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Lagrangian theory of structure formation in pressure–supported cosmological fluids

Susanne Adler and Thomas Buchert

Theoretische Physik, Ludwig-Maximilians-Universität, Theresienstrasse 37, D-80333 München, Germany

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Abstract. The Lagrangian theory of structure formation in cosmological fluids, restricted to the matter model "dust", provides successful models of large–scale structure in the Universe in the laminar regime, i.e., where the fluid flow is single–streamed and "dust"–shells are smooth. Beyond the epoch of shell–crossing a qualitatively different behavior is expected, since in general anisotropic stresses powered by multi–stream forces arise in collisionless matter. In this paper we provide the basic framework for the modeling of pressure–supported fluids, restricting attention to isotropic stresses and to the cases where pressure can be given as a function of the density. We derive the governing set of Lagrangian evolution equations and study the resulting system using Lagrangian perturbation theory. We discuss the first–order equations and compare them to the Eulerian theory of gravitational instability, as well as to the case of plane–symmetric collapse. We obtain a construction rule that allows to derive first–order solutions of the Lagrangian theory from known first–order solutions of the Eulerian theory and so extend Zel'dovich's extrapolation idea into the multi–streamed regime. These solutions can be used to generalize current structure formation models in the spirit of the "adhesion approximation".

Key words: cosmology: large-scale structure of Universe – cosmology: theory – methods: analytical – instabilities – hydrodynamics – gravitation

1. Introduction

Zel'dovich (1970, 1973) initiated the use of Lagrangian coordinates for the construction of analytical models of cosmic structure formation. He pointed out that the solution of a force–free continuum of "dust" (i.e. pressureless and non–gravitating matter), given in terms of trajectories of continuum elements (see, e.g., Zel'dovich & Myshkis 1973), can be rescaled to give the correct solution for gravitational instability in the linear limit. He suggested to follow those trajectories further up to the epoch when shell–crossing singularities in the fluid flow develop. Doroshkevich et al. (1973) confirmed the validity of Zel'dovich's approximation also within the full self–gravitating system of equations. To be more precise they performed a self–consistency check in one of the four Newtonian field equations (see Buchert 1989 for a discussion and extension of this self–consistency argument). It was also known (Zentsova & Chernin 1980) that Zel'dovich's ansatz provides a subclass of exact solutions in the case of plane–symmetric collapse on a Friedmann background. In (Buchert 1989) it was then shown (going back to an earlier work on the fully Lagrangian formulation of the basic equations by Buchert & Götz 1987) that his ansatz also provides a subclass of exact 3D solutions for a special class of initial gravitational potentials that are composed of surfaces of vanishing Gaussian curvature. These solutions feature maximally anisotropic, locally one–dimensional collapse supporting Zel'dovich's original discussion of the formation of "pancakes" as collections of volume elements which degenerate into small sheets and later enclose a three–stream system of the flow. These findings may be restated in the framework of a Lagrangian perturbation theory in which Zel'dovich's model is recovered as a subclass of the first–order solutions (Buchert 1992; for reviews see Bouchet et al. 1995, Buchert 1996, Sahni & Coles 1996, Ehlers & Buchert 1997 and references therein).

The formation of singularities (Arnol'd et al. 1982) is a consequence and drawback of the matter model which allows fluid elements to overtake freely. On the other hand, N–body simulations show (e.g. Doroshkevich et al. 1980) that N–stream systems create additional multi–stream forces which hinder the central fraction of fluid elements to escape from high–density regions; as a result they fall back onto the central region performing oscillations about the center: an internal hierarchy of nested caustics is formed that continues to create new structures down to smaller scales (Gurevich & Zybin 1995).

A phenomenological model based on Burgers' equation was then suggested to overcome the drawback of "dust" models by inventing a Laplacian forcing that acts in the desired way (Gurbatov et al. 1989). This model features "adhesion" of already formed structures and so models the action of self–gravity of multi–stream systems. A recent suggestion of how to derive this "adhesion approximation" from kinetic theory of self–gravitating collisionless systems shows that it is indeed possible to obtain adhesion–type models by taking into account multi–stream forces (Buchert & Domínguez 1998). However, these models form a wider class in the sense that multi–stream forces

Send offprint requests to: T. Buchert (buchert@stat.physik.uni-muenchen.de)

are, in general, anisotropic unlike in the case of Burgers' equation. Moreover, equations involving a Laplacian are expected for the gravitational field strength and not, as is the case for the "adhesion model", for the velocity.

Inspite of the insight gained by deriving the "adhesion model" from first principles, we shall, in the present work, adopt the restricting assumptions that the multi–stream force acts isotropically (in the sense of an idealization), and that the dynamical pressure due to multi–stream systems can be represented as a function of the density. For these assumptions we derive the general Lagrangian evolution equations and expand them to first order using the Lagrangian perturbation approach.

A comment concerning the isotropic approximation of the multi–stream force is in order: our investigation covers pressure–supported fluids as they are studied in hydrodynamics. Here we do not restrict ourselves to the description of matter models which are supported by the usual thermodynamical pressure (i.e., by short–ranged interactions). Rather we follow an approach that we consider to be relevant for collisionless systems: a dynamical pressure force arises due to the action of a "multi–dust" region on the bulk motion. As remarked above such a forcing is in general anisotropic and the evolution equations governing the anisotropic stresses involve all velocity moments of the distribution function (Buchert & Domínguez 1998). Hence, a closure condition is needed which, especially for self–gravitating fluids (e.g. Bertschinger 1993), is not obvious and cannot be formulated along the lines of collisional systems. At the current stage of knowledge about this subtle problem we advance the phenomenological closure condition that multi–stream stresses may be idealized by a scalar function p that is given in terms of the density of the continuum. This opens the possibility to immediately access the multi–streamed regime phenomenologically. For example, Buchert & Domínguez (1998) found that the assumption of small velocity dispersion singles out a relationship $p \propto \varrho^{5/3}$ that is solely based on the dynamical equations; the matter model $p \propto \varrho^2$ yields, together with the restriction to parallelity of peculiar–velocity and –acceleration, the "adhesion approximation" (Gurbatov et al. 1989). Other relationships are put into perspective by Buchert et al. (1999); particular matter models may lead to soliton states which, due to their persistence in time, could dominate the architecture of cosmic structure.

Although the isotropic idealization is simple–minded, it may actually be a good approximation in practice, as has been recently found for high–density regions in numerical simulations (Colombi, priv.comm.).

We proceed as follows: In Sect. 2 we develop the Lagrangian theory for pressure–supported fluids and illuminate the structure of the basic equations by the plane–symmetric case; in Sect. 3 we then move to some detailed studies of the basic equations: a perturbative treatment allows to derive a first-order evolution equation for fluid displacements. Note that this linear equation in Lagrangian space embodies Eulerian nonlinearities. This allows us to formulate an extrapolation rule that extends solutions of the Eulerian theory of gravitational instabilities into the nonlinear and multi–streamed regime. This rule rests on the spirit of Zel'dovich's extrapolation idea that led to the celebrated "Zel'dovich approximation" (Zel'dovich 1970, 1973). We also discuss the special case where the gravitational collapse is plane–symmetric on an isotropic background cosmology. We summarize the results and provide an outlook in Sect. 4.

2. The Lagrange–Newton–System with pressure

2.1. The Euler–Newton–System with pressure

Eulerian coordinates are the most widely employed ones in physics. Here we use $\boldsymbol{x}$ to define the position that a fluid element occupies at the moment t in a non–rotating Eulerian coordinate system. If we consider $\boldsymbol{x}$ and t as independent variables, the total time derivative of any vector field $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t)$ equals:

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \frac{\partial \boldsymbol{a}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{a} \quad . \tag{1}$$

The field $\boldsymbol{v}(\boldsymbol{x}, t)$ corresponds to the mean velocity in a kinetic picture. Eulerian coordinates are used to describe fields, e.g. the spatial mass distribution $\varrho(\boldsymbol{x}, t)$. In this description we are not interested in the fluid elements that produce this density and we do not know their trajectories. In Sect. 2.2 we introduce Lagrangian coordinates which convey this information.

The basic system of equations describing a self–gravitating medium is the so–called Euler–Newton–System. We add the accelerative term $-\frac{\nabla p}{\varrho}$, which takes into account the isotropic part of the velocity dispersion tensor. We have the following evolution equations for the mass density ϱ and the velocity $\boldsymbol{v}$ ($\boldsymbol{b}$ denotes the acceleration field):

$$\frac{\partial \varrho}{\partial t} \;=\; -\boldsymbol{\nabla} \cdot (\varrho \boldsymbol{v}) \quad , \tag{2a}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} \;=\; -(\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v} + \boldsymbol{b} \quad , \text{where } \boldsymbol{b} := \boldsymbol{g} - \frac{\boldsymbol{\nabla} p}{\varrho} \quad . \tag{2b}$$

The gravitational field strength $\boldsymbol{g}$ is a solution of the (Newtonian) field equations:

$$\boldsymbol{\nabla} \times \boldsymbol{g} \;=\; \boldsymbol{0} \quad , \tag{2c}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} \;=\; \Lambda - 4\pi G \varrho \quad , \tag{2d}$$

where Λ is the cosmological constant and G the Newtonian gravitational constant. The evolution equation for ϱ represents the conservation of mass (continuity equation). The second equation (for $\boldsymbol{v}$) is Euler's equation and describes the balance of momentum. In a kinetic picture (2b) is known as Jeans' equation.

To complete the Euler–Newton–System with pressure we need an "equation of state" that relates p with the dynamical variables ϱ and $\boldsymbol{v}$: we employ the assumption $p = \alpha(\varrho)$ with $\alpha' := \frac{\partial \alpha}{\partial \varrho}$. Hence, the field equations become

$$\boldsymbol{0} = \boldsymbol{\nabla} \times \boldsymbol{g} \;=\; \boldsymbol{\nabla} \times \boldsymbol{b} + \boldsymbol{\nabla} \times \frac{\boldsymbol{\nabla} p}{\varrho}$$

$$\;=\; \boldsymbol{\nabla} \times \boldsymbol{b} + \frac{1}{\varrho}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} p) + \frac{1}{\varrho^2}\boldsymbol{\nabla} p \times \boldsymbol{\nabla} \varrho$$

$$= \nabla \times \boldsymbol{b} \quad ; \tag{3a}$$

$$\begin{aligned}\Lambda - 4\pi G\varrho &= \nabla \cdot \boldsymbol{g} = \nabla \cdot \boldsymbol{b} + \nabla \cdot \frac{\nabla p}{\varrho} \\ &= \nabla \cdot \boldsymbol{b} + \nabla p \cdot \nabla \frac{1}{\varrho} + \frac{1}{\varrho}\Delta p \\ &= \nabla \cdot \boldsymbol{b} + (\alpha'' - \frac{\alpha'}{\varrho})\frac{(\nabla\varrho)^2}{\varrho} + \frac{\alpha'}{\varrho}\Delta\varrho \quad . \end{aligned} \tag{3b}$$

The equation for the curl of $\boldsymbol{g}$ does not change with the additional accelerative term for that particular class of "equations of state".

2.2. Transformation tools

Lagrangian coordinates are assigned to fluid elements and do not change along the flow lines: we choose the positions $\boldsymbol{X}$ of the elements at a time t_0 as coordinates ($\boldsymbol{X} = \boldsymbol{x}|_{t_0}$); thus we have indexed each fluid element. If we use $\boldsymbol{X}$ and t as independent variables, the total time derivative becomes $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t}|_{\boldsymbol{X}}$, i.e., the convective term disappears in this description. In order to connect the Lagrangian coordinates with the Eulerian ones, we introduce the position vector field $\boldsymbol{f}$,

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t) \ , \quad \text{where} \quad \boldsymbol{X} := \boldsymbol{f}(\boldsymbol{X}, t_0) \quad . \tag{4}$$

For later convenience we introduce the symbols "," and "|" in order to distinguish Eulerian and Lagrangian differentiation, e.g. $\frac{\partial v_a}{\partial x_b} =: v_{a,b}$ and $\frac{\partial f_a}{\partial X_b} =: f_{a|b}$. ∇ denotes the Eulerian and $\nabla_{\mathbf{0}}$ the Lagrangian nabla operator. In the following we also apply the summation convention.

The Jacobian matrix, which reflects the deformation of a volume element, is of key importance for the transformation of coordinates. Here we use the notation:

$$J := \det J_{ik} = \det f_{i|k} \quad . \tag{5}$$

In order to change the Eulerian differentiation to one with respect to Lagrangian coordinates, we have to define the inverse transformation of (4):

$$\boldsymbol{X} = \boldsymbol{h}(\boldsymbol{x}, t) \quad , \quad \boldsymbol{h} \equiv \boldsymbol{f}^{-1} \quad .$$

Remembering the definition of the adjoint matrix, $\mathrm{ad}J_{jk} = \frac{1}{2}\varepsilon_{klm}\varepsilon_{jpq}f_{p|l}f_{q|m}$, we get

$$h_{k,j} = J^{-1}_{jk} = \frac{1}{J}\mathrm{ad}J_{jk} = \frac{1}{2J}\varepsilon_{klm}\varepsilon_{jpq}f_{p|l}f_{q|m} \quad ; \tag{6}$$

ε_{klm} denotes the totally antisymmetric Levi-Civita tensor; note the useful relation $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ ($\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ else). In order to shorten the equations it is convenient to use functional determinants:

$$\begin{aligned}\mathcal{J}(A, B, C) &:= \frac{\partial(A, B, C)}{\partial(X_1, X_2, X_3)} := \begin{vmatrix} \frac{\partial A}{\partial X_1} & \frac{\partial B}{\partial X_1} & \frac{\partial C}{\partial X_1} \\ \frac{\partial A}{\partial X_2} & \frac{\partial B}{\partial X_2} & \frac{\partial C}{\partial X_2} \\ \frac{\partial A}{\partial X_3} & \frac{\partial B}{\partial X_3} & \frac{\partial C}{\partial X_3} \end{vmatrix} \\ &= \varepsilon_{klm}A_{|k}B_{|l}C_{|m} \, . \end{aligned} \tag{7}$$

Because of the properties of the Levi–Civita tensor, or the definition of a determinant as a multilinear and alternating map, respectively, we have some tools that make work easier, e.g.:

$$\begin{aligned}\mathcal{J}(A + D, B, C) &= \mathcal{J}(A, B, C) + \mathcal{J}(D, B, C) \\ \mathcal{J}(A, B, C) &= -\mathcal{J}(A, C, B) \\ \mathcal{J}(A, A, C) &= 0 \\ \mathcal{J}(A \cdot D, B, C) &= D \cdot \mathcal{J}(A, B, C) + A \cdot \mathcal{J}(D, B, C) \, . \end{aligned}$$

Therefore, we can transform any tensor $a_{i,j}$ as follows:

$$\begin{aligned} a_{i,j} &= a_{i|k}h_{k,j} = a_{i|k}\frac{1}{2J}\varepsilon_{klm}\varepsilon_{jpq}f_{p|l}f_{q|m} \\ &= \frac{1}{2J}\varepsilon_{jpq}\mathcal{J}(a_i, f_p, f_q) \quad . \end{aligned} \tag{8}$$

For further details on Lagrangian evolution equations the reader may consult the introductory tutorial of Buchert (1996) and the review by Ehlers & Buchert (1997).

2.3. The Lagrange–Newton–System with pressure

Starting with the Euler–Newton–System (with pressure) we obtain the Lagrange–Newton–System with the help of the transformation $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t)$.

As the continuity equation (2a) represents the conservation of mass $M = \int_{D_t} \varrho \mathrm{d}^3x = \int_{D_{t_0}} \varrho J \mathrm{d}^3X$ within any domain D_t that may change in time, i.e.,

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}t} = 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_{t_0}} \varrho J \mathrm{d}^3X = \int_{D_{t_0}} \frac{\mathrm{d}}{\mathrm{d}t}(\varrho J)\mathrm{d}^3X \\ &= \int_{D_t} \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\varrho J)}{J}\mathrm{d}^3x \, , \end{aligned}$$

we conclude with $\varrho_0 := \varrho(\boldsymbol{X}, t_0)$, $J_0 := J(\boldsymbol{X}, t_0)$:

$$\varrho J =: C(\boldsymbol{X}) = \varrho_0 J_0 \quad . \tag{9}$$

For our definition of Lagrangian coordinates (4) we have $J_0 = \det \frac{\mathrm{d}x_i}{\mathrm{d}X_j}|_{t_0} = 1$; nevertheless we will use $C(\boldsymbol{X})$ for the time being in order to allow for a later relabelling of trajectories.

According to the definition of $\boldsymbol{f}$ the Eulerian equation (2b) reads:

$$\boldsymbol{b} = \ddot{\boldsymbol{f}}(\boldsymbol{X}, t) \quad . \tag{10}$$

As we have the exact integrals (9, 10), ϱ and $\boldsymbol{v}$ are no longer dynamical variables in the Lagrangian picture; $\boldsymbol{f}$ attains the status of the only dynamical variable, if we demand that p is a functional of $\boldsymbol{f}$, which is obviously true for the assumption $p = \alpha(\varrho)$.

Applying the first integral (9) and the transformation of the Eulerian differentiation (8) to the equations for $\boldsymbol{g}$ (2c, 2d) we obtain the Lagrange–Newton–System with pressure, i.e. a set of four coupled nonlinear partial differential equations for the trajectory field $\boldsymbol{f}$. We first obtain:

$$\begin{aligned} 0 = (\nabla \times \boldsymbol{g})_h &= (\nabla \times \ddot{\boldsymbol{f}})_h + \frac{1}{\varrho^2}(\nabla p \times \nabla\varrho)_h \\ &= \varepsilon_{hji}\ddot{f}_{i,j} + \frac{\varepsilon_{hji}}{\varrho^2}\varrho_{,i}p_{,j} \end{aligned}$$

$$= \frac{1}{J}\mathcal{J}(\ddot{f}_i, f_i, f_h) + \frac{1}{2J^2\varrho^2}\varepsilon_{jpq}\mathcal{J}(\varrho, f_h, f_j)\mathcal{J}(p, f_p, f_q)\ , \quad (11a)$$

$$\begin{aligned}\Lambda - 4\pi G\varrho &= \nabla\cdot\boldsymbol{g} = \nabla\cdot\ddot{\boldsymbol{f}} + \nabla\cdot\frac{\nabla p}{\varrho} \\ &= \frac{1}{2J}\varepsilon_{jpq}\mathcal{J}(\ddot{f}_j, f_p, f_q) \\ &+ \frac{1}{2J}\varepsilon_{jpq}\mathcal{J}(\frac{1}{2J\varrho}\varepsilon_{jkl}\mathcal{J}(p, f_k, f_l), f_p, f_q) \\ &= \frac{1}{2J}\varepsilon_{jpq}\mathcal{J}(\ddot{f}_j, f_p, f_q) \\ &- \frac{1}{2J^3\varrho}\mathcal{J}(p, f_p, f_q)\mathcal{J}(J, f_p, f_q) \\ &- \frac{1}{2J^2\varrho^2}\mathcal{J}(p, f_p, f_q)\mathcal{J}(\varrho, f_p, f_q) \\ &+ \frac{1}{2J^2\varrho}\mathcal{J}(\mathcal{J}(p, f_p, f_q), f_p, f_q)\ . \end{aligned} \quad (11b)$$

In the special case $p = \alpha(\varrho)$ we get our final set of equations:

$$0 = (\nabla\times\ddot{\boldsymbol{f}})_h = \varepsilon_{hji}\ddot{f}_{i,j} = \frac{1}{J}\mathcal{J}(\ddot{f}_i, f_i, f_h) \quad , \quad (12a)$$

$$\begin{aligned}\Lambda - 4\pi G\varrho &= \frac{1}{2J}\varepsilon_{jpq}\mathcal{J}(\ddot{f}_j, f_p, f_q) \\ &- \frac{\alpha'}{2J^3\varrho}\mathcal{J}(\varrho, f_p, f_q)\mathcal{J}(J, f_p, f_q) \\ &- \frac{\alpha'}{2J^2\varrho^2}\mathcal{J}(\varrho, f_p, f_q)\mathcal{J}(\varrho, f_p, f_q) \\ &+ \frac{\alpha'}{2J^2\varrho}\mathcal{J}(\mathcal{J}(\varrho, f_p, f_q), f_p, f_q) \\ &+ \frac{\alpha''}{2J^2\varrho}\mathcal{J}(\varrho, f_p, f_q)\mathcal{J}(\varrho, f_p, f_q) \quad . \end{aligned} \quad (12b)$$

This system of equations can be closed with the help of the exact integral (9).

2.4. Annotations

Substituting the integral (9) into (12b) we obtain:

$$\begin{aligned}&\Lambda - 4\pi G\frac{C}{J} \\ &= \frac{1}{2J}\varepsilon_{jpq}\mathcal{J}(\ddot{f}_j, f_p, f_q) \\ &+ \frac{\alpha'}{2CJ^2}\mathcal{J}(\mathcal{J}(C, f_p, f_q), f_p, f_q) \\ &- \frac{\alpha'}{2J^3}\mathcal{J}(\mathcal{J}(J, f_p, f_q), f_p, f_q) \\ &+ \frac{1}{2CJ^3}\mathcal{J}(C, f_p, f_q)\mathcal{J}(C, f_p, f_q)\left(\alpha'' - \frac{J}{C}\alpha'\right) \\ &- \frac{1}{2J^4}\mathcal{J}(C, f_p, f_q)\mathcal{J}(J, f_p, f_q)\left(2\alpha'' + \frac{J}{C}\alpha'\right) \\ &+ \frac{C}{2J^5}\mathcal{J}(J, f_p, f_q)\mathcal{J}(J, f_p, f_q)\left(\alpha'' + \frac{2J}{C}\alpha'\right)\ . \end{aligned} \quad (13)$$

Let us now introduce the following approximation which is common in cosmology: consider the definition $\varrho(\boldsymbol{X}) = \varrho_{\rm H}(1 + \delta(\boldsymbol{X}))$, where $\varrho_{\rm H}$ denotes the background density of the mean matter distribution and δ the density contrast. At the time $t = t_0$ we have $\varrho_0(\boldsymbol{X}) = \varrho_{\rm H0}(1 + \delta_0(\boldsymbol{X}))$. As the density contrast is numerically very small at the time of recombination (i.e. $t_0 = t_{rec}$), $\varrho_0 = \varrho_{\rm H0}$ is a useful approximation. Therefore we get $C_{|i} = (\varrho_0 J_0)_{|i} =: C_{{\rm H}|i} = 0$ (for an alternative exact argumentation see Appendix A), and we use $C_{\rm H}$ instead of C furtheron to emphasize its homogeneity. The Lagrange–Newton–System then may be written in a much simpler form, since all functional determinants with C disappear:

$$\mathcal{J}(\ddot{f}_i, f_i, f_h) = 0 \quad , \quad (14a)$$

$$\begin{aligned}&\Lambda - 4\pi G\frac{C_{\rm H}}{J} \\ &= \frac{1}{2J}\varepsilon_{jpq}\mathcal{J}(\ddot{f}_j, f_p, f_q) \\ &- \frac{\alpha'}{2J^3}\mathcal{J}(\mathcal{J}(J, f_p, f_q), f_p, f_q) \\ &+ \frac{C_{\rm H}}{2J^5}\mathcal{J}(J, f_p, f_q)\mathcal{J}(J, f_p, f_q)\left(\alpha'' + \frac{2J}{C_{\rm H}}\alpha'\right)\ . \end{aligned} \quad (14b)$$

Looking at Eq. (12b) we conclude that a (not necessarily physically relevant) possibility that simplifies Eq. (14b) further is the following:

$$\alpha'' = -\frac{2J}{C_{\rm H}}\alpha' = -\frac{2}{\varrho}\alpha'\ , \quad \text{i.e.}\ , \quad p = c_1\frac{1}{\varrho} + c_2 \quad . \quad (15)$$

This "equation of state" plays a special role as will become clear in Sect. 3.2.

2.5. Reduction to planar symmetry

In order to learn more about the structure of Eqs. (12a, 12b) it is useful to look at restrictions in symmetry: we restrict $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t)$ to planar symmetry, i.e.,[1]

$$f_1 = f_1(X_1, t) \qquad f_2 = X_2 \quad , \quad f_3 = X_3 \quad ; \quad (16)$$

the only direction in which motion takes place is the X_1–direction; f_2 and f_3 are constant. Consequently, there are no velocities and accelerations in the X_2– and X_3–directions (in particular $g_2 = g_3 = 0$). So, J simplifies to

$$J = \frac{\partial x_1}{\partial X_1} = f_{1|1} \quad .$$

The four field equations that we have in 3D are reduced to the equation for the divergence of g_1,

$$\Lambda - 4\pi G\varrho = \Lambda - 4\pi G\frac{\varrho_0}{f_{1|1}} = g_{i,i} = \frac{g_{1|1}}{f_{1|1}} \quad ;$$

the curl of $\boldsymbol{g}$ vanishes identically. With the help of

$$g_1 = b_1 + \frac{p_{,1}}{\varrho} = \ddot{f}_1 + \frac{p_{|1}}{\varrho f_{1|1}} = \ddot{f}_1 + \frac{p_{|1}}{\varrho_0}$$

[1] In this subsection we use $J_0 = 1$, hence $C(\boldsymbol{X}) = \varrho_0(\boldsymbol{X})$.

we get

$$\Lambda f_{1|1} - 4\pi G\varrho_0 = \ddot{f}_{1|1} + (\frac{p_{|1}}{\varrho_0})_{|1} \quad .$$

Let us define $G_{1|1} := \Lambda - 4\pi G\varrho_0$ ($G(X_1) := g(X_1,t_0)$) in order to cast the last equation into a total divergence,

$$(\ddot{f}_1 - \Lambda f_1 - G_1 + \Lambda X_1 + \frac{p_{|1}}{\varrho_0})_{|1} = 0 \quad ;$$

hence,

$$\ddot{f}_1 - \Lambda f_1 = G_1 - \Lambda X_1 - \frac{p_{|1}}{\varrho_0} \quad ,$$

neglecting an irrelevant function of time. With the relation $p = \alpha(\varrho)$ the pressure–force term becomes

$$\frac{p_{|1}}{\varrho_0} = \frac{\alpha'}{\varrho_0}\left(\frac{\varrho_0}{f_{1|1}}\right)_{|1} = \alpha'\left(\frac{G_{1|11}}{(G_{1|1}-\Lambda)f_{1|1}} - \frac{f_{1|11}}{f_{1|1}^2}\right) \quad .$$

Altogether we arrive at

$$\ddot{f}_1 - \Lambda f_1 \;=\; G_1 - \Lambda X_1 + \alpha'\left(\frac{f_{1|11}}{f_{1|1}^2} - \frac{G_{1|11}}{(G_{1|1}-\Lambda)f_{1|1}}\right) \; , \quad (17)$$

or (with ignorance of dimensions)

$$\ddot{f}_1 - \Lambda f_1 = G_1 - \Lambda X_1 + \frac{\alpha'}{f_{1|1}}\left(\ln\left(\frac{f_{1|1}}{G_{1|1}-\Lambda}\right)\right)_{|1} \quad . \quad (18)$$

For the case $\Lambda = 0$ and $\alpha' = const.$ Götz (1988) has shown that (18) can be mapped to the Sine–Gordon equation, which is a well–studied equation that admits soliton solutions.

3. Lagrangian perturbation approach

3.1. The perturbation ansatz and linearization

In standard cosmology we invoke a homogeneous deformation of the continuum: $f_{\mathrm{H}i} = a_{ij}(t)X_j$ that is isotropic: $a_{ij}(t) = a(t)\delta_{ij}$; thus, with $\boldsymbol{f}_{\mathrm{H}} = a(t)\boldsymbol{X}$ and the homogeneous density $\varrho_{\mathrm{H}} := \frac{C_{\mathrm{H}}}{a^3}$ applied to the Lagrange–Newton–System with pressure we are left with

$$3\frac{\ddot{a}}{a} = \Lambda - 4\pi G\frac{C_{\mathrm{H}}}{a^3} \quad . \quad (19)$$

The first integral of this equation is Friedmann's equation:

$$\frac{\dot{a}^2 + const.}{a^2} = \frac{8\pi G\frac{C_{\mathrm{H}}}{a^3} + \Lambda}{3} \quad . \quad (20)$$

In order to describe structure formation we consider small ($\mathcal{O}(\varepsilon)$) deviations $\boldsymbol{p}(\boldsymbol{X},t)$ from this homogeneous and isotropic motion; we use the ansatz

$$\boldsymbol{f}(\boldsymbol{X},t) \;=\; a(t)\boldsymbol{X} + \boldsymbol{p}(\boldsymbol{X},t) \quad , \quad (21)$$

and suppose that perturbation theory is justifiable. The assumption that the perturbations are smooth leads us to $|p_{i|j}| \ll a$ and $|\ddot{p}_{i|j}| \ll \ddot{a}$. Inserting our ansatz (21) into the first equations (14a) we have

$$\begin{aligned} 0 \;&=\; (\boldsymbol{\nabla}\times\boldsymbol{g})_h = \frac{1}{J}\mathcal{J}(\ddot{f}_i, f_i, f_h) \\ &=\; \mathcal{J}(\ddot{a}X_i + \ddot{p}_i, aX_i + p_i, aX_h + p_h) \\ &=\; \ddot{a}a\mathcal{J}(X_i, p_i, X_h) + \ddot{a}\mathcal{J}(X_i, p_i, p_h) + a^2\mathcal{J}(\ddot{p}_i, X_i, X_h) \\ &+\; a\mathcal{J}(\ddot{p}_i, X_i, p_h) + a\mathcal{J}(\ddot{p}_i, p_i, X_h) + \mathcal{J}(\ddot{p}_i, p_i, p_h) \, . \end{aligned} \quad (22)$$

In the first–order approximation we neglect quadratic and cubic terms, hence:

$$\ddot{a}a\mathcal{J}(X_i, p_i, X_h) + a^2\mathcal{J}(\ddot{p}_i, X_i, X_h) \;=\; 0 \;\; \text{or equivalently,}$$

$$\boldsymbol{\nabla_0}\times\left(\ddot{\boldsymbol{p}} - \frac{\ddot{a}}{a}\boldsymbol{p}\right) \;=\; \boldsymbol{0} \quad . \quad (23a)$$

Linearizing the equation for the divergence of $\boldsymbol{g}$ we start with (14b), i.e., with the assumption $C = C_{\mathrm{H}}$ (compare Appendix A). The following terms appear during the calculation:

$$\begin{aligned} J \;&=\; a^3 + a^2 p_{i|i} + \mathcal{O}(\varepsilon^2) \\ \frac{1}{J} \;&=\; \frac{1}{a^3(1+\frac{1}{a}p_{i|i}) + \mathcal{O}(\varepsilon^2)} \\ &=\; \frac{1}{a^3}(1 - \frac{1}{a}p_{i|i}) + \mathcal{O}(\varepsilon^2) \\ \frac{1}{J^n} \;&=\; \frac{1}{a^{3n}}(1 - \frac{n}{a}p_{i|i}) + \mathcal{O}(\varepsilon^2) \\ \varepsilon_{jpq}\mathcal{J}(\ddot{f}_j, f_p, f_q) \;&=\; 6\ddot{a}a^2 + 4\ddot{a}ap_{i|i} + 2a^2\ddot{p}_{i|i} + \mathcal{O}(\varepsilon^2) \\ \mathcal{J}(J, f_p, f_q) \;&=\; \varepsilon_{apq}a^4 p_{i|ia} + \mathcal{O}(\varepsilon^2) \\ \mathcal{J}(\mathcal{J}(J, f_p, f_q), f_p, f_q) \;&=\; 2a^6 p_{i|ijj} + \mathcal{O}(\varepsilon^2) \\ \mathcal{J}(J, f_p f_q)\mathcal{J}(J, f_p, f_q) \;&=\; \mathcal{O}(\varepsilon^2) \end{aligned}$$

(remember: $\varepsilon_{abc}\varepsilon_{abc} = 6$ and $\varepsilon_{abc}\varepsilon_{abd} = 2\delta_{cd}$). In the end we find:

$$\begin{aligned} 0 \;=\; &-\Lambda + 4\pi G\frac{C_{\mathrm{H}}}{a^3} - 4\pi G\frac{C_{\mathrm{H}}}{a^4}p_{i|i} + 3\frac{\ddot{a}}{a} - \frac{\ddot{a}}{a^2}p_{i|i} + \frac{1}{a}\ddot{p}_{i|i} \\ &-\frac{\alpha'}{a^3}p_{i|ijj} + \mathcal{O}(\varepsilon^2) \quad . \end{aligned} \quad (23b)$$

(The calculation without using $C = C_{\mathrm{H}}$ may be found in (Adler 1998).) In comparison to the matter model "dust" (Buchert 1992) there is only one additional term $\frac{\alpha'}{a^3}p_{i|ijj} = \frac{\alpha'}{a^3}\Delta_{\boldsymbol{0}}(p_{i|i})$. Inserting the zero–order equation (19) (thus neglecting backreaction of the first–order displacements on the background solution) yields

$$0 = -4\pi G\frac{C_{\mathrm{H}}}{a^4}p_{i|i} - \frac{\ddot{a}}{a^2}p_{i|i} + \frac{1}{a}\ddot{p}_{i|i} - \frac{\alpha'}{a^3}p_{i|ijj} \quad .$$

Introducing the comoving displacement field $\boldsymbol{P} := \frac{\boldsymbol{p}}{a(t)}$ representing the inhomogeneous deformation that is scaled with the expansion, we get:

$$0 = -4\pi G\frac{C_{\mathrm{H}}}{a^3}P_{i|i} + \ddot{P}_{i|i} + 2\frac{\dot{a}}{a}\dot{P}_{i|i} - \frac{\alpha'}{a^2}P_{i|ijj} \quad .$$

This equation can be written as a total divergence:

$$\boldsymbol{\nabla_0}\cdot\left(\ddot{\boldsymbol{P}} + 2\frac{\dot{a}}{a}\dot{\boldsymbol{P}} - 4\pi G\frac{C_{\mathrm{H}}}{a^3}\boldsymbol{P} - \frac{\alpha'}{a^2}\Delta_0\boldsymbol{P}\right) = 0 \quad .$$

Writing Eq. (23a) also in terms of the scaled displacement field, and also inserting the zero–order solution, we arrive at our main result:
For vanishing harmonic parts (see Ehlers & Buchert 1997) we obtain the following final set of equations that comprises the linear perturbation theory for the scaled displacement field $\boldsymbol{P} = \boldsymbol{P}^L + \boldsymbol{P}^T$ ($\boldsymbol{\nabla_0} \times \boldsymbol{P}^L = \boldsymbol{0}$; $\boldsymbol{\nabla_0} \cdot \boldsymbol{P}^T = 0$) :

$$\ddot{\boldsymbol{P}}^T + 2\frac{\dot{a}}{a}\dot{\boldsymbol{P}}^T \;=\; \boldsymbol{0} \tag{24a}$$

$$\ddot{\boldsymbol{P}}^L + 2\frac{\dot{a}}{a}\dot{\boldsymbol{P}}^L - 4\pi G\frac{C_{\rm H}}{a^3}\boldsymbol{P}^L \;=\; \frac{\alpha'}{a^2}\boldsymbol{\Delta_0}\boldsymbol{P}^L \quad . \tag{24b}$$

Remark: Remember that $\alpha' = \alpha'(\varrho)$ where ϱ depends on the deformation $\boldsymbol{P}$; thus, the r.h.s. of Eq. (24b) is not linear in $\boldsymbol{P}^L$. So, for given $p = \alpha(\varrho)$, $\alpha'\boldsymbol{\Delta_0}\boldsymbol{P}^L$ has to be linearized with respect to $\boldsymbol{P}^L$ in the context of linear perturbation theory, i.e. $\alpha' = const$. This fact amounts to a major difference compared to the "dust" case: in the latter, the dependence on ϱ completely disappears in the equations and the exact density may be obtained for any perturbative solution. Here, the density is implicitly linearized and this fact has already been used during the above derivation.

3.2. Comparison with plane–symmetric inhomogeneities on a Friedmann background universe

Let us start with trajectories, which are scaled with the expansion $\boldsymbol{F} = \frac{\boldsymbol{f}}{a(t)}$. Reduction to planar symmetry now means:

$$F_1 = X_1 + P_1(X_1,t)\, , \; F_2 = X_2\, , \; F_3 = X_3 \quad . \tag{25}$$

The functional determinant J reduces to

$$J = a^3(1+P_{1|1}) \quad ,$$

and, thus, J is independent of the coordinates X_2 und X_3. With the simplification $C = C_{\rm H}$ we derive from (14b)

$$\begin{aligned}\Lambda - 4\pi G\frac{C_{\rm H}}{J} \;&=\; \frac{2a^2\ddot{a}}{J}(1+P_{1|1}) \\ &+\; \frac{a^2}{J}(\ddot{a}(1+P_{1|1}) + 2\dot{a}\dot{P}_{1|1} + a\ddot{P}_{1|1}) \\ &-\; \frac{\alpha'}{J^3}a^4 J_{|11} + \frac{C_{\rm H}}{J^5}(\alpha'' + \frac{2J}{C_{\rm H}}\alpha')a^4 J_{|1}J_{|1}\, ,\end{aligned}$$

i.e.,

$$\begin{aligned}\Lambda \;-\; 4\pi G\frac{C_{\rm H}}{a^3(1+P_{1|1})} &= \\ 3\frac{\ddot{a}}{a} + 2\frac{\dot{a}}{a}\frac{\dot{P}_{1|1}}{(1+P_{1|1})} &+ \frac{\ddot{P}_{1|1}}{1+P_{1|1}} - \frac{\alpha'}{a^2(1+P_{1|1})^3}P_{1|111} \\ &+ \frac{C_{\rm H}}{a^5(1+P_{1|1})^5}(\alpha'' + \frac{2(1+P_{1|1})a^3}{C_{\rm H}}\alpha')P_{1|11}P_{1|11}\, .\end{aligned}$$

Using the zero–order Eq. (19) and multiplying with $(1+P_{1|1})$ yields

$$\left(\ddot{P}_1 + 2\frac{\dot{a}}{a}\dot{P}_1 - 4\pi G\frac{C_{\rm H}}{a^3}P_1\right)_{|1} = \frac{\alpha'}{a^2(1+P_{1|1})^2}P_{1|111}$$

$$-\frac{C_{\rm H}}{a^5(1+P_{1|1})^4}(\alpha'' + \frac{2(1+P_{1|1})a^3}{C_{\rm H}}\alpha')P_{1|11}P_{1|11} \quad .$$

The r.h.s. of this equation can be written as a total derivative; hence we get for vanishing harmonic parts,

$$\ddot{P}_1 + 2\frac{\dot{a}}{a}\dot{P}_1 - 4\pi G\frac{C_{\rm H}}{a^3}P_1 = \frac{\alpha'}{a^2(1+P_{1|1})^2}P_{1|11} \quad . \tag{26}$$

By linearizing this last equation we recover Eq. (24b) for one-dimensional inhomogeneities. Writing the coefficient on the r.h.s. as a function of ϱ,

$$\chi(\varrho) = \frac{\alpha'(\varrho)}{a^2(1+P_{1|1})^2} = \frac{\alpha'(\varrho)}{a^2}\left(\frac{\varrho}{\varrho_{\rm H}}\right)^2 \quad ,$$

we see that we get a linear equation in the case of the special "equation of state" (15).

Let us finally restrict our problem to the limit of no expansion $a(t) = a(t_0)$. We have $\varrho_{\rm H} = C_{\rm H}$ and $f_1 = a(t_0)(X_1 + P_1)$ from (25). Therefore,

$$\ddot{f}_1 - 4\pi G\frac{C_{\rm H}}{a(t_0)^3}(f_1 - a(t_0)X_1) = \frac{\alpha'}{f_{1|1}^2}f_{1|11} \quad .$$

If we additionally appreciate that (19) now reads $\Lambda = 4\pi G C_{\rm H}$ and $G_{1|1} = \Lambda - 4\pi G\varrho_{\rm H0} = 0$, we recover Eq. (18) in the limit of the static Einstein universe.

3.3. Comparison with Eulerian linear theory

It is a standard procedure in cosmology to scale the Eulerian coordinates with the homogeneous and isotropic expansion ($\boldsymbol{v}_{\rm H} = H\boldsymbol{x}$ with Hubble's function $H := \frac{\dot{a}}{a}$), i.e. to use the following transformation of coordinates

$$\boldsymbol{q} = \boldsymbol{Q}(\boldsymbol{x},t) = \frac{\boldsymbol{x}}{a(t)} \quad . \tag{27}$$

The comoving differential operators $(\boldsymbol{\nabla_q})_i = \frac{\partial}{\partial q_i}$ and $\frac{\partial}{\partial t}\;|_{\boldsymbol{q}}$ become

$$\boldsymbol{\nabla} = \frac{1}{a(t)}\boldsymbol{\nabla_q}$$

$$\frac{\partial}{\partial t}\Big|_{\boldsymbol{x}} = \frac{\partial}{\partial t}\Big|_{\boldsymbol{q}} - H\boldsymbol{q}\,\boldsymbol{\nabla_q} \quad .$$

The deviations from the homogeneous fields are called

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{q},t) \;&:=\; \boldsymbol{v}(\boldsymbol{q},t) - \boldsymbol{v}_{\rm H}(\boldsymbol{q},t) \quad && \text{peculiar-velocity,} \\ \boldsymbol{w}(\boldsymbol{q},t) \;&:=\; \boldsymbol{g}(\boldsymbol{q},t) - \boldsymbol{g}_{\rm H}(\boldsymbol{q},t) \quad && \text{peculiar-acceleration and} \\ \delta(\boldsymbol{q},t) \;&:=\; \frac{\varrho(\boldsymbol{q},t) - \varrho_{\rm H}(t)}{\varrho_{\rm H}(t)} \quad && \text{density contrast,}\end{aligned}$$

where $\boldsymbol{v}_{\rm H} = \dot{a}\boldsymbol{q}$ and $\boldsymbol{g}_{\rm H} = \ddot{a}\boldsymbol{q}$ denote the Hubble–velocity and the Hubble–acceleration. We obtain the Euler–Newton–System with pressure for the peculiar quantities by making use of the homogeneous solutions (see Peebles 1980):

$$\dot{\delta} + \frac{1}{a}(1+\delta)\boldsymbol{\nabla_q}\cdot\boldsymbol{u} \;=\; 0 \quad , \tag{28a}$$

$$\dot{\boldsymbol{u}} + H\boldsymbol{u} \;=\; \boldsymbol{w} - \frac{1}{a}\frac{\boldsymbol{\nabla_q}\, p}{\varrho_{\rm H}(1+\delta)} \quad , \tag{28b}$$

$$\boldsymbol{\nabla_q} \times \boldsymbol{w} \;=\; \boldsymbol{0} \quad , \tag{28c}$$

$$\boldsymbol{\nabla_q} \cdot \boldsymbol{w} \;=\; -4\pi G \varrho_{\rm H} \delta a \quad . \tag{28d}$$

In the following we assume that all peculiar quantities and their derivatives are small ($\mathcal{O}(\varepsilon)$). Using perturbation theory to the first order in ε we get the linearized form of the equations:

$$\frac{\partial}{\partial t}\Big|_{\boldsymbol{q}} \delta + \frac{1}{a}\boldsymbol{\nabla_q} \cdot \boldsymbol{u} \;=\; \mathcal{O}(\varepsilon^2) \tag{29a}$$

$$\frac{\partial}{\partial t}\Big|_{\boldsymbol{q}} \boldsymbol{u} + H\boldsymbol{u} \;=\; \boldsymbol{w} - \frac{1}{a}\frac{\boldsymbol{\nabla_q}\, p}{\varrho_{\rm H}} + \mathcal{O}(\varepsilon^2) \tag{29b}$$

$$\boldsymbol{\nabla_q} \times \boldsymbol{w} \;=\; \boldsymbol{0} \tag{29c}$$

$$\boldsymbol{\nabla_q} \cdot \boldsymbol{w} \;=\; -4\pi G \varrho_{\rm H} \delta a \quad . \tag{29d}$$

The linearized equation for the evolution of δ can be calculated by applying $-\frac{1}{a}\boldsymbol{\nabla_q}$ to the second equation and inserting the first and forth equations:

$$\frac{\partial^2}{\partial t^2}\Big|_{\boldsymbol{q}} \delta + 2{\rm H}\frac{\partial}{\partial t}\Big|_{\boldsymbol{q}} \delta - 4\pi G \varrho_{\rm H} \delta = \frac{\Delta_{\boldsymbol{q}}\, p}{a^2 \varrho_{\rm H}} = \left(\frac{c_s}{a}\right)^2 \Delta_{\boldsymbol{q}}\, \delta \,; \tag{30}$$

where we have set $p = \alpha(\varrho)$, and $\alpha' = \frac{\partial p}{\partial \varrho} =: c_s^2$ is defined in terms of the "speed of sound" c_s as is usual in a hydrodynamical medium. Here, we have $\alpha' = const.$, i.e., $p \propto \varrho$, in order to obtain a linear equation in δ.

4. Discussion of results and outlook

In the last section we have derived three different equations:

– From the linearization of the Lagrange–Newton–System with pressure we have for the longitudinal part of the inhomogeneous deformation $\boldsymbol{P}(\boldsymbol{X},t) = \frac{\boldsymbol{p}(\boldsymbol{X},t)}{a(t)}$:

$$\ddot{\boldsymbol{P}}^L + 2\frac{\dot a}{a}\dot{\boldsymbol{P}}^L - 4\pi G \varrho_{\rm H} \boldsymbol{P}^L = \frac{\alpha'}{a^2}\boldsymbol{\Delta_0} \boldsymbol{P}^L \quad ; \tag{31a}$$

remember $\varrho_{\rm H} := \frac{C_{\rm H}}{a^3}$.

– The plane–symmetric solutions on a Friedmannian background obey:

$$\ddot P_1 + 2\frac{\dot a}{a}\dot P_1 - 4\pi G \varrho_{\rm H} P_1 = \frac{\alpha'}{a^2(1+P_{1|1})^2} P_{1|11} \quad . \tag{31b}$$

– The Eulerian linear theory in comoving coordinates $\boldsymbol{q} = \frac{\boldsymbol{x}}{a(t)}$ leads to a partial differential equation for the density contrast $\delta(\boldsymbol{q},t)$:

$$\frac{\partial^2}{\partial t^2}\Big|_{\boldsymbol{q}} \delta + 2\frac{\dot a}{a}\frac{\partial}{\partial t}\Big|_{\boldsymbol{q}} \delta - 4\pi G \varrho_{\rm H} \delta = \frac{\alpha'}{a^2}\Delta_{\boldsymbol{q}} \delta \quad . \tag{31c}$$

According to the equivalence of the equations (31a) and (31c) up to the time derivative operators $\frac{\rm d}{{\rm d}t} = \frac{\partial}{\partial t}|_{\boldsymbol{X}}$ and $\frac{\partial}{\partial t}|_{\boldsymbol{q}}$, we see that, with the help of the already known results for the density contrast in the Eulerian linear theory (e.g. Haubold et al. 1991), solutions for the inhomogeneous deformation $\boldsymbol{P}^L$ in the Lagrangian linear theory can be constructed. In the case $p = 0$ a class of exact 3D solutions has been found with the help of this method (see: Buchert 1989). But, in contrast to the case $p = 0$, this is not to be expected here, since extrapolation of the solution of the linearized equations does not yield an exact solution of the planar problem as well; this is easy to see from Eq. (31b): the pressure term produces (except in the special case (15)) nonlinear terms in $\boldsymbol{P}$ already for plane–symmetric inhomogeneities.

We emphasize the special role played by the "equation of state" (15): $p = \frac{c_1}{\varrho} + c_2$.

For further applications we can proceed as follows: We look at solutions $\delta^\ell(\boldsymbol{q},t)$ of the differential equation (31c) and use the instruction $\boldsymbol{q} \;\longmapsto\; \boldsymbol{X}$, $\delta^\ell \longrightarrow P_i^{\rm L}$ to construct solutions $P_i^{\rm L}(\boldsymbol{X},t)$ of the differential equations (31a). As the Lagrangian description implicitly respects nonlinearities, our construction rule allows to build nonlinear models of structure formation that take dynamical pressure forces into account. Since solutions to Eq. (31a) constitute the Lagrangian extrapolation of the Eulerian linear theory, they are built in the same spirit as Zel'dovich's approximation in the case $p = 0$ (see: Zel'dovich 1970, 1973; Buchert 1989). Viewed together with the derivation of the "adhesion approximation" by Buchert & Domínguez (1998), these solutions may be used as first approximations to adhesive gravitational clustering in the weakly nonlinear regime.

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Appendix A: alternative argument for the approximation $C_{|i} = 0$

We want to give an alternative argumentation for the approximation $C_{|i} = (\varrho_0 J_0)_{|i} = 0$. Let us introduce new curvilinear coordinates $\boldsymbol{Y}$ by

$$\boldsymbol{Y} := \boldsymbol{A}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{\Psi}(\boldsymbol{X}) \quad . \tag{A1}$$

Below we shall consider (initially) small deviations $\boldsymbol{\Psi}$ from $\boldsymbol{X}$, restricting our argument to the linear approximation. We choose the new coordinates in such a way that

$$C(\boldsymbol{Y}) = \varrho_0(\boldsymbol{Y}) J_0(\boldsymbol{Y}) = const. =: C_{\rm H} \,; \quad C_{\rm H} = \varrho_{\rm H0} \quad .$$

Thus, taking the conservation of mass into account, we have

$${\rm d}m_0 = \varrho_0(\boldsymbol{X}){\rm d}^3 X = \varrho_{\rm H0}{\rm d}^3 Y \quad .$$

Writing $\varrho_0(\boldsymbol{X}) = \varrho_{\rm H0}(1+\delta_0(\boldsymbol{X}))$, the map $\boldsymbol{A}$ is defined by

$$(1+\delta_0(\boldsymbol{X}))\det\left(\frac{\partial X_i}{\partial Y_j}\right) = 1 \quad .$$

Then we get for $|\boldsymbol{\Psi}| = \mathcal{O}(\varepsilon)$ and small derivatives:

$$1 + \delta_0(\boldsymbol{X}) = \det\left(\frac{\partial Y_j}{\partial X_i}\right) = \det\left(\delta_{ij} + \frac{\partial \Psi_j}{\partial X_i}\right)$$

$$= 1 + \boldsymbol{\nabla_0} \cdot \boldsymbol{\Psi}(\boldsymbol{X}) + \mathcal{O}(\varepsilon^2) \quad .$$

Thus, to first order in ε we have:

$$\delta_0(\boldsymbol{X}) = \boldsymbol{\nabla_0} \cdot \boldsymbol{\Psi}(\boldsymbol{X}) \quad ,$$

the particles are initially displaced according to the density contrast. With $\boldsymbol{\nabla_0} \cdot \boldsymbol{W} = -4\pi G \varrho_{\rm H0} \delta_0 a(t_0)$ ((29d) at the time $t = t_0$) $\boldsymbol{\Psi}$ is defined with the help of the peculiar–acceleration $\boldsymbol{w}$ at time t_0:

$$\boldsymbol{\nabla_0} \cdot \boldsymbol{\Psi} = -\frac{1}{4\pi G \varrho_{\rm H0} a(t_0)} \boldsymbol{\nabla_0} \cdot \boldsymbol{W} \; , \; \boldsymbol{W} := \boldsymbol{w}(\boldsymbol{X}, t_0) \; . \quad \text{(A2)}$$

To enforce the property $C_{|i} = 0$, we simply have to relabel the trajectories:

$$\boldsymbol{X} \longrightarrow \boldsymbol{Y} , \quad \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{Y}, t) \text{ with } \boldsymbol{x}|_{t_0} = \boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{\Psi}(\boldsymbol{X}) \quad .$$

Note that Zel'dovich's original discussion of his approximation (Zel'dovich 1970, 1973) as well as subsequent work were all using the coordinates $\boldsymbol{Y}$. We stress, however, that the above argument also involves an approximation and is only consistent within the first–order solutions; compare the discussion in (Buchert 1989).

Appendix B: transformed vector–identities

The following equations comprise a useful collection of formulas; similar expressions may arise in calculations employing Lagrangian coordinates. We list these transformed vector–identities here, because they may be helpful for further considerations.

$$\begin{aligned} 0 &= (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} k))_h = \frac{1}{J} \mathcal{J}(k_{,i}, f_i, f_h) \\ &= \frac{1}{2J^2} \varepsilon_{ipq} \mathcal{J}(\mathcal{J}(k, f_p, f_q), f_i, f_h) \\ &- \frac{1}{2J^3} \varepsilon_{ipq} \mathcal{J}(k, f_p, f_q) \mathcal{J}(J, f_i, f_h) \quad ; \end{aligned}$$

$$\begin{aligned} 0 &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{T}) = (\frac{1}{J} \mathcal{J}(T_p, f_p, f_q))_{,q} \\ &= \frac{1}{2J^2} \varepsilon_{ihq} \mathcal{J}(\mathcal{J}(T_p, f_p, f_q), f_i, f_h) \\ &- \frac{1}{2J^3} \varepsilon_{ihq} \mathcal{J}(T_p, f_p, f_q) \mathcal{J}(J, f_i, f_h) \quad ; \end{aligned}$$

$$\begin{aligned} 0 &= (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{T}) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{T}) + \Delta \boldsymbol{T})_h \\ &= (\frac{1}{J} \varepsilon_{hji} \mathcal{J}(T_r, f_r, f_i))_{,j} \\ &- (\frac{1}{2J} \varepsilon_{jpq} \mathcal{J}(T_j, f_p, f_q))_{,h} \\ &+ (\frac{1}{2J} \varepsilon_{ipq} \mathcal{J}(T_h, f_p, f_q))_{,i} \\ &= \frac{1}{J^2} \mathcal{J}(\mathcal{J}(T_r, f_r, f_i), f_i, f_h) \\ &- \frac{1}{J^3} \mathcal{J}(T_r, f_r, f_i) \mathcal{J}(J, f_i, f_h) \\ &- \frac{1}{4J^2} \varepsilon_{hrt} \varepsilon_{jpq} \mathcal{J}(\mathcal{J}(T_j, f_p, f_q), f_r, f_t) \\ &+ \frac{1}{4J^3} \varepsilon_{hrt} \varepsilon_{jpq} \mathcal{J}(J, f_r, f_t) \mathcal{J}(T_j, f_p, f_q) \\ &+ \frac{1}{2J^2} \mathcal{J}(\mathcal{J}(T_h, f_r, f_t), f_r, f_t) \\ &- \frac{1}{2J^3} \mathcal{J}(J, f_r, f_t) \mathcal{J}(T_h, f_r, f_t) \quad . \end{aligned}$$

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