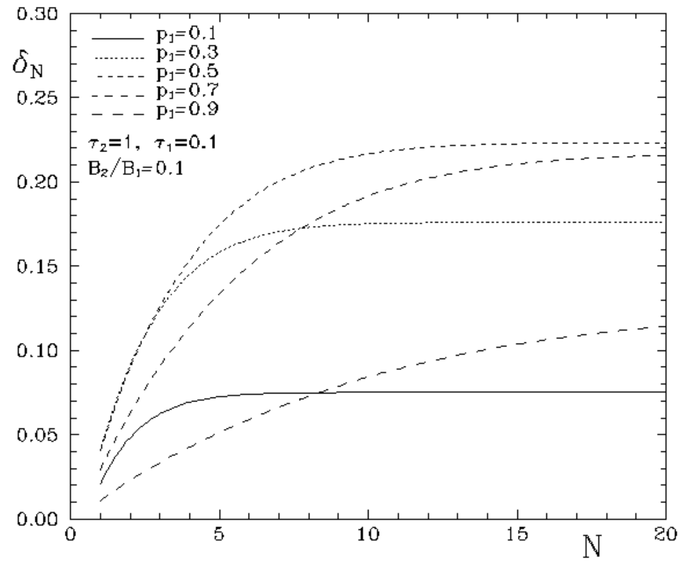

Fig. 2. The function δ_N of N for various p_1 for an atmosphere with the brighter component less opaque than the fainter ($B_1 > B_2$ and $\tau_1 < \tau_2$). Now the values of δ_N for $p_1 = p_2 = 0.5$ are largest.

fluctuations are due only to the difference in absorption of the emerging radiation. Now $\delta_1 = 0$, and $K/\langle I_1 \rangle = \alpha$, which represent the minimal values of these quantities for a given set of τ_i, as compared to other cases. Thus the fluctuations in the observed intensity are also the lowest. In the special case in which the equality of J_i follows from the equality of B_i (and hence τ_i), we are led back to the homogeneous atmosphere. In this particular case, when τ_i are also equal, δ_N is obviously zero, otherwise δ_N is zero only for $N = 1$, and increases monotonically with an increase of N to its asymptotic value, $\delta_\infty = \left(\beta - \alpha^2\right) / (1 - \beta)$, resulting from Eq. (13).

In order to make an impression on the run of δ_N with N for any values of B_i and τ_i, we consider the results of calculations concerning the simplest problem of $n = 2$. As was stated above, the conclusions at which we arrive may be readily generalized to cover more complicated problems. Particular attention will be paid to the behaviour of δ_N with respect to N for $p_1 = p_2 = 0.5$, which may be regarded as the discrete and extremely schematic model of the continuous-valued problem with symmetrical probability distributions characterizing the physical properties of an atmosphere. The values of δ_N for $p_1 = p_2$ fall typically between those evaluated for large and small probabilities (see Fig. 1) (here we exclude the non-interesting situations in which p is close to zero or unity, which collapse to the homogeneous problem). The only exception shown in Figs. 2–4 concerns the case in which J_1 approaches J_2, and this will be discussed below.

Depending on the values of the parameters given by Eq. (12), the function δ_N can exhibit a broad variety of different behaviours. It may decrease or increase monotonically with N, or exhibit an initial decrease followed by an increase for greater values of N. To facilitate further discussion, we note that the symmetry of the problem with respect to simultaneous exchange $B_2 \leftrightarrow B_1$ and $\tau_2 \leftrightarrow \tau_1$, allows us to limit the discussion to the case $B_1 \geq B_2$. It is expedient to distinguish among others the situation in which the structural elements radiate an equal amount of energy (i.e., when $J_1 = J_2$, or $\gamma = J_2/J_1 = 1$). This situation is unreachable if the condition $B_1 > B_2$ is satisfied together with inequality $\tau_1 > \tau_2$. In this case δ_N is a monotonically decreasing function of N and goes to a nonzero limit as $N \to \infty$.

Fig. 1 shows that for $(B_i,\, \tau_i)$ fixed the values of δ_N are smaller when the bright component is more probable. With increasing τ_1 this function becomes steeper, while an increase in the values of both of τ_i not violating the inequality $\tau_1 > \tau_2$ leads to a smaller limit of δ_N as $N \to \infty$. As might be expected, with an increase of γ from 0 to 1, (i.e., with decreasing contrast between B_i), the RelMSD becomes smaller (see first three graphs of Fig. 3a). The behaviour of δ_N with respect to N is altered essentially when one of inequalities, $B_1 > B_2$ or $\tau_1 > \tau_2$, changes its sign (see Fig. 2). This corresponds to the case when the brighter component is less opaque than the fainter. We also observe that now the values of δ_N for $p_1 = p_2 = 0.5$ are the largest. We see from Fig. 3a that for $\tau_1 > \tau_2$, close to $\gamma = 1$, δ_N becomes smaller for any value of N, and alters its behaviour by turning into a monotonically increasing function of N. When $\tau_1 < \tau_2$, δ_N changes from an increasing function of N for $\gamma < 1$ to a monotonically decreasing function for $\gamma > 1$ (Fig. 3b).

Figs. 4a and b illustrate the relationship between δ_N and τ_2 for $B_1 < B_2$ and $B_1 > B_2$, respectively. The minimum attained by δ_N at $J_1 \approx J_2$ is clearly seen. It is noteworthy that depending on whether the inequality $B_1 > B_2$ or $B_1 < B_2$ holds, the behaviour of δ_N as $\tau_2 \to \infty$ is different for large values of N. The quantitative analysis of numerical results described above as well as their application to prominences will be given in Sect. 5.

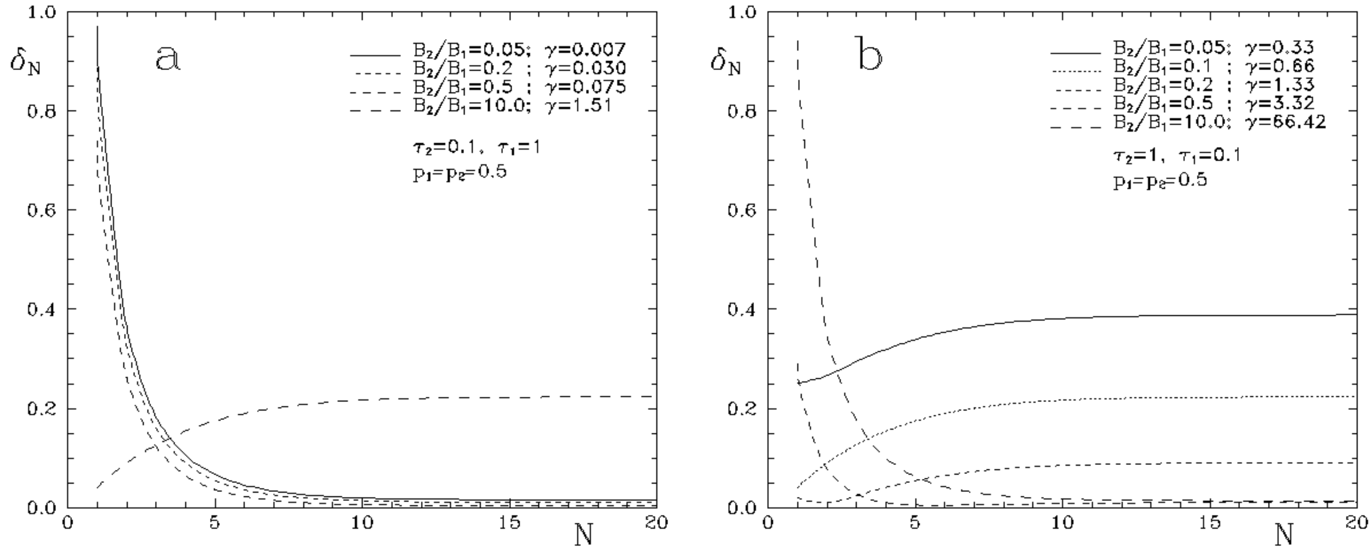

Fig. 3a and b. The function δ_N of N for various values of the ratio B_2/B_1: **a** $\tau_2/\tau_1 = 0.1$, **b** $\tau_2/\tau_1 = 10$. Depending on whether $\gamma < 1$ or $\gamma > 1$ ($\gamma = J_2/J_1$) the behaviour of the function δ_N is different.

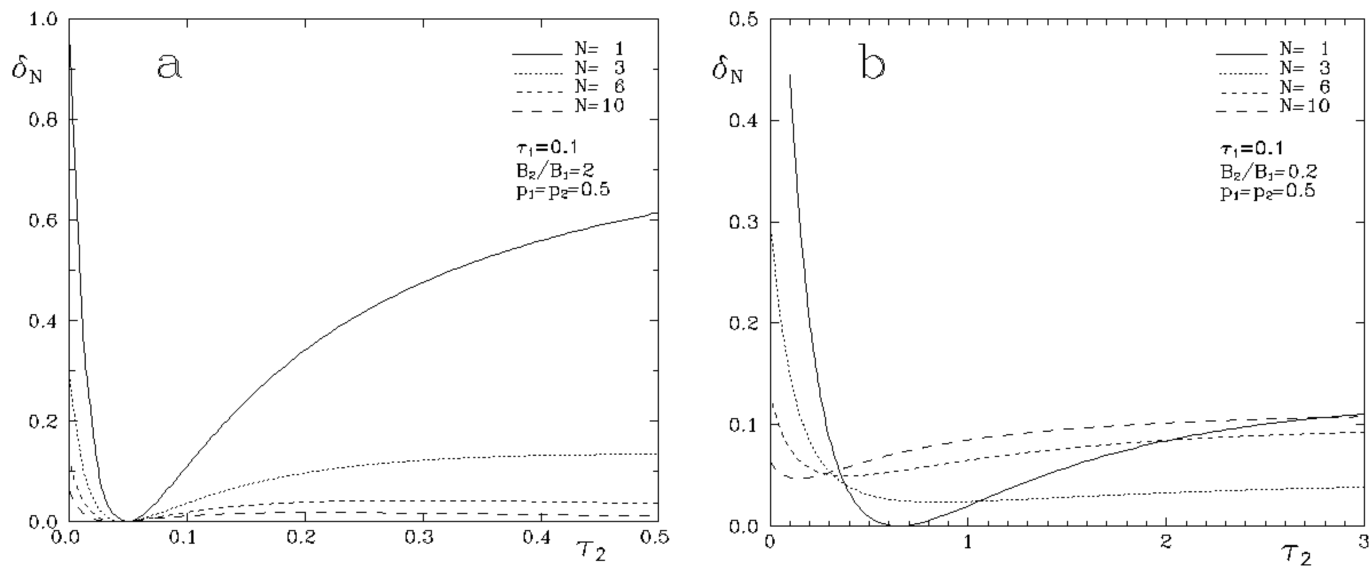

Fig. 4a and b. The function δ_N of τ_2 for various N: **a** $B_2 > B_1$, **b** $B_2 < B_1$. The minimum attained by δ_N at $\gamma \approx 1$ is discernible. The different behaviour of δ_N for large τ_2 in the case **a** as compared to that in **b** is noteworthy.

3. Spectral line profiles

As a consequence of Eq. (4), we consider a particular problem that concerns the profiles of spectral lines formed in an LTE atmosphere with randomly distributed inhomogeneities. For simplicity of exposition we consider the case of two species ($n = 2$) of the structural elements. Let B_i be the possible values of the source function, and τ_i be the optical thickness in the centre of the spectral line. We suppose that each element radiates within a spectral line with the Doppler profile $\phi(x) = \exp\left(-x^2\right)$ for the normalized absorption coefficient (here x is the dimensionless frequency denoting the displacement from the centre of the line measured in the Doppler widths). Because of the absence of scattering, and hence effects of the frequency redistribution, the averaging process may be obviously performed for each frequency separately so that applied to our problem, Eq. (4) can be rewritten as follows

$$\langle I_N(x)\rangle = L_N(x)\langle I_1(x)\rangle. \tag{17}$$

Our immediate objective is to compare the profiles of the spectral lines formed in an atmosphere with randomly varied physical properties (referred to as the case (c) in Figs. 5, 6) with those formed in atmospheres with given constant values of B_1, τ_1 and B_2, τ_2 (cases (a), (b), respectively). On the other hand, it is well-known that difficulties encountered in solving stochastic astrophysical problems often lead one to replace them by the proper deterministic problem with preliminarily averaged, in some sense, random physical parameters describing the me-

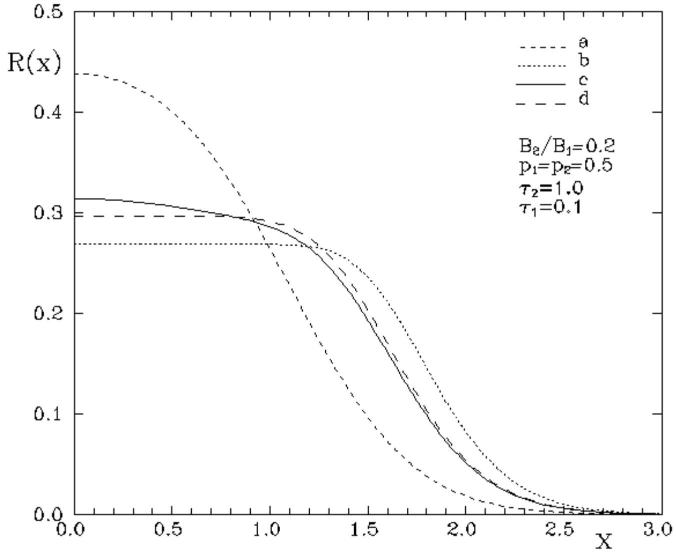

Fig. 5. The normalized profiles of the LTE spectral line for the four formulations of the problem discussed in the text: (a) homogeneous atmosphere with $B_1 = \text{const}$, and optical thickness $N\tau_1$; (b) homogeneous atmosphere with $B_2 = \text{const}$, and optical thickness $N\tau_2$; (c) atmosphere with randomly varied values of B and τ; (d) the profile obtained as a result of preliminarily averaging the random quantities. The profile for the case (d) fits satisfactory the real profile of the spectral line (c).

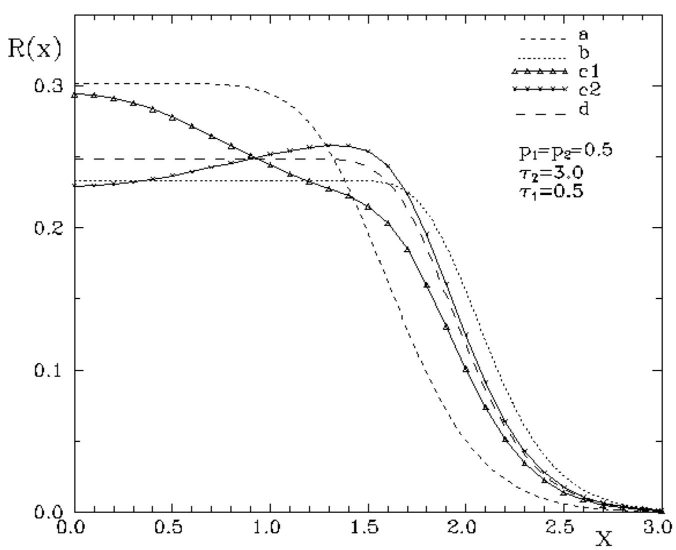

Fig. 6. The same as in Fig. 5 for the values of parameters indicated. The real profiles (c1 – $B_2/B_1 = 0.2$, c2 – $B_2/B_1 = 5$) differ fundamentally from those obtained by considering the deterministic formulation of the problem.

dium. It is of interest from this point of view to compare the solution of the problem posed in such a way with the exact solution of the stochastic problem. With this in mind we give in what follows also the profiles of the line formed in an atmosphere with averaged physical characteristics, $\langle B \rangle$ and $\alpha = \langle \exp(-\tau) \rangle$ (case (d)).

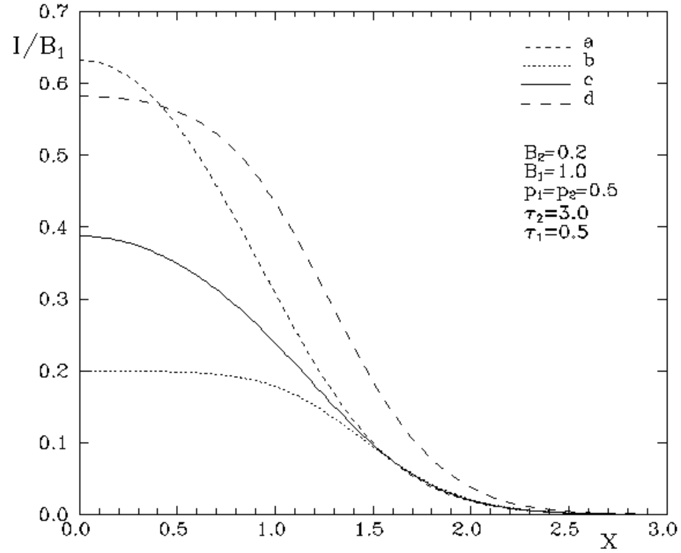

Fig. 7. The intensity of radiation emerging from an atmosphere for the same cases as in Fig. 5. It is seen that averaging the random characteristics of an atmosphere may lead to essentially erroneous results.

Figs. 5, 6 show the normalized line profiles, $R(x)$, calculated for the four formulations of the problem in case of $N = 20$. It is seen that for relatively small values of τ_1 and τ_2 (Fig. 5), the profile obtained by preliminarily averaging the random characteristics of an atmosphere fits satisfactory the exact solution of the stochastic problem. Both of these profiles lay between those corresponding to deterministic problems. However, this is not the case when one of the possible values of the optical thickness is large. This is illustrated by Fig. 6, where two profiles relevant to the randomized problem are given: for $B_2/B_1 = 0.2$ (case c1), and for $B_2/B_1 = 5$ (case c2). We see that these profiles differ fundamentally in their shapes from those obtained by solving the deterministic problems. It is interesting to note that profiles found by averaging random quantities may appear to be erroneous quantitatively as well as qualitatively. Specifically, large discrepancies with respect to the real situation may arise for relatively small numbers of structures, N. Fig. 7 demonstrates such an example for the intensities $I(x)$ emerging from an atmosphere, in some conventional units. The roughness of the result corresponding to case (d) is striking.

4. The NLTE atmosphere

As it was pointed out in the outset of the paper, the Non-LTE atmosphere differs significantly from that in LTE due to multiple scatterings which establish coupling between various volumes of the medium. Now the averaging can no longer be performed by parts as for LTE, since the mean intensity of radiation emerging from any part of an atmosphere is altered when adding a new layer (element) to it. As in Paper II, we confine ourselves by considering pure scattering (destruction coefficient $\epsilon = 0$), for which the theory is simplified at great extent. We assume again that the time scale of variations in the structure of the atmosphere is much greater than the average time of the photon's travel in it. In determining the statistical characteristics of radiation emerging from an inhomogeneous Non-LTE atmosphere, in Paper II we used Ambartsumian's method of "addition of layers". We obtained explicit expressions for the RelMSD relevant to several particular small numbers, N, of structural elements. With increasing N this method becomes, however, too cumbersome to be applied, which stems from the high non-linearity of equations derived for outgoing intensity (see Eqs. 3 and 4 of Paper II). This disadvantage prevents the complete comparison with the LTE case.

To give more insight into the problem, we shall adopt in this section a new approach based on the concept of the photon escape probability. This enables one to obtain a closed-form analytical expression for the RelMSD for arbitrary N. We introduce the quantity $P(t,\tau)$ defined as the probability that a photon, randomly scattered at the optical depth t of a medium of optical thickness τ, will eventually escape from it through the boundary $t=0$. As is known (see, e.g., Sobolev 1963, Chapt. 6), in the case of conservative scattering,

$$P(t,\tau) = (1+\tau-t)/(2+\tau). \tag{18}$$

Particularly,

$$P(0,\tau) = (1+\tau)/(2+\tau), \ P(\tau,\tau) = 1/(2+\tau), \tag{19}$$

by virtue of which Eq. (18) may be recast as

$$P(t,\tau) = P(0,\tau) - tP(\tau,\tau). \tag{20}$$

If the atmosphere contains energy sources radiating $B = \mathrm{const}$ in all directions, the intensities of outgoing radiation integrated over the line are given by

$$f = 2B\int_0^\tau P(t,\tau)\,\mathrm{d}t = B\tau. \tag{21}$$

Consider now an inhomogeneous scattering atmosphere consisting of N structural elements. As above, the elements will be regarded as layers, the physical properties of which are described by two parameters: the optical thickness τ, and the power, $2B$, released by embedded energy sources. The latter is assumed constant within each individual element. The physical conditions in the layers undergo random variations in such a way that the pair of parameters (B,τ) takes only one of two possible values, (B_i,τ_i) $(i=1,2)$, each occurring with probability p_i. We shall treat the idealized one-dimensional problem of the transfer of radiation through such atmosphere and focus our attention on the statistical characteristics of the emerging radiation, $\langle I_N\rangle$ and δ_N.

Let the atmosphere be composed of k layers of the first kind (i.e., described by parameters B_1,τ_1), and $N-k$ layers of the second kind (with parameters B_2,τ_2). There obviously exist $C_N^k = N!/k!\,(N-k)!$ different configurations of a given composition, each occurring with probability $p_1^k p_2^{N-k}$. For pure scattering all the radiation energy generated in the medium escapes it, so that the total amount of energy radiated by such an atmosphere is $2\,[kf_1+(N-k)\,f_2]$, where $f_i = B_i\tau_i$ have the same physical meaning as the quantities J_i, for the LTE problem discussed above.

For odd N, some configurations are symmetrical about the middle of atmosphere, i.e., the distribution of layers of different kinds, being referenced from each boundary, is the same. The intensities outgoing from each boundary of such atmosphere are equal to $kf_1+(N-k)\,f_2$. For non-symmetrical configurations the intensity $I_{N,k}^{(m)}$ emerging from the boundary $t=0$ of atmosphere may be represented in the form

$$I_{N,k}^{(m)} = kf_1 + (N-k)\,f_2 + \Delta I_{N,k}^{(m)}, \tag{22}$$

where the superscript m is introduced to indicate the dependence on the arrangement of layers of the different types within the atmosphere. The explicit expression for $\Delta I_{N,k}^{(m)}$ found in Appendix A (see Eq. A8) is

$$\Delta I_{N,k}^{(m)} = \tau_1\tau_2 P(T_k)\left[k(N-k) - 2U^{(m)}\right](B_1-B_2), \tag{23}$$

where $P(T_k) \equiv P(T_k,T_k)$; $T_k = k\tau_1 + (N-k)\,\tau_2$ is the optical thickness of the atmosphere, and $U^{(m)}$ are certain integers in the range $[0, k\,(N-k)]$ with clear combinatorial meanings explained in Appendix A. When $U^{(m)} = a \equiv k\,(N-k)/2$, i.e., $\Delta I_{N,k}^{(m)} = 0$, we return to the just-considered case of the symmetrical distribution. The other values of $U^{(m)}$ are distributed in pairs about a in such a way that the quantity $k(N-k)-2U^{(m)}$ takes values of the form $\pm A_m$ (where A_m, as we shall see later, are integers from the interval $[0,\ k\,(N-k)]$) with each of signs corresponding to the intensity emerging from one of two boundaries of the atmosphere with a non-symmetrical distribution of layers of different types. It is evident that for every non-symmetrical configuration there exists a certain companion configuration of the same composition but with an inverted distribution of layers for which $\Delta I_{N,k}^{(m)}$ has the opposite sign.

In calculating the mean intensity $\langle I_N\rangle$, the terms $\Delta I_{N,k}^{(m)}$ disappear to yield

$$\begin{aligned}\langle I_N\rangle &= \sum_{k=0}^{N} p_1^k p_2^{N-k} \sum_{m=1}^{C_N^k} I_{N,k}^{(m)} \\ &= \sum_{k=0}^{N} C_N^k\,[kf_1+(N-k)\,f_2]\,p_1^k p_2^{N-k} \\ &= N\,(p_1f_1+p_2f_2) = N\langle I_1\rangle.\end{aligned} \tag{24}$$

This result has been already used in Paper II, and differs fundamentally from its counterpart equation for LTE (cf. Eq. 4).

In contrast to the mean intensity, it is somewhat more difficult and lengthy to derive a closed-form expression for the RelMSD, since the integers $U^{(m)}$ do not disappear. Even so, the needed summations may be performed, as is shown in Appendix B, to derive

$$\delta_N = \frac{p_1p_2}{N\langle I_1\rangle^2}\left[(f_1-f_2)^2 + \omega_N(\tau_1,\tau_2)\,(B_1-B_2)^2\right], \tag{25}$$

where

$$\begin{aligned}\omega_N(\tau_1,\tau_2) &= \frac{N^2-1}{3}\tau_1^2\tau_2^2 \sum_{k=0}^{N-2} C_{N-2}^k p_1^k p_2^{N-k-2} P^2(T_{k+1}) \\ N &= 2,3,...\end{aligned} \tag{26}$$

Some special forms of Eq. (25) for small values of N were presented in Paper II. The direct calculation shows particularly that $\omega_1 = 0$, and

$$\delta_1 = \frac{p_1 p_2}{\langle I_1 \rangle^2} \left(f_1 - f_2\right)^2 . \tag{27}$$

We see that the RelMSD, as presented in Eq. (25), is a sum, in which only the second item depends on the relative proportion of layers of different types and their arrangement in the atmosphere. It is also seen that, depending on the individual properties of components, either of the two terms in brackets may become dominant. Examining Eq. (26), we observe that with increasing N the quantity ω_N remains bounded from above by the limiting value $(1/3)\,\tau_{\max}^2$ (where $\tau_{\max} = \max\left(\tau_1, \tau_2\right)$). So we obtain an important assertion: *the RelMSD of the intensity of radiation emerging from the multicomponent randomly inhomogeneous NLTE atmosphere decreases with an increase of the number N of components, tending to 0 as $1/N$*. It should be noted that this assertion holds under general assumptions concerning the physical properties of structural elements, which was not the case in LTE.

We see that δ_N depends merely on the ratio of B_1 and B_2. In the special cases, (i)–(iii) discussed in Sect. 2 for LTE, the expression for δ_N simplifies to a great extent.

(i) Let $\tau_1 = \tau_2 = \tau$, then Eq. (25) takes the form

$$\delta_N = \frac{\delta\left(B\right)}{N} \left[1 + \frac{N^2-1}{3}\left(\frac{\tau}{N\tau + 2}\right)^2\right]. \tag{28}$$

Now δ_N is a function of B_2/B_1, p_2/p_1, and τ. For optical thickness τ sufficiently small, the effect of scatterings is negligible, and $\delta_N = \delta\left(B\right)/N$. In this special case there is no difference between LTE and NLTE (cf., Eq. 15) since in both cases all the quanta radiated in the atmosphere escape it. The dependence of δ_N on τ is not important for large N, considering that the term in brackets only varies from unity to $(4/3) - \left(1/3N^2\right)$ as τ changes from 0 to ∞.

(ii) If $B_1 = B_2$, Eq. (25) simplifies to result $\delta_N = \delta\left(\tau\right)/N$, where $\delta\left(\tau\right) = \left(\langle \tau^2 \rangle / \langle \tau \rangle^2\right) - 1$.

(iii) Suppose that all the structural elements radiate equal amounts of energy, i.e., $f_1 = f_2$. Then we have $\delta_N = \omega_N(\tau_1, \tau_2)\delta(B)/N$, and the range of variation in the RelMSD is as much as $\left[\left(N\tau_{\max} + 2\right) / \left(N\tau_{\min} + 2\right)\right]^2$ times greater than that in the case (i).

(iv) It is important for the further discussion concerning the H Ly-α line to consider the special case in which the components of a medium are supposed to be optically thick ($\tau_1, \tau_2 \gg 1$). Letting $p_1 = p_2 = 1/2$, for simplicity, Eqs. (25) and (26) take a much more simple form. Now $P(T_k) = [k\tau_1 + (N-k)\tau_2]^{-1}$, and for relatively large N, from Eq. (26), we find that $(1/3)\,\tau_{\min}^2 < \omega_N < (1/3)\,\tau_{\max}^2$. Excluding from the treatment the less interesting case of the quasi-homogeneous atmosphere when $f_1 \approx f_2$, and $B_1 \approx B_2$, in place of Eq. (25), one may write

$$\delta_N \approx \frac{1}{N}\left[\delta\left(f\right) + \varpi\delta\left(B\right)\right], \tag{29}$$

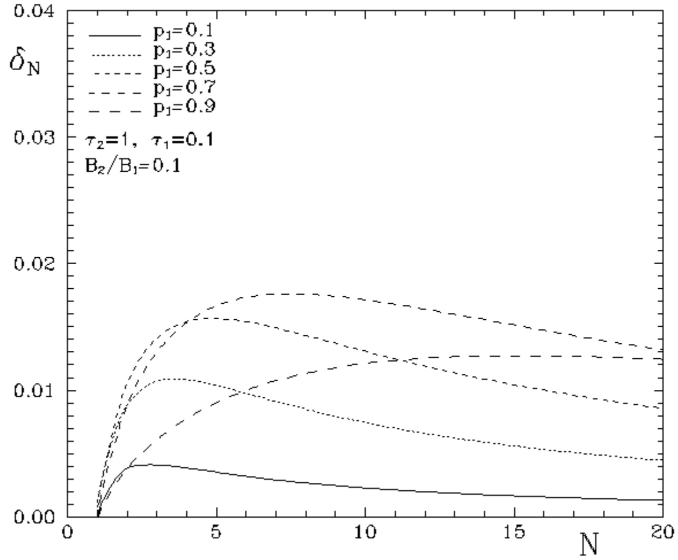

Fig. 8. The function δ_N of N for various p_1 and indicated values of other parameters (analogue of Fig. 2 for the Non-LTE atmosphere).

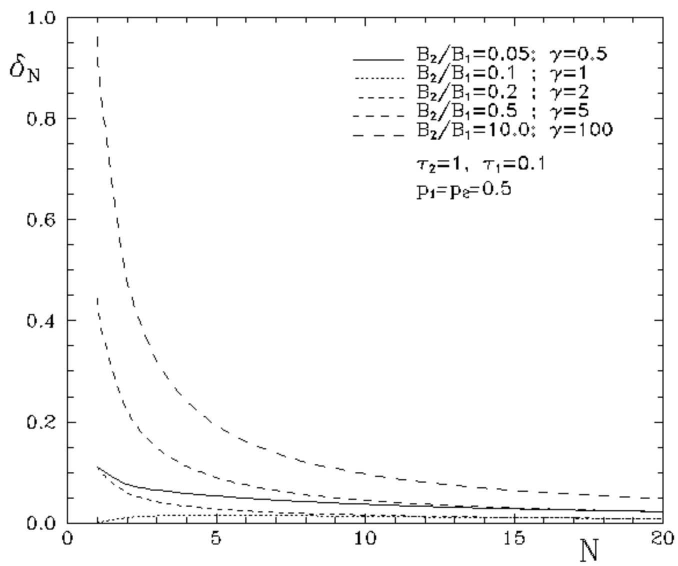

Fig. 9. The relationsip between δ_N and N for various values of the ratio B_2/B_1. In contrast to the LTE atmosphere, the RelMSD is a monotonically decreasing function of N as long as $\gamma\;(= f_2/f_1)$ is not too close to 1 (case $B_2/B_1 = 0.1$).

where ϖ is a certain constant from the interval $[(1/3)(\tau_{\min}/\tau_{\max})^2, (1/3)]$. We shall use this result in Sect. 5 below.

The numerical results based on Eq. (25) for δ_N are given in Figs. 8–10, which are the Non-LTE analogues of Figs. 2, 3b and 4b, respectively. For convenience of comparison, the values of parameters in both cases are chosen the same. The conclusions we draw in comparing the theoretical values of δ_N for the LTE and Non-LTE multicomponent atmospheres may be summarized as follows: in general, the RelMSD for a Non-LTE atmosphere can be less as well as greater than that for one in LTE. The greater δ_N for Non-LTE are observed only for relatively small N when $B_1 > B_2, \tau_1 > \tau_2$ (or $B_1 < B_2, \tau_1 < \tau_2$, in view of the symmetry of the problem at hand). In this case the difference between δ_N for LTE and Non-LTE is only quantitative, so we limit ourselves by plotting in Figs. 8, 9 the graphs that illustrate the case $B_1 > B_2, \tau_1 < \tau_2$, with the only exception being the last curve in Fig. 9. Fig. 8 illustrates the situation when the RelMSD for Non-LTE is much smaller as compared to that for LTE, even for small N. In addition, as N increases, $\delta_N \rightarrow 0$ for the conservatively scattering atmosphere, which is not the case in LTE. This fact is important for any values of parameters involved, and leads to smaller δ_N for Non-LTE as long as the number N of components is sufficiently large (cf. Figs. 10 and 4b). Note also that normally δ_N is a monotonically decreasing function of N (if $\gamma = f_2/f_1$ is not too close to 1 (see Fig. 9)), while it is not the case in LTE for rather wide range of values of γ.

The final comment to be made in connection of the model problem considered in this section is that we did not specify the origin of the initial energy sources. In fact, these sources may be partially due to the external radiation incident on the atmosphere from outside. This is the case when considering the effect of the photospheric radiation incident onto prominence, which is of particular importance for the H Ly-α line. In the case of pure scattering adopted in our model problem, the contribution of the external radiation will be the same for each individual structural element, i.e., B_i will increase by the same value, so that $B_2/B_1 \rightarrow 1$. As it follows from Eq. (25), this, in turn, leads to a decrease of the RelMSD, that is the external radiation tends to smooth the actual fluctuations in brightness. The quantitative estimation of the contribution of the incident intensity is afield of the theory we developed and must be derived from other reasonings as an input parameter.

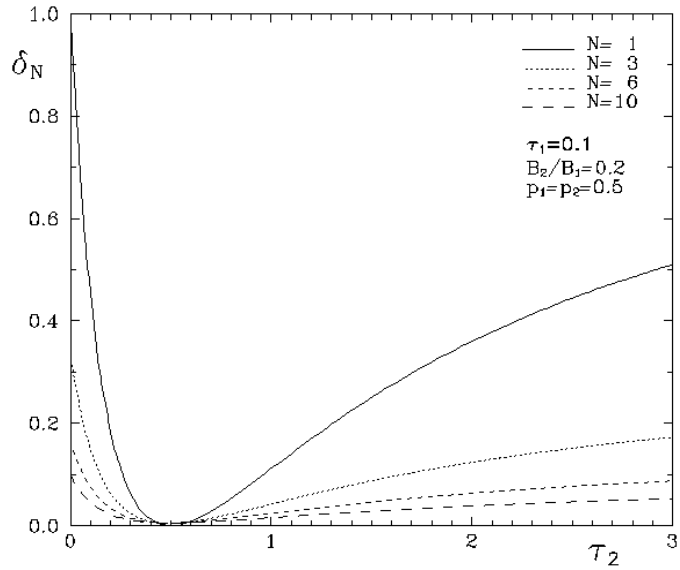

Fig. 10. The same as in Fig. 4b for the Non-LTE atmosphere.

5. Analysis of the observational data

The spatial variations in the EUV line intensities have been analysed in Paper I. We attempted to explain some specific features of these by randomness in the number of threads along a line-of-sight. The predictions of the theory we developed are essentially in agreement with the observational data. It is only natural to surmise that the brightness fluctuations exhibited by prominences are due, at least partially, to physical inhomogeneities in the emitting medium.

In this section we shall avail ourselves of the results obtained above to elucidate the role of this latter effect. The prominences, raster-images of which we use, are listed in Paper I. To begin, we briefly recall several conclusions of Paper I that are needed for the further discussion. The statistical analysis shows that, for a given region, the brightness fluctuations are usually the least for the H Ly-α line. Among different lines formed by a similar mechanism, the more opaque lines are characterized by smaller fluctuations. This relationship particularly applies to the lines of the Lyman series. For the same line the smallest variations are usually observed in the bright regions of a prominence. This correlation is especially well pronounced for the H Ly-α line. We note also that the largest RelMSD are specific to the lines Mg X λ625 Å and O IV λ554 Å, which exhibit strong absorption in the Lyman continuum. Let us examine each of the investigated lines separately.

The line H Ly-α. The optical thickness of prominences in this line is, in general, very large ($\sim 10^4 - 10^5$), and may in some cases reach values up to $\sim 10^7$–10^8 (see e.g., Morojenko 1984, Tandberg-Hanssen 1995). Under such conditions the effects of the multiple scattering in the strong resonance line become critical. For typical values of temperature ($T \sim 10^4$ K), and the electron density ($n_\mathrm{e} \sim 10^{10} - 10^{11}$ cm^{-3}) we find that $n_\mathrm{e}C_{21} \ll A_{21}$ (for the rates of spontaneous and collisional deexcitation the standard notations are used), so that the photon destruction coefficient $\epsilon = n_\mathrm{e}C_{21}/\left(n_\mathrm{e}C_{21} + A_{21}\right)$ is of the order of $10^{-3} - 10^{-4}$. Thus the assumption of the conservative scattering made in this paper is believed to be a fair approximation in determining the integrated intensity of outgoing line-radiation.

The high opacity of prominences in Ly-α suggests that the brightness fluctuations in this line must be smaller on the average than those in the other EUV lines. The RelMSD for the Ly-α line is of the order of 0.02–0.06 for the bright regions, and rises in passing to the fainter regions up to values of order 0.2 and even greater. A typical example of such behaviour is presented below in Table 1. In Paper I, our study of the brightness fluctuations was based on the assumption that they are due only to randomness in the line-of-sight number of threads. Adopting Poisson's law for the variation of N, we inferred that the mean number of threads must be 20–50 in the bright regions of prominences, and of about 3–8, in the faint regions. Making allowance for an additional source of the brightness fluctuations namely, physical inhomogeneities, the aforementioned estimates will be obviously changed so that greater values of N are needed now to explain a given observed value of the RelMSD. We will proceed to discuss this point in more detail.

Suppose that the two particular sources of fluctuation act jointly. Then the standard procedure of averaging over the Poisson distribution being applied to the case (iv) considered in the preceding section yields

$$\delta_{\overline{N}} = \frac{1}{\overline{N}}\left[1 + \delta\left(f\right) + \varpi\delta\left(B\right)\right], \tag{30}$$

where $\overline{N}$ is Poisson's mean of the line-of-sight number of threads. For a homogeneous atmosphere $\delta\left(f\right) = \delta\left(B\right) = 0$, and we obtain $\delta_{\overline{N}} = 1/\overline{N}$, thus returning to the case discussed in Paper I. Eq. (30) shows to what extent the random inhomogeneities in an atmosphere affect the observed value of δ. The actual scale of variations in the power of the internal energy sources and optical thickness of threads is unknown a priori, though it is reasonable to expect that this must be relatively small in the bright regions, and larger for the faint top parts of prominences. A typical example of the latter is region V in Table 1. For the bright regions the spread in f within the order of magnitude leads to $\delta\left(f\right) + \varpi\delta\left(B\right) \approx 0.7$–$0.8$, and in place of the previous estimates (20–50) of $\overline{N}$, we obtain $\overline{N} \sim 35$–90. Similar estimation for the faint and top parts of prominences yields $\overline{N} \sim 5$–14. It must be noted however that the use of Eq. (30) in this latter case is not quite substantiated, as some of the assumptions underlying this formula (e.g., high opacity of threads) may be failed. Other plausible causes of large dispersion in the faint regions are a large spread in the physical characteristics of the threads and a non-uniform probability distribution of inhomogeneities of different scale. We recall that, according to our theory, the presence of a small portion of bright components in an ensemble of weakly radiating threads leads to a sharp increase in the observed value of the RelMSD. As was shown in Paper II, such a composition of differently radiating elements is characterized by a kind of asymmetrical Poisson-like profile for the probability distribution, which is actually observed. These arguments suggest that the above estimates for the faint parts of prominences may be considered as the lower limit to the real values of $\overline{N}$.

It is of particular interest to compare our results with those derived by others. Schmahl and Orrall (1979) inferred by analysing the Lyman continuum absorption that there must be at least 4–10 cool threads along a typical line-of-sight. Fontenla and Rovira (1985) have constructed Non-LTE models of prominence threads and computed profiles and absolute intensities of the lines of the Lyman series and the Lyman continuum from an ensemble of threads. They found that the minimum number of threads along a line-of-sight ranges from 10 to 100. We see that their estimates agree with those stated in this paper.

It is easily seen that the treated problem may be inverted in the sense that having reliable estimates for the number of threads, one can evaluate the scale of variations in the local physical characteristics of structural elements of prominences.

*The lines*C II λ1336 Å, C III λ 977 Å, O VI λ1032 Å. These lines are formed in the prominence-corona interface (PCI) at $3 \cdot 10^4$ K$\leq T \leq 3 \cdot 10^5$ K, and are thought to be optically thin (with the only exception perhaps the line C II λ1336 Å) thus providing for a wealth of information on prominence structure. However, the observational data and the proper theory for these lines are characterized now by a wide variety of features, which offer difficulties in attempting to interpret unambiguously the information obtained. We shall consider here some properties of the brightness variations which are common, to a degree, for different prominences. Typical values of the spatially averaged intensity and RelMSD for the PCI-lines and H Ly-α are presented in Table 1. The data refer to a prominence with a clear-cut height-variation. Sequences I, III and V, and likewise II, IV and V, indicate progressively higher regions above the limb. Each of these regions encompasses 2–3 rows of pixels. In spite of the difference in brightness, regions I and II (as well as III and IV), refer on the average to the same height (the data concerning the line O VI λ1032 Å are tabulated after subtracting the background from the corona).

Table 1 shows that the intensities of all the lines decrease on the average with height. Some deviations from this relationship, which is usually the case for relatively non-extended and inhomogeneous prominences, may result from a scarcity of the volume of statistics. Comparing the regions I and V, for instance, we find the largest drop in $\langle I\rangle$ for Ly-α (by a factor 10.2), while in the lines C II λ1336 Å, C III λ977 Å, and O VI λ 1032 Å the mean intensity decreases only by factors of 6.5, 3.6, and 2.9 times, respectively. It is notable that these factors decrease with an increase in the temperature of line formation. The data we present may be used to estimate the relative dimensions of regions radiating in a given PCI-line, which ultimately allows to get a rough picture of the temperature gradient in the prominence-corona transition zone. By assuming that the second energetic level of an atom is excited predominantly by electron collisions from the ground state, we may express the line intensity by (Dupree 1972)

$$I_\nu = h\nu N_{\rm el}\int R_{\rm i} n_{\rm e} C_{12} {\rm d}s, \tag{31}$$

where $N_{\rm el}$ is the total density of a given element, $R_{\rm i}$ is the fractional ion concentration. The integration is performed along a line-of-sight over the path where a given spectral line is formed. Being written for the optically thin resonance lines, Eq. (31) remains valid in the conservative case for the opaque lines (like H Ly-α) in so far as all the radiation energy released in the medium escapes it. Under the condition of pressure equilibrium, $n_{\rm e}^{\rm (pr)}kT^{\rm (pr)} = n_{\rm e}^{\rm (cor)}kT^{\rm (cor)}$, the usual procedure of evaluating the integral in Eq. (31) (see e.g., Pottasch 1964, Dupree 1972) yields for the relative sizes of zones radiating in the proper lines: Δs(O VI λ1032 Å) : Δs (C III λ977 Å) : Δs(C II λ1336 Å) : Δs(H Ly-α) $= 3.04 : 0.33 : 1.31 : 1$ for the bright region I, and $10.7 : 0.96 : 2.09 : 1$, for the top region V. The relatively large range of the O VI line formation zone deserves attention but might be overestimated due to difficulties in correcting the coronal background.

Some interesting relationships are observed when treating the behaviour of the brightness fluctuations. As has been already mentioned, there exists a pronounced correlation between the mean intensity of a region and the RelMSD for the H Ly-α line: the brighter the region under consideration, the smaller

Table 1. Typical values for the mean intensity (erg cm^{-2} sec^{-1}st^{-1}) and the RelMSD for different regions of a prominence.

lines		I	II	III	IV	V
Ly-α	$\langle I\rangle$	47526	33952	24513	13055	4647
Ly-α	δ	0.017	0.036	0.078	0.110	0.330
C II 1336 Å	$\langle I\rangle$	318.2	220.0	221.4	145.2	49.87
C II 1336 Å	δ	0.057	0.075	0.137	0.216	0.562
C III 977 Å	$\langle I\rangle$	452.3	274.1	362.5	347.2	127.4
C III 977 Å	δ	0.051	0.077	0.060	0.049	0.331
O VI 1032 Å	$\langle I\rangle$	291.1	200.6	164.8	136.0	100.3
O VI 1032 Å	δ	0.096	0.064	0.150	0.041	0.191

the value of δ. This correlation is valid for all prominences. Table 1 shows a similar correlation between $\langle I\rangle$ and δ for the C II λ1336 Å line, while for the other lines this effect is less pronounced. On the other hand, the variations in $\langle I\rangle$ and δ for the line C II λ1336 Å are definitely correlated with those for Ly-α. It is noteworthy that this effect does not depend on whether or not the changes are due to the difference in height. These two lines have similar behaviour in many respects which may be explained by strongly correlated variations in the emission measure. The similarity between variations in δ suggests that these variations may be a result (at least partially) of changes in the line-of-sight number of threads, provided that the parts of the PCI radiating in the line C II λ1336 Å are present around each thread. This proposition is supported by the possibility of rendering a good fit to the observed values of δ for the line C II λ1336 Å by taking the same numbers of threads along a line-of-sight as those for the line H Ly-α. Indeed, let the number N of structural elements to be a random quantity distributed according to the Poisson law. Averaging expressions for $\langle I_N\rangle$, $\langle I_N^2\rangle$ (Eqs. 4 and 8) over this law yields

$$\delta_{\overline{N}} = \frac{M_{\overline{N}}}{L_{\overline{N}}^2}\,(1+\delta_1) + 2\frac{KA_{\overline{N}}}{\langle I_1\rangle L_{\overline{N}}^2} - 1, \tag{32}$$

where $L_{\overline{N}} = [1-\exp\left(-\left(1-\alpha\right)N\right)]\,/\,(1-\alpha)$, $M_{\overline{N}} = [1-\exp(-(1-\beta)N)]/(1-\beta)$, $A_{\overline{N}} = (L_{\overline{N}} - M_{\overline{N}})/(\alpha-\beta)$.

Now taking, for instance, $p_1 = 0.8, p_2 = 0.2$ (again, we suppose the presence of a small portion of relatively bright elements in the radiating gas), $\tau_1 = 0.005$, $\tau_2 = 0.01$ ($\langle\tau\rangle = 0.006$), and $(B_2/B_1) = 20$, we find from Eq. (32) that $\delta(\overline{N}=7) = 0.58$, and $\delta\left(\overline{N}=70\right) = 0.051$. This allows one a rough idea of the total optical thickness τ_0 in the line C II λ1336 Å: $\tau_0 = 0.04$ for faint and rarified regions, and $\tau_0 = 0.42$, for bright and dense regions. Variations in B_2/B_1 and τ_i within reasonable limits lead only to minor changes in the estimated values of $\overline{N}$.

To proceed to the lines O VI λ1032 Å and C III λ977 Å, notice first that for an optically thin atmosphere when $(1-\alpha)N \ll 1$, Eq. (32) simplifies to

$$\delta_{\overline{N}} = \frac{1}{\overline{N}}\left[1+\delta\left(f\right)\right], \tag{33}$$

where we took into account that $L_{\overline{N}} = M_{\overline{N}} = \overline{N}$, $A_{\overline{N}} = (1/2)\,\overline{N}^2$, $K = \langle I_1\rangle = \langle f\rangle$.

The correlation between $\langle I\rangle$ and $\delta_{\overline{N}}$ for the mentioned lines is not so pronounced as in the case of H Ly-α and C II λ1336 Å. If we start from the assumption that the lines O VI λ1032 Å and C III λ977 Å are optically thin, then we can use Eq. (33) to evaluate the number $\overline{N}$. The consistency with the results obtained above will be attained by admitting that inhomogeneities of various amplitudes are equally probable in the radiating area. If this is not the case, Eq. (33) leads, under otherwise identical conditions, to larger δ, which is not actually observed. The most probable explanation for this discrepancy is that the PCI regions radiating in these lines are present mostly around the bulk of threads. This concerns primarily to the line O VI λ 1032 Å. The line C III λ977 Å occupies, in all connections, a somewhat intermediate place between C II λ 1336 Å and O VI λ1032 Å.

*The lines*O IV λ554 Å *and* Mg X λ625 Å. Additional information on the structural pattern may be obtained by studying the brightness fluctuations of prominences in these lines, which as it has been said, are essentially larger than those for other lines. In some cases the RelMSD for the O IV and Mg X lines takes values of the order of 0.4–0.6 and higher. This is easy to understand remembering that these lines are formed in the PC transition sheaths and corona, whereas, laying inside the Lyman continuum ($\lambda < 912$ Å), they undergo absorption in a different medium namely, in the cool central parts of threads. Hence the observed fluctuations in brightness result from inhomogeneities in these two media.

Let I_0 be the intensity of the line radiation incident from the region of its formation behind the prominence and I be the observed intensity attenuated in the cores of threads. Then one can write $I = I_0 Q$, where $Q \equiv \exp\left(-\tau_0\right)$ is the opacity, and τ_0 is the total optical thickness of the absorbing matter along a line-of-sight. All the three quantities introduced are obviously random. Since the emission and absorption of the line radiation occur in the different media, it is reasonable to suppose that these two processes (thereby the quantities I_0 and Q) are statistically independent. This allows us to express the RelMSD, $\delta\left(I\right)$, of the observed intensity in terms of similar quantities for I_0 and Q. Indeed, it is easily seen that

$$\begin{aligned} 1+\delta\left(I\right) &= \frac{\langle (I_0 Q)^2\rangle}{\langle I_0 Q\rangle^2} = \frac{\langle I_0^2\rangle\langle Q^2\rangle}{\langle I_0\rangle^2\langle Q\rangle^2} \\ &= \left[1+\delta\left(I_0\right)\right]\left[1+\delta\left(Q\right)\right], \end{aligned} \tag{34}$$

and consequently

$$\delta(I) = \delta(I_0) + \delta(Q) + \delta(I_0)\,\delta(Q). \tag{35}$$

Knowledge of $\delta(I)$ and $\delta(I_0)$ appearing in Eq. (35) allows us to infer the values of $\delta(Q)$. Let us go further into this point by considering, for instance, the line O IV λ554 Å. Since the temperature of formation of this line ($\log T = 5.3$) is close to that for the lines O VI λ1032 Å ($\log T = 5.5$) and C III λ977 Å ($\log T = 4.95$), it is reasonable expect that $\delta(I_0)$ for the O IV line does not differ strongly from the observed values of $\delta(I)$ for the mentioned lines. The typical values of $\delta(I)$ for the lines O VI λ1032 Å and C III λ977 Å are ranged from 0.01 to 0.1 (see Paper I). The similar values for the line O IV λ 554 Å vary within fairly wide limits between 0.1 and 0.6. In any case, estimates of $\delta(Q)$ based on Eq. (35), ~ 0.1–0.2 for the bright regions and ~ 0.3–0.4 for the faint regions, may be considered realistic.

Adopting the multithread model, we can use this result to infer the individual properties of structural elements. For simplicity, let us suppose, as above, that the absorbing matter consists of only two types of threads characterized by the optical thickness τ_i and opacity $q_i = \exp(-\tau_i)$ $(i = 1, 2)$, each occurring with the probability p_i. The total number N of the encountered threads will be assumed at this stage to be fixed, while the numbers k and $N - k$ of each kind of threads are allowed to be random. Then it is seen that

$$\langle\tau_0\rangle = \sum_{k=0}^{N} C_N^k p_1^k p_2^{N-k}\left[k\tau_1 + (N-k)\,\tau_2\right] = N\langle\tau\rangle, \tag{36}$$

$$\langle Q\rangle = \sum_{k=0}^{N} C_N^k \left(p_1 q_1\right)^k \left(p_2 q_2\right)^{N-k} = \langle q\rangle^N, \tag{37}$$

where $\langle\tau\rangle$ and $\langle q\rangle$ are the mean values of the optical thickness and the opacity of a single thread.

Similarly one may obtain

$$\delta(\tau_0) \equiv \delta_N(\tau) = \delta_1(\tau)/N, \tag{38}$$

$$\delta(Q) \equiv \delta_N(q) = \left[1 + \delta_1(q)\right]^N - 1, \tag{39}$$

where $\delta_1(\tau) = p_1 p_2 (\tau_1 - \tau_2)^2 / \langle\tau\rangle^2$ and $\delta_1(q) = p_1 p_2 (q_1 - q_2)^2 / \langle q\rangle^2$.

It is to be noted that although Eqs. (36)–(39) are derived by assuming the presence of only two types of structural elements $(n = 2)$, the final results remain valid for an arbitrary number, n, of element types. In this latter case the probability of a certain configuration of threads is subject to the polynomial law. Finally, if N is allowed to be random, being distributed according to Poisson's law Eqs. (36)–(39) yield

$$\langle\tau_0\rangle = \overline{N}\langle\tau\rangle, \tag{40}$$

$$\langle Q\rangle = \exp\left(-\overline{N}\left(1 - \langle q\rangle\right)\right), \tag{41}$$

$$\delta_{\overline{N}}(\tau) = \left[1 + \delta_1(\tau)\right]/\overline{N}, \tag{42}$$

$$\delta_{\overline{N}}(q) = \exp\left(\overline{N}\langle(1-q)^2\rangle\right) - 1. \tag{43}$$

Eqs. (36)–(43) show that, for fixed physical parameters of elements, the RelMSD of the total optical thickness decreases with N (or $\overline{N}$) whereas the quantities $\delta_N(q)$ and $\delta_{\overline{N}}(q)$ increase (cf. Eqs. 38, 42 and 39, 43). This effect is in agreement with the observed increase of the normalized fluctuations in the brightness of the O IV λ554 Å line when passing from bright regions to the faint ones. The compliance of the results predicted by Eqs. (36)–(43) with those obtained above in considering other lines may be checked by taking the obtained estimates of N to solve Eqs. (40)–(43) with respect to τ_i $(i = 1, 2)$. Considering, for instance, the faint regions and letting $\overline{N} = 6$–8 for $\langle\tau_0\rangle = 1$, and $\delta_{\overline{N}}(q) = 0.3$–$0.4$, we find that $\tau_1 \approx 0.01$ and $\tau_2 \approx 0.6$–0.9 if $p_1 = 0.8$ and $p_2 = 0.2$. The dispersion in τ agrees with that used above in treating the line Ly-α. The estimate τ_0(O IV λ554 Å) > 5.2 obtained by Schmahl et al. (see Schmahl et al. 1974) apparently applies to the dense, bright region of the prominence. In principle, agreement with observational values of the RelMSD can be accomplished in this case by taking $\overline{N} \sim 70$–100. However, the dispersion in τ that results is unrealistically small. This can be explained by possible deviations from the Poisson law or by relinquishing certain assumptions underlying our model problem.

6. Concluding remarks

The model problems we have considered for addressing the statistical characteristics of radiation formed in stochastic atmospheres provide considerable insight into the structure of prominences. In two important cases, the transparent LTE atmosphere and the opaque Non-LTE atmosphere, the theory predicts that $\langle I\rangle \sim \overline{N}$, and $\delta_{\overline{N}} \sim 1/\overline{N}$, so that $\delta_{\overline{N}} \sim 1/\langle I\rangle$. The fact that such a relationship is observed may be regarded as an evidence that the bright and faint regions of prominences differ with each other primarily by the mean number of constituent threads along a line-of-sight. This effect would not be observed if the contrasts were the result of variations in the optical thicknesses of individual threads. Being applied to the lines H Ly-α and C II λ 1336 Å, the theory leads to values of $\overline{N}$ consistent with those found by the other authors. We believe that these estimates are fairly close to the actual values of $\overline{N}$, though consideration of any new source of fluctuations may alter the theoretical results presented. An example of such sources is the filling factor, which is also a random quantity capable of affecting the amplitudes of fluctuations.

The present theory may be generalized in various directions, the most important of which are consideration of the continuous analogues of the structure types as well as the non-conservative version of the Non-LTE problem and the case in which random variations of inhomogeneities are intercorrelated. This will be the subject of our future investigations.

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Appendix A: the derivation of Eq. (23)

For fixed N and k, and an arbitrary arrangement of layers (marked by the superscript m), the outgoing intensity may be expressed in terms of $P\left(t,T_k\right)$ as follows

$$I_{N,k}^{(m)} = 2B_1 \sum_{j=1}^{k} \int_{t_j^{(m)}}^{\bar{t}_j^{(m)}} P\left(t,T_k\right) \mathrm{d}t + 2B_2 \sum_{j=1}^{N-k} \int_{t_j^{\prime(m)}}^{\bar{t}_j^{\prime(m)}} P\left(t,T_k\right) \mathrm{d}t, \tag{A1}$$

wherein $T_k = k\tau_1 + (N-k)\,\tau_2$ is the optical thickness of the atmosphere. The limits of integrations represent the optical depths that correspond to boundary planes of a given layer. These are random quantities and depend on the realized particular ordering of components in the atmosphere. It should be seen that $\bar{t}_j^{(m)} - t_j^{(m)} = \tau_1$, and $\bar{t}_j^{\prime(m)} - t_j^{\prime(m)} = \tau_2$.

By making use of Eq. (20), the integrals in Eq. (A1) are evaluated immediately to give

$$\int_{t_j^{(m)}}^{\bar{t}_j^{(m)}} P\left(t,T_k\right) \mathrm{d}t = \int_{t_j^{(m)}}^{\bar{t}_j^{(m)}} \left[P\left(0,T_k\right) - tP\left(T_k,T_k\right)\right] \mathrm{d}t = \tau_1 \left[1 - P\left(T_k,T_k\right)\left(1+\zeta_j^{(m)}\right)\right],$$

$$\int_{t_j^{\prime(m)}}^{\bar{t}_j^{\prime(m)}} P\left(t,T_k\right) \mathrm{d}t = \tau_2 \left[1 - P\left(T_k,T_k\right)\left(1+\zeta_j^{\prime(m)}\right)\right], \tag{A2}$$

where $\zeta_j^{(m)} = \left(\bar{t}_j^{(m)} + t_j^{(m)}\right)/2$, $\zeta_j^{\prime(m)} = \left(\bar{t}_j^{\prime(m)} + t_j^{\prime(m)}\right)/2$ are the optical depths which correspond to the middles of the jth layer (counting from the boundary $t=0$ of the atmosphere) of the first and second type, respectively. Now Eq. (A1) yields

$$I_{N,k}^{(m)} = 2kf_1 + 2\,(N-k)\,f_2 - 2P\left(T_k\right) f_1 \left(k + \sum_{j=1}^{k} \zeta_j^{(m)}\right) - 2P\left(T_k\right) f_2 \left(N-k + \sum_{j=1}^{N-k} \zeta_j^{\prime(m)}\right), \tag{A3}$$

where for brevity one of the arguments in $P\left(T_k,T_k\right)$ is omitted. Taking into account the second of Eqs. (19) one may write

$$I_{N,k}^{(m)} = kf_1 + (N-k)\,f_2 + P\left(T_k\right) f_1 \left(kT_k - 2\sum_{j=1}^{k} \zeta_j^{(m)}\right) + P\left(T_k\right) f_2 \left((N-k)\,T_k - 2\sum_{j=1}^{N-k} \zeta_j^{\prime(m)}\right). \tag{A4}$$

The coordinates $\zeta_j^{(m)}$ and $\zeta_j^{\prime(m)}$ can be represented in the form

$$\zeta_j^{(m)} = \left(j - \frac{1}{2}\right)\tau_1 + u_j^{(m)}\tau_2,$$

$$\zeta_j^{\prime(m)} = \left(j - \frac{1}{2}\right)\tau_2 + v_j^{(m)}\tau_1 \tag{A5}$$

wherein $u_j^{(m)}$ and $v_j^{(m)}$ are the numbers of the components of the opposite type preceding a given layer. Thus we have

$$2\sum_{j=1}^{k} \zeta_j^{(m)} = k^2\tau_1 + 2U^{(m)}\tau_2,$$

$$2\sum_{j=1}^{N-k} \zeta_j^{\prime(m)} = (N-k)^2\,\tau_2 + 2V^{(m)}\tau_1 \tag{A6}$$

with the quantities

$$U^{(m)} = \sum_{j=1}^{k} u_j^{(m)},\, V^{(m)} = \sum_{j=1}^{N-k} v_j^{(m)} \tag{A7}$$

representing the total numbers of layers of the opposite type preceding all the layers of a given type. These numbers admit an alternative interpretation that is convenient for further discussion. This rests on the concept of 'transposition' widely used in the combinatorial analysis. By transposition we mean here any exchange, or swap, of two adjacent layers of different kinds. Let us agree to call 'direct' the arrangement consisting of two series such as all the layers of the first structure type precede those of the second type. The opposite arrangement, obtained by a simple inversion of two series, will be referred to as the 'inverted' order. It is easily seen that, for a certain random realization, the quantity $U^{(m)}$ is nothing more than the total number of transpositions needed to establish the direct ordering of layers, while $V^{(m)}$ is the total number of transpositions performed in obtaining the inverted order. On the other hand, the number of transpositions transforming the direct order into the inverted one is $k\,(N-k)$ so that we may write

$$U^{(m)} + V^{(m)} = k\,(N-k)\,. \tag{A8}$$

Now utilizing Eqs. (A6), (A8) in Eq. (A4), we find

$$I_{N,k}^{(m)} = kf_1 + (N-k)\,f_2 + \tau_1\tau_2 P\left(T_k\right)\left[k\,(N-k) - 2U^{(m)}\right](B_1 - B_2)\,. \tag{A9}$$

Thus, we obtained the requisite explicit expression for $\Delta I_{N,k}^{(m)}$ (cf., Eq. 22) that allows us to elucidate the characteristic features of the emerging intensity. The layers of the same sort are obviously indistinguishable, so that for fixed N and k there exist $C_N^k = N!/k!\,(N-k)!$ different configurations. The values of $\Delta I_{N,k}^{(m)}$ (and then of $I_{N,k}^{(m)}$) are determined completely by the discrete quantity $U^{(m)}$ (or $V^{(m)}$), and obey the Fermi-Dirac statistics (see, e.g., Feller 1957). It follows from Eq. (A8), that the values taken by $U^{(m)}$ must lie between 0 and $k\,(N-k)$. The configurations with $U^{(m)} = a \equiv k\,(N-k)/2$ (i.e., $\Delta I_{N,k}^{(m)} = 0$)

are symmetrical. The other values of $U^{(m)}$ are grouped in pairs in such a way that the quantity $k\left(N-k\right)-2U^{(m)}$ is of the form $\pm A_m$,where A_m are integers from the interval $[0,k\left(N-k\right)]$. These correspond to the non-symmetrical configurations which are grouped in pairs (see Sect. 4).

Appendix B: the RelMSD δ_N for the NLTE athmosphere

To proceed to the derivation of an explicit expression for δ_N, we employ Eqs. (24) and (A1) to write

$$\delta_N = \frac{1}{N^2\langle I_1\rangle^2}\left[\sum_{k=0}^{N} p_1^k p_2^{N-k}\sum_{m=1}^{C_N^k}\left(I_{N,k}^{(m)}\right)^2 - N^2\langle I_1\rangle^2\right]. \tag{B1}$$

By virtue of Eq. (A8) the internal sum in the right-hand side of Eq. (B1) takes the form

$$\sum_{m=1}^{C_N^k}\left(I_{N,k}^{(m)}\right)^2 = C_N^k\left\{\left[kf_1+\left(N-k\right)f_2\right]^2+\Omega_{N,k}\right\}, \tag{B2}$$

where

$$\Omega_{N,k} = 4\left(\tau_1\tau_2\right)^2 P^2\left(T_k\right)\left(B_1-B_2\right)^2 \times\left[\frac{1}{C_N^k}\sum_{m=1}^{C_N^k}\left(a-U^{(m)}\right)^2\right]. \tag{B3}$$

The bracketed term in Eq. (B3) may be rewritten as follows

$$\frac{1}{C_N^k}\sum_{m=1}^{C_N^k}\left(a-U^{(m)}\right)^2 = \sum_{j=0}^{a}\varphi_j\left(a-j\right)^2, \tag{B4}$$

wherein the coefficients $\varphi_j \le 1$ have a simple probabilistic meaning: φ_j is the probability that the particular distribution of layers may be converted into the direct distribution as a result of exactly j transpositions. The rigorous evaluation of the sum Eq. (B4) is of interest from the point of view of combinatorial analysis and may become the subject of a separate treatment. Nevertheless, the direct calculation of this sum for successively large values of N allows us to establish that

$$\frac{1}{a}\sum_{j=0}^{a}\varphi_j\left(a-j\right)^2 = \frac{N+1}{12}. \tag{B5}$$

Thus, in place of Eq. (B3) we find that

$$\Omega_{N,k} = \frac{N+1}{3}a\left(\tau_1\tau_2\right)^2 P^2\left(T_k\right)\left(B_1-B_2\right)^2. \tag{B6}$$

Substituting Eq. (B2) into Eq. (B1), we shall use the following easily checked identity

$$\sum_{k=0}^{N} C_N^k p_1^k p_2^{N-k}\left[kf_1+\left(N-k\right)f_2\right]^2 - N^2\left(p_1f_1+p_2f_2\right)^2 = Np_1p_2\left(f_1-f_2\right)^2 \tag{B7}$$

to obtain

$$\delta_N = \frac{p_1p_2}{N\langle I_1\rangle^2}\left[\left(f_1-f_2\right)^2+\omega_N\left(\tau_1,\tau_2\right)\left(B_1-B_2\right)^2\right], \tag{B8}$$

where

$$\omega_N\left(\tau_1,\tau_2\right) = \frac{1}{N}\sum_{k=1}^{N}C_N^k p_1^{k-1}p_2^{N-k-1}\Omega_{N,k} = \frac{N^2-1}{3}\tau_1^2\tau_2^2\sum_{k=0}^{N-2}C_{N-2}^k p_1^k p_2^{N-k-2}P^2\left(T_{k+1}\right). \tag{B9}$$

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