Astron. Astrophys. 342, 863–866 (1999)

ASTRONOMY AND ASTROPHYSICS

Research Note

Leaky and resonantly damped flux tube modes reconsidered

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Received 20 April 1998 / Accepted 3 November 1998

Abstract. In this research note the results for the eigenfrequencies of the uniform and non-uniform magnetic flux tubes of Stenuit et al. (1998) are reconsidered. In that paper it is shown that the eigenfrequencies may have a damping rate due to two mechanisms causing a loss of energy. In non-uniform flux tubes the eigenmodes can be damped by resonant absorption. The other mechanism is leakage of wave energy into the surroundings, which can occur for both uniform and non-uniform flux tubes.

We point out that the dispersion relations obtained by Stenuit et al. are correct for leaky and undamped non-leaky modes, but are not correct for resonantly damped non-leaky modes.

Key words: Magnetohydrodynamics (MHD) – Sun: oscillations – Sun: photosphere – Sun: sunspots

1. Introduction

The linear MHD spectrum of a system consisting of a non-uniform magnetic flux tube and its infinite non-magnetic surroundings was studied by Stenuit, Keppens & Goossens (1998, hereafter SKG).

SKG allowed for complex eigenfrequencies. For a uniform tube complex frequencies are related to leaky modes. Leaky modes are characterized by an external solution that carries energy away from the tube. Due to this acoustic wave leakage in the external medium these modes are damped. For inhomogeneous tubes the wave motions can also be damped by resonant absorption. Global oscillations of the sytem may now resonantly couple to Alfvén and/or slow continuum modes at certain magnetic surfaces. These resonances act as a sink of energy. Again the solutions are damped and the frequencies are complex.

In SKG both eigenfrequencies and optimal driving frequencies are determined for uniform and non-uniform flux tubes. The impedance matching conditions between the different wave fields inside and outside the tube at the boundary of the tube resulted in one dispersion relation for the eigenfrequenies and one for the optimal driving frequencies. Eigenfrequencies were selected by the requirement that there is no incoming wave field. The spectrum of optimal driving frequencies is obtained by imposing the condition that there is no outgoing wave field.

At first sight this selection of eigenfrequencies and optimal driving frequencies looks obvious. However, in this note we demonstrate that the choice of the external boundary condition applied in SKG is incorrect when we look for modes with an oscillation frequency below the external cut-off frequency.

2. Mathematical approach

We refer to SKG for the details of the mathematical approach for solving the eigenvalue problem. Since the equilibrium quantities are only depending on the radial coordinate, we can Fourier analyze with respect to θ and z. Hence all perturbed quantities are assumed to be proportional to

$$\exp[i(k_z z + m\theta - \omega t)],$$

where m and k_z are the azimuthal and the axial wave number respectively and ω denotes the frequency. For a 1D cylindrical plasma the linearized ideal MHD equations can be reduced to a set of two first-order differential equations for the radial component of the Lagrangian displacement ξ_r and the total pressure perturbation P_1:

$$D\frac{d(r\xi_r)}{dr} = C_1 r\xi_r - C_2 r P_1, \quad (1)$$

$$D\frac{dP_1}{dr} = C_3\xi_r - C_1 P_1, \quad (2)$$

where $D = \rho(c^2 + v_A^2)(\omega^2 - \omega_C^2)(\omega^2 - \omega_A^2)$. The sound speed and the Alfvén speed are defined as $c^2 = (\gamma p)/\rho$ and $v_A^2 = B^2/\rho$ where the ratio of specific heats $\gamma = 5/3$, as usual. ω_A and ω_C denote the Alfvén and cusp frequency respectively.

As can be seen from the expression of the coefficient D, the ideal MHD equations are singular at the positions r_A or r_C where $\omega_A(r_A) = \omega$ or $\omega_C(r_C) = \omega$.

As explained in SKG, the combination of the numerical integration of the ideal MHD equations away from any singularity and the jump conditions to cross the singularity leads to the internal solutions for ξ_r and P_1 at the boundary of the tube and thus to the transmitted impedance, needed to match the internal to the external solution.

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The external region is uniform and non-magnetic. Thus the waves in the surroundings are accurately described by the ideal HD equations (1–2) with constant coefficients. The solutions to these equations can be written in terms of Hankel functions as:

$$P_{1e} \;=\; \alpha_1 H_m^{(1)}(k_{\perp e} r) + \alpha_2 H_m^{(2)}(k_{\perp e} r), \tag{3}$$

$$\xi_{re} \;=\; \frac{k_{\perp e}}{\rho_{0e}\omega^2}[\alpha_1 H_m^{(1)'}(k_{\perp e} r) + \alpha_2 H_m^{(2)'}(k_{\perp e} r)], \tag{4}$$

where P_{1e} and ξ_{re} denote the Eulerian perturbation of the total pressure and the radial component of the Lagrangian displacement in the external region. ρ_{0e} is the density outside the flux tube, and $k_{\perp e}$ is the radial external wavenumber. $H_m^{(1)}$ and $H_m^{(2)}$ are the Hankel functions of the first and second kind respectively, and the prime on these symbols denotes the derivative with respect to their argument.

3. The physically relevant solution

The argument of the Hankel functions contains the external radial wavenumber $k_{\perp e}$. The prescription to obtain $k_{\perp e}$ is:

$$k_{\perp e} = \sqrt{\frac{\omega^2}{c_e^2} - k_z^2} = \sqrt{(\frac{\omega_r^2 - \omega_i^2}{c_e^2} - k_z^2) + \frac{2i\omega_r\omega_i}{c_e^2}}. \tag{5}$$

Since we allow for complex frequencies, this external radial wavenumber may be complex. To remove the double-valuedness of the complex root, SKG chose the signs for $Re(k_{\perp e})$ and $Im(k_{\perp e})$ so that $Re[\omega k_{\perp e}^*] \geq 0$, where the asterisk denotes the complex conjugate.

Since we are dealing with stable modes where $Im(\omega) \leq 0$, we are able to choose the root such that:

$$-\frac{\pi}{2} \leq arg(k_{\perp e}) \leq 0. \tag{6}$$

This choice implies that $Re(k_{\perp e}) \geq 0$ and $Im(k_{\perp e}) \leq 0$. This is the same root as was taken in SKG, except for the case where ω is real. Then SKG chose the root where $Im(k_{\perp e}) \geq 0$. Once the root for $k_{\perp e}$ is chosen, we can look for the physically relevant external solution for the different kind of modes.

A mode is leaky when its eigenfunction in the external medium corresponds to a propagating wave, carrying energy away from the tube. This means that its oscillation frequency has to lie above the external cut-off frequency. A mode is non-leaky when its oscillation frequency is lying below the external cut-off frequency so that the corresponding wave field cannot propagate within the external medium. In combination with the fact that a mode can be damped (or not) by resonant absorption, this leads to four different types of eigenmodes as far as their damping is concerned. These four types are: (i) NL & NR, (ii) L & NR, (iii) NL & R and (iv) L & R. Here L and R denote leaky and resonantly damped respectively, and N(=non) denotes that the effect is absent. We can easily demonstrate where SKG went wrong by considering the transition from NL & NR to NL & R modes. Therefore let us first consider a body mode (a non-leaky mode) in a uniform tube with its oscillation frequency lying within the interval determined by the external and internal Alfvén frequency, but out of the interval determined by the external and internal cusp frequency. Since the tube is uniform, there is no Alfvén continuum and hence no resonant coupling possible. And, since the external solution of a body mode is non-propagating, there is no damping mechanism, so that the eigenfrequency must be purely real. Thus $k_{\perp e}^2 = \frac{\omega^2}{c_e^2} - k_z^2 \leq 0$. Now we take the root as prescribed by (6). In order to determine the physically relevant external solution we look at the asymptotic behaviour of $H_m^{(1)}$ and $H_m^{(2)}$ as $r \to \infty$, which is as follows

$$H_m^{(1)} \;\sim\; \sqrt{\frac{2}{\pi k_{\perp e} r}} e^{ik_{\perp e} r},$$

$$H_m^{(2)} \;\sim\; \sqrt{\frac{2}{\pi k_{\perp e} r}} e^{-ik_{\perp e} r}.$$

Hence the external solution must be described by the second Hankel function, since the evanescent solution is the only physical one. Note that the sign of the root is the opposite as the one taken in SKG. We also have chosen the other kind of Hankel function as external solution. Since $H_m^{(1)}(ze^{\pi i}) = -e^{-m\pi i} H_m^{(2)}(z)$, this choice leads to the same solution as in SKG. When we replace the true discontinuity at the boundary of the flux tube by a small non-uniform transition layer, the body mode resonantly couples to an Alfvén continuum mode at a certain magnetic surface in the transition layer and is damped due to resonant absorption. Therefore the eigenfrequency has a negative imaginary part. Again, we take the root as prescribed by (6). Now we use the time averaged radial energy flux to select the physically relevant solution. The time averaged radial energy flux is defined as (Bray and Loughhead, 1974):

$$F = \frac{Re(P_1^* v_r)}{2} = \frac{Re(-P_1^* i\omega\xi_r)}{2}$$

where v_r is the radial velocity perturbation.

For large values of the radial coordinate r, the radial energy flux that correponds to respectively the first and second Hankel function, behaves as:

$$F^{(1)} \;\sim\; Re[\omega k_{\perp e}^*] e^{-2Im(k_{\perp e})r},$$

$$F^{(2)} \;\sim\; Re[-\omega k_{\perp e}^*] e^{2Im(k_{\perp e})r}.$$

This means that for non-leaky modes the Hankel function of the first kind must be rejected, since there is no physical ground to assume that the resonant coupling would cause the body mode to radiate energy outward. However with the Hankel function of the second kind there is no radial outward energy flux, but there is an energy flux towards the resonance, as one could expect since the resonance acts as a sink of energy.

SKG have chosen the same root for $k_{\perp e}$, but retained the Hankel function of the first kind and rejected the Hankel function of the second kind under the requirement that there is no incoming wave field possible for eigenmodes. Hence SKG did not allow for a negative radial energy flow, asymptotically $\to$ zero. Thus the dispersion relation (18) in SKG is incorrect for resonantly damped non-leaky modes. However their dispersion relation (18) is correct for leaky modes. For leaky modes, the root is again taken as prescribed by (6) and now the Hankel

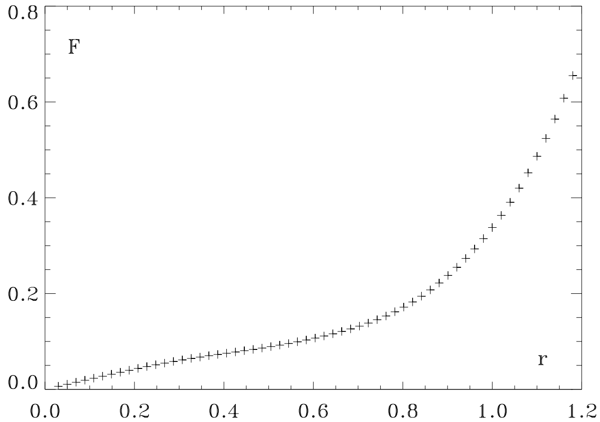

Fig. 1. The radial energy flux as function of the radial coordinate for a L & NR mode.

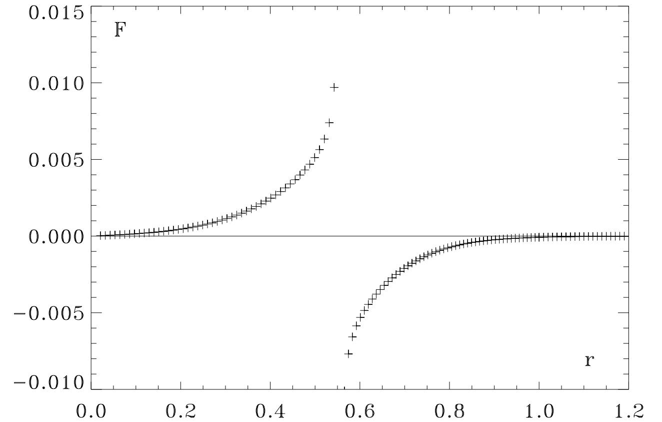

Fig. 2. The radial energy flux as function of the radial coordinate for a NL & R mode.

function of the first kind must be chosen to obtain an outward radial energy flux. An outward radial energy flux is only permissible when the corresponding oscillation frequency lies above the external cut-off frequency.

Hence the dispersion (18) in SKG has to be replaced by two distinct dispersion relations for respectively the leaky and the non-leaky modes:

$$\frac{P_{1,tr}}{\xi_{r,tr}} - \frac{\rho_{0e}\omega^2}{k_{\perp e}} \frac{H_m^{(1)}(k_{\perp e})}{H_m^{(1)'}(k_{\perp e})} = 0 \quad \text{for leaky modes,} \quad (7)$$

$$\frac{P_{1,tr}}{\xi_{r,tr}} - \frac{\rho_{0e}\omega^2}{k_{\perp e}} \frac{H_m^{(2)}(k_{\perp e})}{H_m^{(2)'}(k_{\perp e})} = 0 \quad \text{for non-leaky modes,} \quad (8)$$

where $P_{1,tr}$ and $\xi_{r,tr}$ denote the internal solutions at the flux tube boundary.

4. Results: eigenfunctions and energy fluxes

As explained in SKG, the eigenfunctions for leaky modes have amplitudes that exponentially increase towards infinity, since these modes are damped due to the acoustic wave leakage into the surroundings. As the perturbations are propagating outwards, the amplitudes at large distances correspond to earlier and therefore larger amplitudes within the tube.

As pointed out in Sect. 3 the dispersion relation (18) in SKG has to be replaced by the dispersion relation (8) of the present note for the non-leaky modes. Hence changing the combination of the choice of the sign of root for $k_{\perp e}$ and the kind of Hankel function alters the eigenfunctions shown in Fig. 8 and Fig. 10 in SKG. The amplitudes for $r \to \infty$ must $\to$ zero.

Let us now look at the time averaged radial energy flux as a function of the radial coordinate for the four types of eigenmodes. For type (i) NL & NR eigenmodes, the averaged energy flux is of course zero everywhere. The averaged energy flux for type (ii) L & NR eigenmodes is shown in Fig. 1. It is positive and steadily increases with radial distance due to the leaky character and the corresponding damping. For type (iii) NL & R eigenmodes the time averaged energy flux is shown in Fig. 2. The resonant frequency matching occurs at $r = r_A$. For $r \leq r_A$ the energy flux is positive and for $r \geq r_A$ the flux is negative. There is a flow of energy towards the resonant surface from both sides in complete agreement with our intuitive picture that the resonance is now the only sink of energy present in the system. For type (iv) L & R eigenmodes, the averaged radial energy flux is shown in Fig. 3. It is a combination of the two previous cases. The resonant frequency matching occurs at $r = r_A$. For $r \leq r_A$ the flux is positive. On the other side of the resonance the energy flux is negative for $r \leq 0.43$. In other words, there is a flow of energy from both sides of the resonance towards the resonance. From $r \approx 0.43$, the flux of energy again becomes positive. This value corresponds to the radial coordinate where the local upper cut-off frequency (dividing oscillating from evanescent solutions) becomes smaller than the real part of the eigenfrequency. Thus from this radial coordinate on, the behaviour of the solutions in the external region becomes oscillatory and propagating outwards.

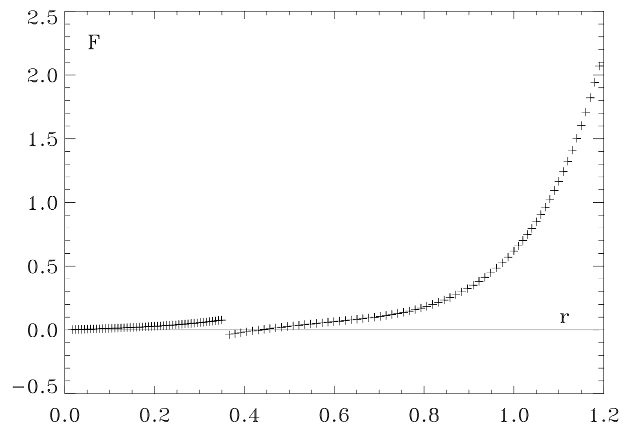

Fig. 3. A typical radial energy flux as function of the radial coordinate for a L & R mode.

5. The cubic mode: a mathematical artefact?

Both in SKG and in Cally (1986) a so-called cubic mode was found. This mode has an almost purely imaginary eigenvalue: a perturbation that simply damps out in time.

The question rises what effect does account for the damping of this mode. In the framework of our interpretation, only two phenomena can validate the damping: a resonance or the leaky character of the mode.

Inspection of the radial flux of energy in this case leads to two possibilities depending on the choice of the sign of the root with respect to the kind of Hankel function. The first possibility is a negative, but finite radial energy flux that → zero for $r \rightarrow \infty$. This would mean that energy is drawn into the tube, but as this cubic mode is also found for uniform tubes, there is no physical reason for this inflow of energy from the external plasma into a uniform magnetic flux tube. The other possibility is that of a positive, exponentially increasing flux that is infinite for $r \rightarrow \infty$. This means that there must be a way to transport the energy outwards. But due to the small real part of the eigenfrequency, the external solution is non-progagating and therefore again this possibility must be ruled out. This means that according to our interpretation, such an eigenmode must be dropped on account of physical interpretation.

If a physical explanation (e.g. some kind of 'diffusion' as Cally mentioned) could be found for the flux to be positively infinite for $r \rightarrow \infty$ for the cubic mode, this same explanation could be used for all non-leaky modes. This would mean that for all of the modes with frequencies below the external cut-off frequency both Hankel functions should be taken into account, leading to an underdetermined problem.

6. Conclusions

In this treatment of the eigenvalue problem for a non-uniform magnetic flux tube, the importance of the correct choice of the sign for the external perpendicular wave number in combination with the correct choice of the Hankel function has been pointed out. This choice must be based on the physical correct boundary conditions of the problem and can be understood by looking at the averaged radial energy flux.

For non-leaky modes with oscillation frequency out of the range of the Alfvén and slow continuum, the eigenfrequencies are real and the averaged energy flux is zero everywhere. Non-leaky modes resonantly coupled to Alfvén and/or slow continuum modes are damped in time. The flow of energy is always directed towards the resonance. Hence for $r \rightarrow \infty$ the flux is negative and tends to zero. This physical fact has led us, in this case, to a combination of $k_{\perp e}$ and the Hankel function opposite to the one made in SKG. Therefore the results presented in SKG for the resonantly damped non-leaky modes should be altered.

In the framework of this interpretation, the cubic mode discussed in Cally (1986) and SKG must be rejected as a mathematical artefact resulting from taking an unphysical boundary condition.

As for the optimal driving frequencies calculated in SKG, a final comment is in order. SKG found some results for optimal driving frequencies below the external cut-off frequency. But in fact, these results do not have any physical relevance, since the flux tube cannot be driven from the external medium by incident waves with oscillation frequency below the external cut-off frequency. The only region that is interesting to look for optimal driving frequencies is between the external cut-off frequency and the maximum of the Alfvén continuum.

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