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Self-consistent Alfvén-wave transmission and test-particle acceleration at parallel shocks

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Abstract. The transmission of Alfvén waves at parallel shocks is treated self-consistently, i.e., taking the pressure and energy flux exerted by the waves into account when determining the shock's gas compression ratio. The resulting test-particle acceleration and transport parameters at the shock are shown to be functions of five parameters: the Alfvénic Mach number of the shock, the upstream plasma beta, and the cross helicity, magnetic amplitude, and power-spectral index of the upstream waves. In a large region of parameter space, the scattering-center compression ratio is significantly larger than the gas compression ratio. For some conditions, it may get very large values that result in an extremely hard test-particle-energy spectrum of the cosmic-rays accelerated at the shock. This result is qualitatively the same as that obtained by the same authors under the "test-wave" approach, i.e., assuming that the waves have no impact on the shock's dynamics. However, the present analysis gives limits to the application of the previous results, and widens the scope of the previous analysis by removing a mathematical singularity in shocks with squared Mach number approaching the test-wave-theory gas compression ratio.

Key words: acceleration of particles – Magnetohydrodynamics (MHD) – shock waves – turbulence – ISM: cosmic rays

1. Introduction

Recently, Vainio & Schlickeiser (1998) (hereafter VS) examined small-amplitude Alfvén wave transmission in parallel shock waves and its implications on shock-acceleration of particles (Axford et al. 1977, Krymsky 1977, Bell 1978, Blandford & Ostriker 1978). They showed that the most important parameter controlling the spectral index of the shock-accelerated particles, the scattering-center compression ratio, may get values considerably larger than the gas compression ratio at shocks with a low Alfvénic Mach number. This implies that parallel astrophysical shock waves are capable of producing power-law particle-energy spectra with spectral index $\Gamma < 2$, values too small to be reached by the conventional test-particle shock-acceleration theories (e.g., Drury 1983).

In their study, VS assumed that the waves have no impact on the compression ratio of the shock. Their study, however, revealed interesting results at small Alfvénic Mach numbers, where the waves have finite pressure at least in the downstream region. Here, we study the transmission of Alfvén waves taking into account the pressure and energy flux of the waves in the shock-structure equation, i.e., the Rankine–Hugoniot relations. It is shown that this *self-consistent* wave transmission approach is important for obtaining the correct value of the scattering-center compression ratio at the limit $M^2 \to 4$–3β, where $M = u_1/V_{\rm A}$ is the Alfvénic Mach number of the shock, $\beta = c_{\rm s}^2/V_{\rm A}^2$ is the plasma beta, and u_1, $c_{\rm s}$ and $V_{\rm A}$ are the upstream background plasma speed, sound speed, and Alfvén speed, respectively. At this limit, the downstream wave field is standing in the shock frame and, thus, producing an infinite scattering-center compression ratio in the "test-wave" transmission model. This singularity is not a physical one and, as will be shown in this paper, is removed by the correct consideration of the wave effects.

The transmission of finite-amplitude Alfvén waves through a plane shock wave was first studied by Scholer & Belcher (1971) and later by McKenzie & Bornatici (1974). The difference between the two approaches was that McKenzie and Bornatici used spatially averaged values of the downstream wave-momentum and wave-energy fluxes, whereas Scholer and Belcher used values calculated immediately behind the shock front, when determining the shock's compression ratio. We have chosen to work with the Scholer & Belcher -approach, since there are physical reasons supporting their treatment and because analytic expressions for the gas compression ratio as a function of the upstream plasma parameters can be calculated in this approach.

2. Self-consistent Alfvén wave transmission

Considering the continuity of the mass flux, the transverse momentum, and the tangential electric field, VS derived the transmission coefficients for Alfvén waves at parallel shocks (see also McKenzie & Westphal 1969),

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$$T \equiv \frac{\delta B_2'}{\delta B_1} = \frac{r^{1/2}(r^{1/2}+1)}{2}\frac{M+H_{c_1}}{M+r^{1/2}H_{c_1}} \tag{1}$$

$$R \equiv \frac{\delta B_2''}{\delta B_1} = \frac{r^{1/2}(r^{1/2}-1)}{2}\frac{M+H_{c_1}}{M-r^{1/2}H_{c_1}}, \tag{2}$$

where δB_1 is the magnetic field associated with the incident (upstream) Alfvén wave, $\delta B_2'$ and $\delta B_2''$ are the magnetic fields of the transmitted and reflected Alfvén waves, $H_{c_1} = \pm 1$ is the cross helicity of the upstream wave field, $r = \rho_2/\rho_1$ is the gas compression ratio of the shock, and $\rho_{1(2)}$ and the upstream (downstream) mass density of the plasma.

In their calculations, VS assumed that the small-amplitude waves have no impact on the shock dynamics, i.e., that the gas compression ratio is not affected by the upstream wave parameters, δB_1 and H_{c_1}. In this case, the continuity of mass, parallel momentum and energy flux leads to the well-known expression

$$r = \frac{\gamma+1}{\gamma-1+2\beta/M^2} \tag{3}$$

for the gas compression ratio at a parallel shock, i.e., for fixed values of the plasma beta and the ratio of specific heats, γ, the upstream Alfvénic Mach number fully determines r. In the following, we will include the modifications to Eq. (3) from the inclusion of Alfvén-wave pressure and energy flux. Thus, the present model is not a perturbation-theory approach: there is no underlying "unperturbed shock solution" that the wave transmission would be based on but the wave fields are taken into account in all equations they appear in.

2.1. Gas compression ratio in a fast shock

The structure of a parallel shock in presence of Alfvén waves can be obtained as a solution to the jump conditions (Landau & Lifshitz 1960) expressing the continuity of the mass flux, $[\rho u_{\rm n}] = 0$, the continuity of parallel momentum, $[\rho u_{\rm n}^2 + P + P_{\rm w}]$=0, and the continuity of the energy flux, $[\frac{1}{2}\rho u_{\rm n}^3 + \gamma P u_{\rm n}/(\gamma-1) + F_{\rm w}] = 0$, if the jump conditions for the wave pressure, $[P_{\rm w}]$, and wave-energy flux, $[F_{\rm w}]$, are specified. Here, ρ, $u_{\rm n}$, and P are the mass density, normal component of the velocity, and gas pressure of the plasma, respectively. Like VS, we will use a coordinate system, where the shock normal points towards the upstream, and thus $u_{{\rm n}_{1(2)}} = -u_{1(2)}$, where $u_{1(2)} > 0$ is the upstream (downstream) background plasma speed. A rearrangement of the conservation laws gives the gas-pressure ratio at the shock in the form

$$\frac{P_2}{P_1} = \frac{(\gamma+1)r-(\gamma-1)+(\gamma-1)(r-1)\Delta}{(\gamma+1)-(\gamma-1)r}, \tag{4}$$

where

$$\Delta = \frac{r+1}{r-1}\frac{[P_{\rm w}]}{P_1} - \frac{2r}{r-1}\frac{[F_{\rm w}]}{P_1 u_1}. \tag{5}$$

The upstream wave pressure is (neglecting the electric-field contribution that is of the order $(u_1 + H_{c_1}V_{\rm A})^2/c^2$)

$$P_{{\rm w}_1} = \frac{(\delta B_1)^2}{8\pi} \tag{6}$$

and the upstream wave-energy flux is

$$F_{{\rm w}_1} = -\frac{(\delta B_1)^2}{4\pi}(u_1 + H_{c_1}V_{{\rm A}_1}) - \frac{(\delta B_1)^2}{8\pi}u_1, \tag{7}$$

where the first term on the right hand side is the normal component of the Poynting flux, $-\frac{1}{4\pi}(\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}$, and the second term comes from the normal kinetic energy flux, $\frac{1}{2}\rho(\delta\boldsymbol{u})^2 u_{\rm n}$. We have used the relation $\delta\boldsymbol{u} = -H_{\rm c}\,\delta\boldsymbol{B}/(4\pi\rho)^{1/2}$ between the velocity and magnetic field perturbations; this holds for Alfvén waves of degenerate cross-helicity state, $H_{\rm c} = \pm 1$. Note that for the circularly polarized Alfvén waves propagating in one direction, the upstream wave-energy flux and wave pressure are constants. In the downstream region, there are both forward and backward propagating waves present. This means that the downstream wave pressure,

$$P_{{\rm w}_2} = \frac{(\delta\boldsymbol{B}_2^{\rm f} + \delta\boldsymbol{B}_2^{\rm b})^2}{8\pi}, \tag{8}$$

and wave-energy flux,

$$F_{{\rm w}_2} = -\frac{(\delta B_2^{\rm f})^2}{4\pi}(u_2 + V_{{\rm A}_2}) - \frac{(\delta B_2^{\rm b})^2}{4\pi}(u_2 - V_{{\rm A}_2}) - \frac{(\delta\boldsymbol{B}_2^{\rm f} + \delta\boldsymbol{B}_2^{\rm b})^2}{8\pi}u_2, \tag{9}$$

are functions of position and time. Note, however, that the wave pressure and energy flux are constants in the whole plane of the shock on both sides. This justifies our model assumption of a stationary planar shock that does not perform any motion under the perturbation induced by the upstream wave field, and explains why the compressional wave modes are absent from the downstream region.

Two approaches have been used to calculate the wave pressure and wave-energy flux differences at the shock. Scholer & Belcher (1971) used the downstream values calculated immediately behind the shock, where the two wave fields are always in the same phase. McKenzie & Bornatici (1974) chose to use the average values of these quantities taken over the whole downstream region. They claim that this is necessary in order to correctly predict the mean state behind the shock to the order $\delta B_2^{\rm b}\delta B_2^{\rm f}/B_0^2$, where $\boldsymbol{B}_0$ is the ordered magnetic field parallel to the shock normal. It should be noted, however, that they have assumed the coexistence of the two downstream wave fields without any interaction between them, which is a correct solution of linearized MHD equations; this approximation is valid only, if we can neglect all second order terms like $\delta B_2^{\rm b}\delta B_2^{\rm f}/B_0^2$, so the argument of McKenzie & Bornatici does not seem appropriate. On the other hand, the Scholer & Belcher -approach predicts a correct gas compression ratio at the limit $\omega_1 \to 0$ for the upstream wave frequency and correct phase speeds of small-amplitude MHD-perturbations, as we shall see later, whereas the McKenzie & Bornatici -model does not. Therefore, to say the least, we feel that the McKenzie & Bornatici -model cannot be regarded as a more physical approach than the Scholer & Belcher -model, which we choose to work with.

Using the continuity of the transverse momentum, $[\rho u_{\rm n}\boldsymbol{u}_{\rm t} - B_{\rm n}\boldsymbol{B}_{\rm t}/4\pi] = 0$, and the tangential electric field, $[u_{\rm n}\boldsymbol{B}_{\rm t} -$

$B_{\rm n}\boldsymbol{u}_{\rm t}] = 0$, along with the conservation laws introduced earlier, we calculate the parameter Δ in Eq. (4) as

$$\Delta = \frac{[\boldsymbol{B}]^2}{8\pi P_1} = \frac{(\delta B_2 - \delta B_1)^2}{8\pi P_1}, \tag{10}$$

which is in accordance with the Eq. (6–19b) of Boyd & Sanderson (1969) for an oblique MHD shock. Let us denote the normalized upstream wave magnetic field by $b = \delta B_1/B_0$. Using Eqs. (4) and (10) and the continuity of parallel momentum, we derive an equation for the compression ratio,

$$1 - \frac{b^2\gamma}{2\beta}\left(\frac{\delta B_2^2}{\delta B_1^2} - 1\right) + \frac{M^2\gamma}{\beta}\frac{r-1}{r} = \frac{(\gamma+1)r - (\gamma-1)}{(\gamma+1)-(\gamma-1)r}$$
$$+\frac{(\gamma-1)(r-1)}{(\gamma+1)-(\gamma-1)r}\,\frac{b^2\gamma}{2\beta}\left(\frac{\delta B_2}{\delta B_1} - 1\right)^2. \tag{11}$$

In the derivation of Eq. (11) we use $P_1/\rho_1 V_{\rm A_1}^2 = \beta/\gamma$, which holds for an adiabatic gas, $P\rho^{-\gamma} = $ const. Using Eqs. (1–2), we may write for the ratio of the wave amplitudes

$$\frac{\delta B_2}{\delta B_1} = T + R = r\frac{M^2 - H_{\rm c_1}^2}{M^2 - rH_{\rm c_1}^2} = r\frac{M^2-1}{M^2-r}, \tag{12}$$

since $H_{\rm c_1} = \pm 1$ by assumption. In fact, the latter form of the equation holds for a general upstream magnetic field if time derivatives may be neglected (c.f. Eq. (5.46) of Priest 1982). The expression for $\delta B_2/\delta B_1$ can be substituted to Eq. (11). The resulting equation contains only one unknown (r) and it may be solved for a set of input parameters, M, β, γ, and b. Note that the cross-helicity state of the upstream waves, $H_{\rm c_1}$, has no effect on the compression ratio. Combining Eqs. (11) and (12), the problem can be reduced to finding the roots of the cubic equation in r

$$y(r) \equiv (M^2 - r)^2\left\{2r\beta - M^2[\gamma + 1 - (\gamma-1)r]\right\}$$
$$+b^2M^2r\left\{(\gamma-1)r^2 + [M^2(2-\gamma) - (\gamma+1)]r + \gamma M^2\right\}$$
$$= 0. \tag{13}$$

From this general shock equation we can reproduce a number of special cases: the parallel-shock solution (3) immediately results for $b \equiv 0$, i.e., when there are no waves present. The second solution in this case, $r = M^2$, gives the so called *switch-on shock* (e.g., Boyd & Sanderson 1969), where the magnetic field upstream is parallel to the shock normal, but makes a finite angle with the shock normal in the downstream region. This solution is possible only if

$$1 < M^2 < \frac{\gamma + 1 - 2\beta}{\gamma - 1} \overset{\gamma=5/3}{=} 4\text{–}3\beta;$$

otherwise the downstream tangential magnetic field magnitude would have an imaginary value (Priest 1982). In a plasma with $\beta > 1$ the inequality is never satisfied, so parallel shock is the only physical solution in such plasmas, if the upstream tangential magnetic field is zero. Note that Eq. (13) is in accord with Eq. (5.48) of Priest (1982) for the compression ratio in oblique shocks, as expected, since an oblique shock may be regarded as the special case where the upstream wave frequency in the shock frame, $\omega_1 = V_1 k_1 = 0$, where V_1 and k_1 are the phase velocity and wave number of the upstream waves, respectively. In fact, since we neglected all time derivatives from our calculation, we have silently assumed that the wave lengths on both sides of the shock are much larger than the shock thickness, δ, and we don't expect our analysis to carry over to waves with $k^{-1} \sim \delta$.

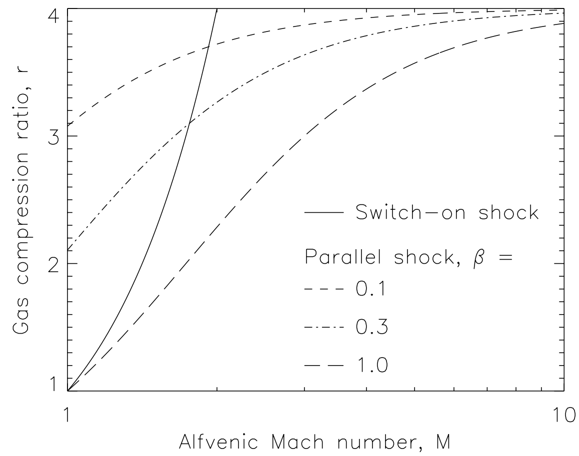

Fig. 1. Gas compression ratio as a function Alfvénic Mach number for shocks with $b = 0$ and $\gamma = 5/3$ for different values of the plasma beta. Note that the switch-on shock solution is physical only under the parallel-shock curve for a given value of β.

2.2. *Case* $\gamma = 5/3$

Let us study the roots of the Eq. (13) in a $\gamma = 5/3$ plasma. We have

$$3y(r) = 2(M^2 - r)^2\left[3r\beta - M^2(4-r)\right] +$$
$$+b^2M^2r\left[2r^2 + (M^2-8)r + 5M^2\right] = 0. \tag{14}$$

Fig. 1 shows the real roots of Eq. (14) in the case of $b \equiv 0$. The solid line gives the switch-on shock solution and the dashed lines give parallel shocks for some values of the plasma beta. Note that the switch-on-shock solution for a given value of plasma beta exists only under the curve giving the respective parallel-shock solution. A fast mode shock is always the one with largest M if r is kept constant; thus, a fast shock is a switch-on shock if $1 < M^2 < 4$–3β and a parallel shock otherwise.

The situation is qualitatively different if there are waves present, i.e., if $b > 0$. We calculate the fast-shock solutions of Eq. (14) for $\beta = 0$ as

$$M^2 = r$$
$$+rb\frac{(5+r)b + \sqrt{(5+r)^2b^2 + 24(r-1)(4-r)}}{4(4-r)}, \tag{15}$$

which are shown in Fig. 2. The key feature of the plot is that all the fast-shock solutions with a finite b lie below the critical curve $r = M^2$ so the downstream flow is always super-Alfvénic and the infinite values for the wave transmission coefficients are avoided on the whole range $M^2 > 1+b^2$, which now defines fast

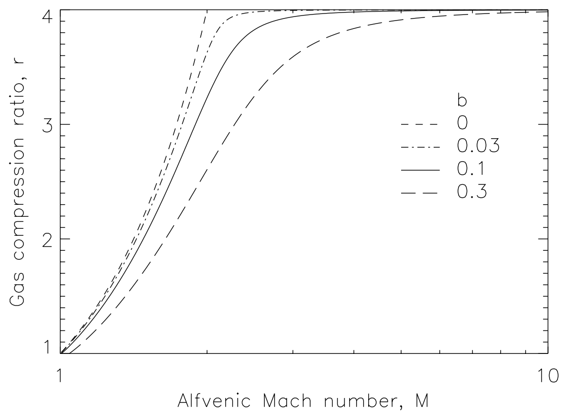

Fig. 2. Gas compression ratio as a function Alfvénic Mach number for fast shocks with $\beta = 0$ and $\gamma = 5/3$ for different values of the upstream wave amplitude, b.

shocks; because the total upstream magnetic field magnitude is (a constant) given by $B^2 = B_0^2(1+b^2)$, the phase speed of the fast MHD wave relative to the upstream plasma flow is given by

$$V_{\rm fast}^2 = \frac{B^2}{4\pi\rho_1} = \frac{B_0^2}{4\pi\rho_1}(1+b^2) = \frac{u_1^2}{M^2}(1+b^2)$$

in a $\beta = 0$ plasma. In fact, putting $r = 1$ in Eq. (13) gives a cubic equation for M^2, where the three roots

$$M^2 = 1, \quad M^2 = \frac{\beta+1+b^2 \pm \sqrt{(\beta+1+b^2)^2 - 4\beta}}{2} \quad (16)$$

may be easily connected with the three MHD wave modes, the Alfvén mode, and the fast (+ sign) and slow (− sign) magnetosonic modes, respectively, for all values of β. This is yet another point in favor of the Scholer & Belcher (1971) calculation of the shock's compression ratio, since different phase speeds for the small-amplitude perturbations are obtained in the McKenzie & Bornatici (1974) approach, if b is finite. Note also that in a $b > 0$ plasma the fast shock is always parallel if the ordered upstream magnetic field is along the shock normal, since so is then the ordered downstream magnetic field. This is because the tangential magnetic field is now totally due to the Alfvén waves on both sides of the shock.

In the general case $\beta \neq 0$, $b \neq 0$, Eq. (14) can be solved in parametric form

$$M^2 = (1+z)r(z) \quad (17)$$

and

$$r(z) = \frac{8z^2(z+1) - 6\beta z^2 - b^2(z+1)(5z-3)}{2z^2(z+1) + b^2(z+1)(z+3)}, \quad (18)$$

where the parameter z generally runs between $-1 < z < \infty$ and for fast shocks between $M_{\rm fast}^2 - 1 < z < \infty$, where $M_{\rm fast}^2$ is the largest one of the roots (16).

3. Cosmic-ray transport and acceleration parameters

Cosmic-ray transport in a homogeneous guiding field can be described by the quasi-linear diffusion–convection equation (Schlickeiser 1989), if the turbulence is weak, i.e., $b^2 < (T^2 + R^2)b^2 < 1$. (We will get back to this point in Sect. 4.) In a degenerate cross-helicity state $H_{\rm c} = \pm 1$, which is considered here in the upstream region ($x > 0$), the transport equation can be written in form (correcting for two typographical sign errors in Eq. (25) of VS)

$$\frac{\partial}{\partial x}\kappa_1 \frac{\partial f}{\partial x} + V_1 \frac{\partial f}{\partial x} - \frac{p}{3}\frac{\partial V_1}{\partial x}\frac{\partial f}{\partial p} = -Q_1(x,p), \quad (19)$$

where x is distance from the shock along the shock normal, p is particle momentum, v is particle velocity, $Q_1(x,p)$ represents sources and sinks, and

$$V_1 = u_1 + H_{\rm c_1} V_{\rm A_1} \quad (>0) \quad (20)$$

is the shock-frame phase speed of the upstream waves. The form of the upstream spatial diffusion coefficient, κ_1, may be calculated if the upstream wave number spectrum is known. We will consider a spectrum of the form

$$I^{\rm L}(k) = \frac{1+\sigma}{2}\frac{\delta B^2\,(q-1)}{k_1}\,\mathcal{U}(k-k_1)\left(\frac{k_1}{k}\right)^q \quad (21)$$

$$I^{\rm R}(k) = \frac{1-\sigma}{2}\frac{\delta B^2\,(q-1)}{k_1}\,\mathcal{U}(k-k_1)\left(\frac{k_1}{k}\right)^q \quad (22)$$

for the left- and right-handed Alfvén waves, respectively, where σ is the magnetic polarization state of the waves, q is their power-law spectral index, $\mathcal{U}(k)$ is the step function, and k_1 is the low cut-off value for the wave numbers. We consider the spectral index of the waves in the range $1 < q < 2$. Thus, we obtain the diffusion coefficient in form (VS)

$$\kappa_1(x,p) = \frac{vR_{\rm L}S_1}{2\pi(q-1)}\left(\frac{B_0}{\delta B_1}\right)^2 (k_1 R_{\rm L})^{q-1} \quad (23)$$

where $R_{\rm L}$ is the Larmor radius of the particle, and

$$S_1 = \frac{4}{(2-q)(4-q)}\frac{1}{1-\sigma^2}. \quad (24)$$

In the downstream region ($x < 0$), the transport equation reads

$$\frac{\partial}{\partial x}\kappa_2 \frac{\partial f}{\partial x} + V_2 \frac{\partial f}{\partial x} - \frac{p}{3}\frac{\partial V_2}{\partial x}\frac{\partial f}{\partial p} + \frac{1}{p^2}\frac{\partial}{\partial p}p^2\theta_2\frac{\partial f}{\partial p} = -Q_2(x,p). \quad (25)$$

Here, the transport and acceleration parameters are the downstream diffusion coefficient, $\kappa_2(x,p)$, that has the boundary condition,

$$\kappa_2(0,p) = \frac{\kappa_1(0,p)}{W_k}; \quad (26)$$

the average shock-frame phase speed of the downstream waves,

$$V_2(k) = u_2 + H_{\rm c_2}^k V_{\rm A_2} \quad (>0); \quad (27)$$

and the momentum diffusion coefficient,

$$\theta_2(x,p) = \vartheta\frac{V_{\rm A_2}^2 p^2}{\kappa_2}, \quad (28)$$

where

$$\vartheta \equiv \frac{2\,R_k^2\,T_k^2\,S_1}{q(q+2)W_k^2}, \tag{29}$$

and

$$T_k \;=\; \frac{r^{1/2}+1}{2r^{1/2}}\left(r\frac{M+H_{c_1}}{M+r^{1/2}H_{c_1}}\right)^{(q+1)/2} \tag{30}$$

$$R_k \;=\; \frac{r^{1/2}-1}{2r^{1/2}}\left(r\frac{M+H_{c_1}}{M-r^{1/2}H_{c_1}}\right)^{(q+1)/2}, \tag{31}$$

give the wave transmission and reflection coefficients at constant wave number k (see VS), and

$$W_k \;=\; T_k^2+R_k^2 \tag{32}$$

$$H_{c_2}^k \;=\; H_{c_1}\frac{T_k^2-R_k^2}{W_k} \tag{33}$$

are the wave intensity increase and the downstream cross helicity at constant wave number, respectively. Since the downstream cross-helicity state is now non-degenerate, the momentum diffusion term in the transport equation is unavoidable (VS).

VS studied the behavior of the cosmic-ray transport and acceleration parameters at the shock due to the transmission of Alfvén waves. They considered two cases for the plasma beta, $\beta = 0.3$ and 1.2 and found that for $\beta < 1$ the shock may be very efficient in accelerating particles as $M^2 \to 4$–3β since, in their model, the squared Mach number approaches the compression ratio at this limit and, hence, the downstream wave field is effectively standing behind the shock. This formally yields an infinite scattering-center compression ratio (VS),

$$r_k = \frac{V_1}{V_2(k)} = r\frac{M+H_{c_1}}{M+r^{1/2}H_{c_2}^k}. \tag{34}$$

This singularity is removed in the present analysis. Figs. 3–5 give the wave amplification factor, the downstream cross helicity, and the scattering-center compression ratio at constant wave number for $b = 0.1$ when the wave pressure effects have been taken into account, i.e., r and M are connected through Eqs. (17–18) instead of using Eq. (3), like was done in (VS). The scattering-center compression ratio still gets substantially larger values than the gas compression ratio, but the values are finite for all shocks with $M^2 > M_{\rm fast}^2$.

When downstream momentum diffusion is neglected, the particle spectrum at the shock is (e.g., Schlickeiser et al. 1993) $dJ/dE \propto p^{-\Gamma}$, where $\Gamma = (r_k+2)/(r_k-1)$. Thus, at the limit $r_k \to \infty$, the power-law spectral index $\Gamma \to 1$. In the analysis of VS, this limit was approached as $M^2 \to 4$–3β. When the wave pressure effects are included this limit is not reached but, nevertheless, Γ may get very close to unity for a small value of b. In Fig. 6, the spectral index Γ with $b = 0.1$ is plotted against the spectral index resulting from conventional shock acceleration theory, where the scattering centers are frozen into the plasma flow, i.e., $\Gamma_{\rm gas} = (r+2)/(r-1)$. The spectral index Γ is quite generally below 1.5 and reducing the value of b will get it closer and closer to the limit $\Gamma = 1$, whereas the spectral index $\Gamma_{\rm gas} > 2$ for a plasma with $\gamma = 5/3$.

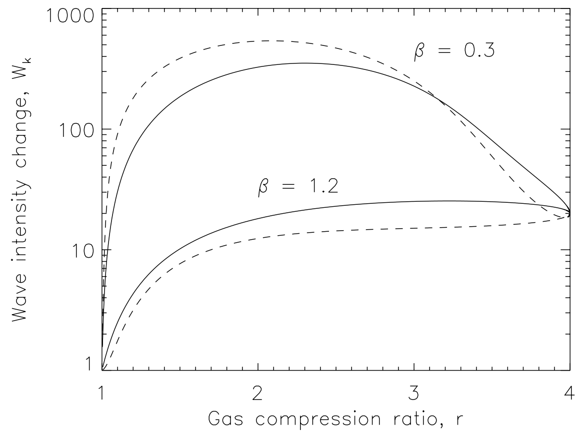

Fig. 3. Total wave amplification factor at constant wave number for a shock with a constant upstream plasma beta. Dashed and solid lines give the results for $H_{c_1} = +1$ and -1, respectively. The magnetic amplitude of the upstream waves is $b = 0.1$ and their spectral index is $q = 1.5$.

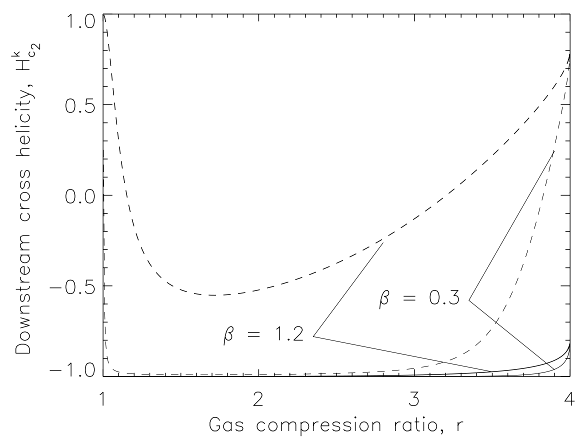

Fig. 4. Downstream cross-helicity at constant wave number for a shock with a constant upstream plasma beta. Dashed and solid lines give the results for $H_{c_1} = +1$ and -1, respectively. The magnetic amplitude of the upstream waves is $b = 0.1$ and their spectral index is $q = 1.5$.

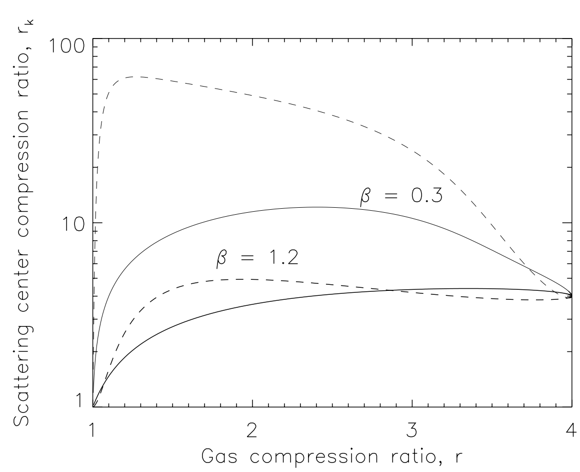

Fig. 5. Scattering center compression ratio for a shock with a constant upstream plasma beta. Dashed and solid lines give the results for $H_{c_1} = +1$ and -1, respectively. The magnetic amplitude of the upstream waves is $b = 0.1$ and their spectral index is $q = 1.5$.

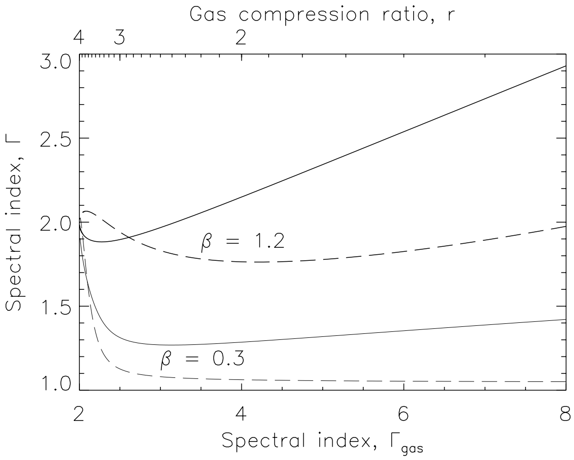

Fig. 6. Cosmic-ray spectral index produced by a shock with a constant upstream plasma beta, neglecting stochastic acceleration in the downstream region. Dashed and solid lines give the results for $H_{c_1} = +1$ and -1, respectively. The magnetic amplitude of the upstream waves is $b = 0.1$ and their spectral index is $q = 1.5$.

It was shown by VS, that the second-order Fermi acceleration in the downstream region is generally less effective than the first-order acceleration by multiple shock crossings. Including the wave pressure effects does not change this result qualitatively, as can be seen in Fig. 7; in the physical case of backward upstream waves (e.g., Bell 1978), the ratio of the characteristic time scales, $\tau_{\rm F}$ and $\tau_{\rm s}$, of the two processes (see VS for details) is,

$$\frac{\alpha(q,v)\;\tau_{\rm F}/\tau_{\rm s}}{(1-\sigma^2)A(q)} = (M+H_{\rm c_1})^2\,\frac{r_k-1}{1+r_k/W_k}\,\frac{r}{15\,r_k}\left(\frac{1}{T_k^2}+\frac{1}{R_k^2}\right). \qquad (35)$$

It will not get smaller values than 5 for $\beta = 0.3$ and $b = 0.1$ and zero magnetic helicity. Here $A(q) = 5q(q+2)(2-q)(4-q)/24$ is a slightly varying function of q, $A(1.5) \approx 1.4$, $A(5/3) \approx 1.0$, and $A(1.8) \approx 0.63$, and $\alpha(q,v) = \left[\partial \log(p^4\kappa^{-1})/\partial \log p\right]/3$, which equals $(1+q)/3$ at non-relativistic particle velocities and $(2+q)/3$ at relativistic velocities, i.e., $2/3 < \alpha < 4/3$.

In Figs. 8 and 9 we give the energetic particle acceleration parameters as a function of the Alfvénic Mach number with $\beta = 0.3$ and two values of $b = 0.01$ and 0.1 for the case of backward upstream waves. As b gets smaller we approach the limiting value of $\Gamma = 1$, whereas at the same time the relative importance of the second-order Fermi acceleration stays small. Including the wave pressure effects to the Alfvén wave transmission problem does not, therefore, change the most important conclusions of VS, but rather gives limits to and somewhat widens the scope of their application.

4. Discussion and conclusions

We have studied the problem of Alfvén-wave transmission at parallel shocks self-consistently, i.e., taking into account the pressure exerted by the waves to the plasma and the result-

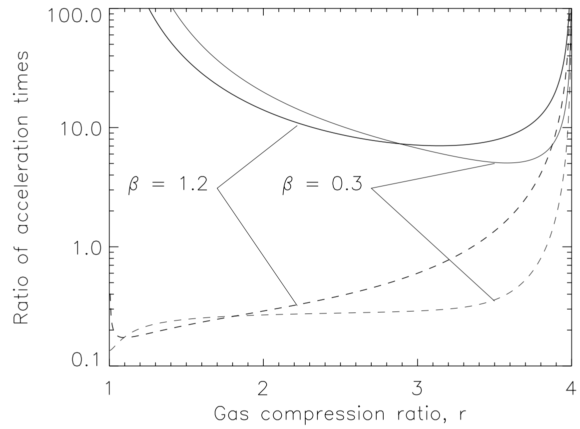

Fig. 7. The ratio of acceleration times for a shock with a constant upstream plasma beta. Dashed and solid lines give the results for $H_{c_1} = +1$ and -1, respectively. The magnetic amplitude of the upstream waves is $b = 0.1$ and their spectral index is $q = 1.5$.

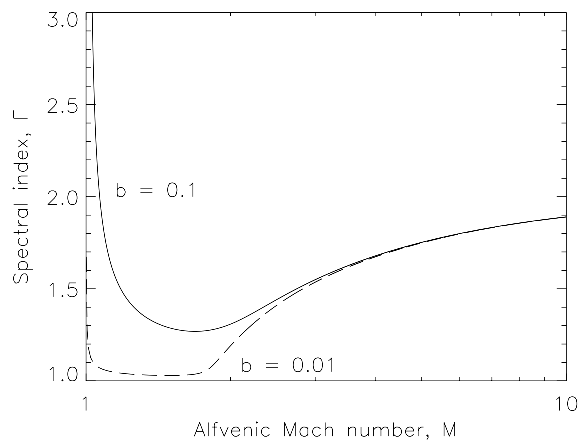

Fig. 8. Cosmic-ray spectral index produced by a shock with an upstream plasma beta $\beta = 0.3$, neglecting stochastic acceleration in the downstream region. The upstream cross-helicity $H_{c_1} = -1$, and the dashed and solid lines give the results for $b = 0.01$ and 0.1, respectively. The spectral index of the upstream waves is $q = 1.5$.

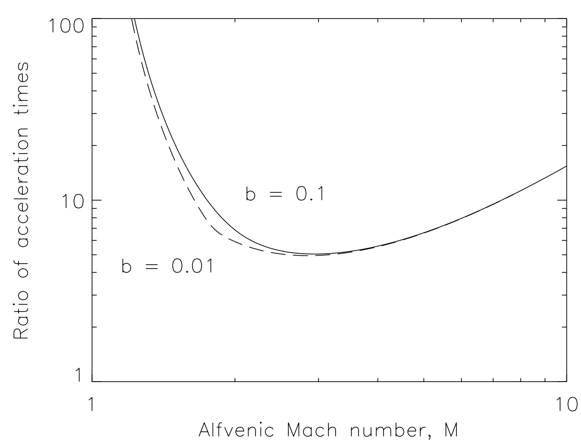

Fig. 9. The ratio of acceleration times for a shock with an upstream plasma beta $\beta = 0.3$. The upstream cross-helicity state is $H_{c_1} = -1$, and dashed and solid lines give the results for $b = 0.01$ and 0.1, respectively. The spectral index of the upstream waves is $q = 1.5$.

ing impact on the shock structure. The problem was previously treated by VS under a test-wave approach, where the pressure and energy flux of the waves was neglected.

We have given an explicit solution for the Alfvénic Mach number of a fast shock as a function of the gas compression ratio in a cold plasma, and a parametric dependence between the two quantities in the finite temperature case. Since there are always low-frequency waves in astrophysical plasmas, we regard the obtained equations for the shock's gas compression ratio as important generalizations of the parallel-shock-structure equation (3), that may still be applied in plasmas with a large plasma beta or a very large Alfvénic Mach number. Note that our treatment holds only for unidirectional circularly polarized upstream wave fields, since variations in the magnitude of the perpendicular magnetic field component may induce motion of the shock front itself from its equilibrium position (see, e.g., Achterberg & Blandford 1986). The situation gets even more complicated, if the upstream wave vectors are not aligned with the shock normal, since the assumption of a planar shock is then probably violated, also. This is why we do not expect our model to be generalized to oblique shocks very easily.

Like VS, we also studied the implications of the wave transmission to the diffusive shock acceleration of test particles, and confirmed their finding that in a low-beta plasma the particle-energy spectra may be significantly harder than predicted by the simplest version of the diffusive shock acceleration theory, where the waves are frozen in to the plasma flow. The hardening of the particle-energy spectrum is most pronounced if (i) the Alfvénic Mach number of the shock is small, and (ii) the magnetic amplitude of the upstream waves relative to the ordered magnetic field magnitude is small. When the physical case of backward upstream waves with a small net magnetic helicity is considered, the acceleration process is dominated by the first-order Fermi acceleration by multiple shock crossings over the second-order, stochastic acceleration in the downstream region. We found that the ratio of acceleration times of these two mechanisms is not very sensitive to the relative amplitude of the upstream waves. Thus, while stochastic acceleration may be very important for the fully non-linear treatment of background plasma, waves and and cosmic rays around the shock, we believe that the spectral index of the shock accelerated cosmic rays is controlled by the first-order Fermi mechanism.

We treated the cosmic-ray transport quasi-linearly, which means that the underlying assumption is that of weak turbulence. In Fig. 10, we can see that the quasi-linear treatment of the particle motion in the downstream region may be questionable for large values of b and at low-Mach-number shocks, in general. We may still, however, regard the particle motion as diffusive in the wave frame, even if the actual form of the downstream spatial and momentum diffusion coefficients are no longer given exactly by their quasi-linear forms. Thus, the main results for our study should hold qualitatively even for values of $(\delta B_2/B_0)^2$ reaching unity.

A more serious problem due to the large downstream wave amplitude arises since we assumed that the downstream fluctuations may be described by a superposition of two oppositely

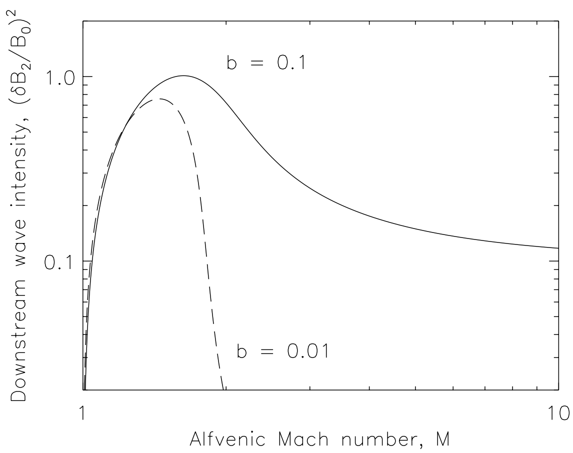

Fig. 10. The relative intensity of the downstream waves for a shock with an upstream plasma beta $\beta = 0.3$. The upstream cross-helicity state is $H_{c_1} = -1$, and dashed and solid lines give the results for $b = 0.01$ and 0.1, respectively.

propagating Alfvén wave fields. However, if the amplitude of one of them is large and the other one is finite, the terms involving the squared perturbation amplitude in the MHD equations may not be neglected anymore, and the linear picture breaks down. In fact, a sinusoidal circularly polarized Alfvén wave with degenerate cross-helicity, although it is an exact solution of the MHD equations for any amplitude δB, is unstable against decay into a backscattered Alfvén wave and a magnetosonic wave traveling in the direction of the parent wave (Sagdeev & Galeev 1969). Cohen & Dewar (1974) and Cohen (1975) showed, however, that if the longitudinal plasma fluctuations are subjected to linear damping (which is not unreasonable), a broad-band spectrum of non-coherent Alfvén waves traveling in one direction is stable, if its power-law spectral index $q > 1$. Thus, the wave spectra that we have studied are probably even overestimating the amplitude of the forward Alfvén waves somewhat away from the shock, since other wave modes than the backward Alfvén wave (which is always the one with the largest amplitude under physically plausible conditions) will gradually decay from the system. This would also occur as a result of downstream stochastic acceleration, which would damp the downstream waves making the minor wave components evanescent from the system (Ko 1992). Anyway, in a low-Mach-number ($M < 2$) shock with backward upstream Alfvén waves, we may estimate the intensity of the downstream forward waves from Eq. (2) as $R^2b^2 < b^2(M-1)^4/16$, and since the backward wave intensity is $T^2b^2 \sim 1$, we may estimate that the ratio $R^2/T^2 \sim b^2/16$, i.e., an order of magnitude smaller than the small parameter b^2. Thus, in such shocks the downstream wave field may regarded as consisting only of backward Alfvén waves. Note also, that if the plasma beta is very large, like in the downstream region of a high-Mach-number shock, the medium may be treated as non-compressional; in this case, the longitudinal fluctuations are negligible, but the gas pressure is fluctuating so that the total pressure (gas+waves) of the system remains constant.

Scholer & Belcher (1971) raised the question about the evolutionarity of their shock solution. They claimed that the low-Mach-number shocks would have a critical Mach number value below which the downstream flow would be super-fast-magnetosonic. We, in contrast, find that when circularly polarized upstream waves with constant magnetic field magnitude are considered, the downstream flow immediately behind a fast shock is always sub-fast-magnetosonic and, thus, the evolutionary condition for the shock is fulfilled. This is simply because one should take into account the transverse magnetic field when computing the phase speed of the small-amplitude fast MHD waves behind the shock, when considering non-compressive upstream fluctuations. It is precisely this field component that removes the problem encountered by the analysis of Scholer and Belcher, who applied the model to linearly polarized waves. The proof of the evolutionarity lies in the fact that our model predicted the same gas compression ratio for the shock as may be obtained for an oblique shock without waves and this, in turn, always satisfies the evolutionary conditions.

The extremely turbulent state of the downstream medium leads also to an observable effect that could be used as a test for the present model. Since the normalized wave intensity, $(\delta B/B_0)^2$, may change drastically from upstream to downstream the model predicts that the degree of linear polarization of the synchrotron radiation of shock-accelerated electrons changes accordingly from large values upstream to small values downstream. This could, at least in principle, be observed by high-resolution radio observations of supernova-remnant shocks. The effect would be most pronounced for shocks with Alfvénic Mach number in range $M = 1$–3, so it would be most likely detectable in slowly expanding supernova remnants. The model predicts simultaneously an electron-energy spectral index close to unity, which means that the synchrotron frequency spectrum should be flat in such objects. Note also, that Kennel et al. (1986) discovered that the spectral index of the accelerated protons in a traveling interplanetary shock on 12 November 1978 was consistent with Alfvénic scattering-center velocities directed away from the Sun in both upstream and downstream regions; since the gas compression ratio in that shock was ≈ 2.7, this is precisely what our model predicts (see the solid curves of Fig. 4). This shock, however, was not a parallel one so the agreement may also be coincidental.

The study of Kennel et al. (1986) also lends support to our assumption of the nature of the upstream turbulence: for the shock that they studied, the upstream turbulence seems to be almost completely non-compressional (see their Fig. 5). Shocks with compressional upstream turbulence are, however, observed as well. Such shocks may exhibit features that cannot be described by our simple model, as described above. Also, as pointed out by VS, fast MHD waves in the downstream region can only propagate away from the shock and, therefore, the presence of compressive modes in the system may lower the effective scattering center compression ratio. One must, therefore, be careful when applying our model to shocks observed in space.

Our current treatment is fully self consistent when it comes to the compression ratio of and transmission of the Alfvén waves through the step-like gas shock at $x = 0$; the cosmic rays have no direct effect on them, since the cosmic-ray distribution is continuous over the shock. We have not analyzed the state of the upstream medium self-consistently, but assumed that it is given – our upstream values refer to the ones measured immediately in front of the shock, not in the far upstream region. However, since the accelerated particles probably generate most of the waves in the upstream medium, a fully self-consistent calculation would take this into account as well as the back-reaction of the upstream waves and the cosmic rays to the upstream plasma flow pattern through their respective pressure gradients and energy fluxes. The self-consistent shock-structure has been calculated by McKenzie & Völk (1982), but since their cosmic-ray treatment was a hydrodynamical one (Parker 1969, Drury & Völk 1981, Axford et al. 1982) they did not obtain any information about the detailed form of the energetic-particle spectrum, and the interesting spectral effects due to the high scattering-center compression ratio discovered by VS and the present paper were missed. One should bear in mind that the hard energy spectra with $\Gamma < 2$ cannot be realized if one assumes the same spectral index to hold for all $p \to \infty$, because of the infinite cosmic-ray energy densities involved. Finite acceleration times t, however, explain why such a hard spectrum may be realized below some cut-off momentum $p_{\max}(t)$. The McKenzie & Völk model, being based on the wave-energy exchange equation of Dewar (1970), also assumed that the amplitudes of the downstream waves do not evolve, but this seems to be an assumption that does not hold generally; whenever the time dependence of wave field cannot be Lorentz-transformed away, we expect the minor wave components to be damped either by plasma effects not included in MHD (or Dewar's equation) or by cosmic rays experiencing second-order Fermi acceleration (see also Jiang et al. 1996). Including these effects in our treatment will provide us with interesting tasks for the future.

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