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# Pseudo-Newtonian models of a rotating black hole field

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**Abstract.** A pseudo-Newtonian description of the gravitational field which yields the equations of motion resembling, as closely as possible, the geodesic equation of general relativity is found for the Kerr spacetime. The potential obtained consists of three parts, interpreted as a purely Newtonian (gravitoelectric) term, a dragging (gravitomagnetic) term, and a space-geometry correction. The accuracy of the pseudo-Newtonian model is studied by a method which compares, systematically, two large sets of trajectories: geodesics in the Kerr spacetime versus test-particle trajectories in the pseudo-Newtonian field. It is suggested that every pseudo-Newtonian model should be submitted to analogous systematic analysis before it is used in astrophysical applications. A modified Newtonian potential which accounts for the frame-dragging effects can be a practical tool in studying stationary accretion discs. Non-stationary configurations are more complicated and we will not suggest to use this approach to such topics in accretion theory as gravitomagnetic oscillations of discs and their relation to quasi-periodic sources.

**Key words:** accretion, accretion discs – black hole physics

## 1. Introduction

Most of theoretical astronomy uses Newtonian theory of gravitation, considering that the effects of general relativity are weak for most astronomical objects. Even in situations where these effects do become important or even dominant, pseudo-Newtonian models have often been applied which aspire to mimic the corresponding relativistic situations. Pseudo-Newtonian approaches aim to save numerical work, to enable analytical solutions where fully relativistic treatment is too cumbersome, and to provide new insights into the relativity theory and its implications.

The present contribution concerns pseudo-Newtonian models developed in order to describe very compact objects which presumably reside in the nuclei of galaxies and in some X-ray binaries (e.g., Rees 1998). In these sources, a key feature is an accretion disc around a rotating black hole. The actual accretion flows are most likely non-stationary, non-axisymmetric, and described by complex local physics. For the sake of simplicity, however, a standard model of disc accretion (see Kato et al. (1998) for a textbook exposition of the accretion theory including its recent advances) provides an ingenious approximation: the standard disc is smooth, axially symmetric, geometrically thin, and characterized by three parameters. It is astrophysically realistic only in a restricted range of its parameters. The pseudo-Newtonian approach has been devised in order to introduce effects of general relativity into accretion models, especially in the case of geometrically thick discs (Paczyński & Wiita 1980). The black hole is treated there as a Newtonian body whose gravitational field is determined by the potential $V_{\rm PW}=-M/(r-2M)$ (geometrized units with $c=G=1$ will be used throughout the paper). This simple expression mimics the gravitational field of a *non-rotating* (Schwarzschild) black hole quite accurately, in particular, it reproduces correctly the radii of the marginally stable and the marginally bound circular orbits of free test particles, $r_{\rm ms}=6M$ and $r_{\rm mb}=4M$. Recently, Abramowicz et al. (1996) proposed a certain rescaling of velocities in the field of $V_{\rm PW}$ to obtain better agreement with corresponding relativistic values. In this way, calculations of the observed spectra of the discs could also be improved. It has been argued, however, both on theoretical grounds (Bardeen 1970) and from observations (Iwasawa et al. 1996; Karas & Kraus 1996), that the central black holes in galactic nuclei would be more likely to rotate rapidly. Thus the main feature that should be included in the pseudo-Newtonian models is the rotation of the central object. In addition, the rotation-induced dragging must be taken into account when studying non-equatorial warped discs around compact objects (Bardeen & Petterson 1975).

The pseudo-Newtonian potential for a non-rotating black hole has been used in numerous works and we recall at least a few of them. For example, Abramowicz et al. (1988) used $V_{\rm PW}$ in their introductory work on slim discs. Okazaki et al. (1987) adopted this approach to describe global trapped oscillations of relativistic accretion discs. Later, Nowak & Wagoner (1991) devised another form of the potential which is better suited for their purpose because it reproduces the epicyclic frequency $\kappa$ with higher accuracy than $V_{\rm PW}$. Szuszkiewicz & Miller (1998) studied the limit-cycle behaviour of thermally unstable flows, also with the help of the pseudo-Newtonian description. On quite a

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different subject, Daigne & Mochkovitch (1997) applied $V_{\rm PW}$ to study the runaway instability in accretion discs which could trigger gamma ray bursts. Very recently, Ruffert & Janka (1998) used $V_{\rm PW}$ in a detailed study of neutron-star mergers and related production of the bursts. It is quite apparent from this short list of different astrophysical applications that the approach has its advantages and enables qualitative (and quantitative, with some caution) studies of different topics concerning black holes. Especially the problems of accretion can be treated in this manner. It is understood that *quantitative results* need always be checked carefully within the full relativistic framework. This is particularly true for time-dependent phenomena (like oscillations and waves in fluids) and for computations of observed spectra. Notice that a textbook overview of the whole subject can be found in Kato et al. (1998).

Now we come to the case of a rotating black hole. At least two topics of current astrophysical interest can immediately be mentioned where the frame-dragging effects around a rotating black hole are important: the problem of oscillations of relativistic accretion discs (see Wagoner 1999 for a recent review), and that of precessing discs in low-mass X-ray binaries (Stella & Vietri 1997) and black-hole binaries (Wei et al. 1998). It is apparent that the whole subject of oscillation modes in relativistic discs calls for detailed investigation (cf. also Marković & Lamb 1998) and a suitable pseudo-Newtonian formulation can be an appropriate tool before embarking on a complete solution. However, unlike the case of stationary gaseous configurations, nonstationary phenomena require a more refined choice of the pseudo-Newtonian model because there are additional quantities apart from the location of marginal circular orbits which must be correctly modelled (e.g. $\kappa$).

Although the ever increasing computational facilities will make practical reasons for the pseudo-Newtonian approach rather old-fashioned in near future, what still remains desirable is a simple expression which would simulate the rotating (Kerr) black hole accurately. This is however difficult to find. For example, the potential suggested by Chakrabarti & Khanna (1992) could be useful in studying thin accretion discs around rotating holes, but it applies only to the equatorial plane and its interpretation is rather unclear (free parameters are fitted in a purely pragmatic manner for a restricted set of trajectories). Another form of the pseudo-Newtonian potential has recently been proposed by Artemova et al. (1996). This pseudo-potential reproduces very well steady circular motion but it ignores all effects which in the true Kerr metric are ascribed to the Lense-Thirring precession and which make test-particle trajectories non-planar. This is also the main reason why, in contrast to the non-rotating case, pseudo-potentials for the Kerr metric have had rather restricted impact. Also, a proper understanding of these generalizations may be as difficult as employing general relativity fully. However, the motivation which stems from attempts to understand and interpret predictions of the relativity theory has not disappeared.

It is the aim of our present contribution to discuss the pseudo-Newtonian modelling of the gravitational field of a rotating (Kerr) black hole, and also to propose how to test such models. One should remember that the idea of the pseudo-Newtonian approach represents a certain mathematical model rather than a rigorously defined approximation such as e.g. the weak-field approximation of the Einstein equations. The model does not aspire to represent any gravitational theory and, in particular, the corresponding potential is not required to satisfy any field equations. (It is exactly the lack of precise definition of the approximation method which leads us to introduce a test of accuracy of the model in the present article.) This fact, however, does not diminish the practical value of Paczyński-Wiita potential and other pseudo-Newtonian models, which apply even to regions with strong gravity and capture qualitative features of motion near the horizon.

We start by writing down the spatial components of the Kerr geodesic equation in Boyer-Lindquist coordinates $x^\mu = (t, r, \theta, \phi)$:

$$\begin{aligned}\Sigma\ddot{r} &= M\Delta\Sigma^{-2}(\Sigma-2r^2)(\dot{t}-a\dot{\phi}\sin^2\theta)^2 \\ &\quad+[(r-M)\Sigma/\Delta-r]\,\dot{r}^2+a^2\dot{r}\dot{\theta}\sin 2\theta \\ &\quad+r\Delta(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\,, \qquad (1)\\ \Sigma\ddot{\theta} &= \{2Mr\Sigma^{-2}[a\dot{t}-(r^2+a^2)\dot{\phi}]^2+a^2\dot{\theta}^2- \\ &\quad a^2\dot{r}^2/\Delta+\Delta\dot{\phi}^2\}\sin\theta\cos\theta-2r\dot{r}\dot{\theta}\,, \qquad (2)\\ \tfrac{1}{2}\Delta\Sigma^2\ddot{\phi} &= Ma(\Sigma-2r^2)(\dot{t}-a\dot{\phi}\sin^2\theta)\dot{r} \\ &\quad+2Mr\Delta[a\dot{t}-(r^2+a^2)\dot{\phi}]\,\dot{\theta}\cot\theta \\ &\quad-\Sigma(\Sigma-2Mr)(r\dot{r}+\Delta\dot{\theta}\cot\theta)\dot{\phi}\,, \qquad (3)\end{aligned}$$

where

$$\Delta = r^2-2Mr+a^2, \quad \Sigma = r^2+a^2\cos^2\theta; \qquad (4)$$

$M$ and $a$ are parameters of the Kerr solution, and the dot denotes differentiation with respect to the affine parameter normalized so that the 4-momentum is $p^\mu = \dot{x}^\mu$.

In the next section, Newtonian equations of test-particle motion in an axially symmetric gravitational field are given. In Sect. 3, we derive a pseudo-Newtonian potential for which these equations get a form very similar to the above geodesic equation in the Kerr spacetime.[1] In Sect. 4, numerical integration is carried out for a large number of trajectories both in the Kerr and in the "pseudo-Kerr" fields. A specific criterion analogous to the method of Lyapunov coefficients is introduced and the rate of divergence of each couple of corresponding trajectories is determined numerically. The mean values of this rate obtained for various combinations of constants of motion indicate the quality (i.e. accuracy, as defined below) of the pseudo-Newtonian approximation.

## 2. Stationary and axisymmetric field

In Euclidean space and in ellipsoidal coordinates $x^i = (r, \theta, \phi)$, related to the Cartesian coordinates $(x, y, z)$ by

$$x = \sqrt{r^2+a^2}\,\sin\theta\cos\phi,$$

[1] For a static and spherically symmetric field (both gravitational and non-gravitational), a similar problem was solved by Jaen (1967) using the Hamilton-Jacobi theory.

$$y = \sqrt{r^2+a^2}\,\sin\theta\sin\phi,$$
$$z = r\,\cos\theta,$$

the kinetic energy of a particle of mass $m$ is

$$T = \frac{p^2}{2m} = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2m} = \frac{1}{2m}\left[\Sigma\left(\frac{\dot{r}^2}{r^2+a^2}+\dot{\theta}^2\right)+(r^2+a^2)\,\dot{\phi}^2\sin^2\theta\right], \quad (5)$$

where $a$ is a non-negative constant of the dimension of length and the dot denotes differentiation with respect to the time parameter $t/m$ (normalized so that the momentum is $\boldsymbol{p}=\dot{\boldsymbol{x}}$). The motion of the particle in a stationary axisymmetric potential $U=U(r,\theta,\dot{r},\dot{\theta},\dot{\phi})$ is described by equations

$$\Sigma\ddot{r} = m^2(r^2+a^2)[(U_{,\dot{r}})^{\cdot}-U_{,r}] -r(a\dot{r}\sin\theta)^2/(r^2+a^2)+a^2\dot{r}\dot{\theta}\sin 2\theta +r(r^2+a^2)(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\,, \quad (6)$$

$$\Sigma\ddot{\theta} = m^2[(U_{,\dot{\theta}})^{\cdot}-U_{,\theta}]+[a^2\dot{\theta}^2-a^2\dot{r}^2/(r^2+a^2) +(r^2+a^2)\dot{\phi}^2]\,\sin\theta\cos\theta-2r\dot{r}\dot{\theta}\,, \quad (7)$$

$$(r^2+a^2)\ddot{\phi} = m(U_{,\dot{\phi}})^{\cdot}\sin^{-2}\theta -2\,[r\dot{r}+(r^2+a^2)\dot{\theta}\cot\theta]\,\dot{\phi}\,, \quad (8)$$

where we denote, for example,

$$(U_{,\dot{r}})^{\cdot} = m\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{r}}\right). \quad (9)$$

## 3. Potential for the Kerr field

Relativistic gravitational fields can be interpreted as consisting of three parts: the Newtonian (also Coulomb or gravitoelectric) component, the dragging (gravitomagnetic) component, and the space-geometry component. The Newtonian component is generated by mass-density and is given essentially by the gradient of the $g^{00}$ component of the spacetime metric. The dragging component is generated by mass-currents and determined by curl of $g_{0i}$. The space-geometry component (determined by $g_{ij}$) has no classical analog. This view and, in particular, the analogy with electromagnetism are most straightforward within the linearized approximation, but can be given a precise and invariant meaning even in strong fields (Thorne et al. 1986; Jantzen et al. 1992, and references cited therein). Let us compose the potential for the Kerr spacetime in a way that acknowledges this approach.

### *3.1. The Newtonian component*

Cutting the Kerr manifold at $r=0$, one can obtain the spacetime which is free of causality-violating regions. The cut leads to jumps in derivatives of the metric at $r=0$ which are interpreted as a thin massive layer. This induced mass spreads over the $r=0$ hypersurface and consists of the attractive singular ring ($z=0$, $\rho_{\rm f}=\sqrt{x^2+y^2}=a$) of infinite positive mass density, spanned by the repulsive disc ($z=0$, $0\le\rho_{\rm f}<a$) of negative mass density. Constructing the causally maximal extension of the Kerr metric, Keres (1967) and Israel (1970) showed that the Newtonian field generated by the massive layer corresponds to the potential[2]

$$V = -Mr/\Sigma. \quad (10)$$

The structure of this field was shown in Semerák (1995; cf. Fig. 2 there) by depicting its lines in the $(\rho_{\rm f}, z)$-plane. We will use the Keres-Israel potential as a scalar "seed" of our model. With $U=V$ given by Eq. (10), Eqs. (6)–(8) read

$$\Sigma\ddot{r} = m^2M\,(r^2+a^2)(\Sigma-2r^2)/\Sigma^2 -r(a\dot{r}\sin\theta)^2/(r^2+a^2)+a^2\dot{r}\dot{\theta}\sin 2\theta +r(r^2+a^2)(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\,, \quad (11)$$

$$\Sigma\ddot{\theta} = \{2m^2Ma^2r/\Sigma^2+a^2\dot{\theta}^2-a^2\dot{r}^2/(r^2+a^2) +(r^2+a^2)\dot{\phi}^2\}\sin\theta\cos\theta-2r\dot{r}\dot{\theta}\,, \quad (12)$$

$$\ddot{\phi} = -2\,[r\dot{r}/(r^2+a^2)+\dot{\theta}\cot\theta]\,\dot{\phi}\,. \quad (13)$$

Separated first integrals of these equations (analogy of Carter's equations for the Kerr spacetime) were found by Israel (1970). It was illustrated in Semerák (1996) that Eqs. (11)–(13) often yield trajectories indiscernible from their exact-Kerr counterparts computed from Eqs. (1)–(3). However, they do not contain the terms linear in velocities which embody the very characteristic feature of the Kerr geometry – the frame-dragging effects.

### *3.2. The dragging component*

In this section, we are led by analogy with classical electrodynamics and by observation that the Kerr dipole-like gravitomagnetic field resembles the Kerr-Newman magnetic field. One thus arrives at the potential which incorporates dragging in the form

$$U = V - \boldsymbol{v}\cdot\boldsymbol{A}, \quad (14)$$

where $\boldsymbol{v}=\boldsymbol{p}/m$ and the vector potential $\boldsymbol{A}$ is given by that of the Kerr-Newman electromagnetic field, $A_i=(0,0,Qra\sin^2\theta/\Sigma)$, with $Q$ (electric charge of the Kerr-Newman centre) replaced by $-2M$ (the extra factor of 2 is ascribed to the tensorial character of gravity and the minus sign to its attractive nature) – i.e., $A_i=(0,0,2Va\sin^2\theta)=(0,0,g_{t\phi})$. Similar (just half) correction for the dragging was considered by Dadhich (1985) for a special case of particles with zero axial angular momentum. The added term $-\boldsymbol{v}\cdot\boldsymbol{A}$ really introduces the desired dragging terms into the Eqs. (11)–(13): they read now

$$\Sigma\ddot{r} = [\text{r.h.s. of Eq. (11)}]-2mM\Sigma^{-2} \times(r^2+a^2)(\Sigma-2r^2)a\dot{\phi}\sin^2\theta\,, \quad (15)$$

[2] Cf. also Qadir (1986) where an electrically charged generalization of this potential was discussed. A somewhat different expression, $V=-(M/a)\,\arctan(a/r)$, was obtained by Krasiński (1980) in construction of a Newtonian model of the Kerr source. For the other result, the potential of an oblate spheroidal homeoid $V=-(M/a)\mathrm{arccot}(r/a)$, see Misra (1970).

$$\begin{aligned}\Sigma\ddot{\theta} &= \text{[r.h.s. of Eq. (12)]} \\ &\quad -2mMr\Sigma^{-2}(r^2+a^2)a\dot{\phi}\sin 2\theta\,, \qquad (16)\\ \tfrac{1}{2}(r^2+a^2)\Sigma^2\ddot{\phi} &= mMa\,[(\Sigma-2r^2)\,\dot{r} \\ &\quad +2r(r^2+a^2)\,\dot{\theta}\cot\theta] \\ &\quad -\Sigma^2[r\dot{r}+(r^2+a^2)\,\dot{\theta}\cot\theta]\,\dot{\phi}\,. \qquad (17)\end{aligned}$$

Note that the contravariant 3-potential obtained by using the inversion of the Kerr 3-metric, $g^{ij}=1/g_{ij}$, is

$$A^i=(0,0,g_{t\phi}/g_{\phi\phi})=(0,0,-\omega_{\rm K}), \qquad (18)$$

where $\omega_{\rm K}=2Mar/\mathcal{A}$ and $\mathcal{A}=\Delta\Sigma+2Mr(r^2+a^2)$; $A^i$ equals the shift vector of Thorne et al. (1986) which stands for the potential of the gravitomagnetic field in this reference.

### 3.3. *The space-geometry component*

The space curvature cannot be understood directly within the Newtonian or electromagnetic analogy. It may only be included by introducing ad hoc corrections into the form (14):

$$\begin{aligned}U &= V-\boldsymbol{v}\cdot\boldsymbol{A}+\frac{(\boldsymbol{v}\cdot\boldsymbol{A})^2}{4V}-\frac{Mr\Sigma}{(r^2+a^2)\Delta}\,\frac{\dot{r}^2}{m^2} \\ &= -\frac{Mr}{\Sigma}\left(1-\frac{\dot{\phi}}{m}a\sin^2\theta\right)^2-\frac{Mr\Sigma}{(r^2+a^2)\Delta}\,\frac{\dot{r}^2}{m^2}\,. \qquad (19)\end{aligned}$$

This potential leads to equations of motion

$$\begin{aligned}\Sigma\ddot{r} &= M\Delta\Sigma^{-2}(\Sigma-2r^2)(m-a\dot{\phi}\sin^2\theta)^2 \\ &\quad +[(r-M)\Sigma/\Delta-r]\,\dot{r}^2+a^2\dot{r}\dot{\theta}\sin 2\theta \\ &\quad +r\Delta(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\,, \qquad (20)\\ \Sigma\ddot{\theta} &= \{2Mr\Sigma^{-2}[am-(r^2+a^2)\dot{\phi}]^2+a^2\dot{\theta}^2 \\ &\quad -a^2\dot{r}^2/\Delta+\Delta\dot{\phi}^2\}\sin\theta\cos\theta-2r\dot{r}\dot{\theta}\,, \qquad (21)\\ \tfrac{1}{2}\Sigma\mathcal{A}\ddot{\phi} &= Ma(\Sigma-2r^2)(m-a\dot{\phi}\sin^2\theta)\,\dot{r} \\ &\quad +2Mr(r^2+a^2)[am-(r^2+a^2)\dot{\phi}]\,\dot{\theta}\cot\theta \\ &\quad -\Sigma^2(r\dot{r}+\Delta\dot{\theta}\cot\theta)\dot{\phi}\,. \qquad (22)\end{aligned}$$

The only point in which Eqs. (20) and (21) differ from the relativistic Eqs. (1)–(2) is that the relativistic variable $\dot{t}$ – given by $(\Delta\Sigma)^{-1}(\mathcal{A}E-2Mar\Phi)$ ($E$ and $\Phi$ stand for the particle's energy and axial angular momentum at infinity) – is replaced by $m$ in the classical model. This distinction reflects, however, the conflict between the very roots of classical physics (where time parameter is universal) and relativity (where proper time and some coordinate time occur, related to each other in a specific way at each point). Notice that Eq. (22) differs from (3) also in several other points.

### 3.4. *Specific features of motion in the pseudo-Kerr potential*

Independence of the Lagrangian $L=T-mU$ on $t$ and $\phi$ implies two constants of motion,

$$\begin{aligned}E &= mv^2/2-mMr/\Sigma \\ &= \frac{1}{2m}\left[\Sigma\left(\frac{\dot{r}^2}{\Delta}+\dot{\theta}^2\right)+\frac{\mathcal{A}}{\Sigma}\dot{\phi}^2\sin^2\theta\right]-\frac{mMr}{\Sigma}\,, \qquad (23)\end{aligned}$$

$$\Phi=\frac{m\mathcal{A}}{\Sigma}\,(\omega-\omega_{\rm K})\sin^2\theta, \qquad (24)$$

where $\omega={\rm d}\phi/{\rm d}t=\dot{\phi}/m$. The Kerr axial angular momentum at infinity contains (again, as in Eqs. (20)–(21)) $\dot{t}$ instead of $m$ ($\omega_{\rm K}$ is the angular velocity with which the field rotates relative to an observer standing at infinity). For large $r$ one obtains the usual forms of the energy and axial angular momentum in the Newtonian central gravitational field:

$$\begin{aligned}E &\simeq (2m)^{-1}(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\phi}^2\sin^2\theta)-mM/r, \qquad (25)\\ \Phi &\simeq r^2\dot{\phi}\sin^2\theta. \qquad (26)\end{aligned}$$

Let us determine the acceleration $a^i$ of an observer orbiting uniformly ($\omega={\rm const}$) at $r={\rm const}$, $\theta={\rm const}$. We will define acceleration as a specific force necessary to keep the observer in the orbit, i.e. as the *minus* acceleration of a *free* particle having $\dot{r}=\dot{\theta}=0$ at a given point. According to Eqs. (20)–(22), we find

$$\begin{aligned}a^r &= \Delta\Sigma^{-3}[M(2r^2-\Sigma)(1-a\omega\sin^2\theta)^2 \\ &\quad -r(\Sigma\omega\sin\theta)^2]\,, \qquad (27)\\ a^\theta &= -\Sigma^{-3}\{2Mr[a-(r^2+a^2)\omega]^2 \\ &\quad +\Delta\Sigma^2\omega^2\}\sin\theta\cos\theta\,, \qquad (28)\\ a^\phi &= 0\,. \qquad (29)\end{aligned}$$

This 3-acceleration differs from the space part of 4-acceleration of the Kerr stationary observer only by the absence of the multiplicative factor $(u^t)^2$ (square of the time-component of the observer's 4-velocity) [cf. Semerák 1993, Eqs. (37)–(39)].

For a static observer ($\omega=0$) one obtains, in particular,

$$a^i=(M/\Sigma^3)\left(\Delta(2r^2-\Sigma),-ra^2\sin 2\theta,0\right). \qquad (30)$$

Hence, the field is repulsive at $0\le r<a|\cos\theta|$ in the sense that the radial component of (30) is negative there.

According to (27)–(29), the stationary observer needs no thrust (i) at $r=a$ on the axis ($\theta=0°,180°$), and (ii) in the equatorial plane if his angular velocity is $\omega=1/(a\pm\sqrt{r^3/M})$. The latter agrees exactly with the Keplerian angular velocity of an equatorial observer orbiting along a circular geodesic in the Kerr spacetime.

Important circular geodesics in the equatorial plane, on the other hand, are not reproduced properly by the potential (19), viz. the equation for marginally stable orbits reads

$$r^3\pm 8a\sqrt{Mr^3}-3a^2r-2Ma^2=0 \qquad (31)$$

and for marginally bound orbits

$$r^2\pm 4a\sqrt{Mr}-a^2=0. \qquad (32)$$

The correct equations have respectively the forms

$$r^2-6Mr\pm 8a\sqrt{Mr}-3a^2=0 \qquad (33)$$

and

$$r^2-4Mr\pm 4a\sqrt{Mr}-a^2=0. \qquad (34)$$

In the Schwarzschild case, for instance, both our equations imply $r=0$.

In any pseudo-Newtonian description of a rotating black hole, the main difficulty is to simulate the presence of the horizon and of the dragging effects with an acceptable precision. Both these phenomena are outside the scope of Newtonian physics. Our potential accounts for the frame-dragging effects, and especially in the intermediate and large distances provides a good fit, whereas it does not describe correctly the innermost region where the horizon and important circular orbits lie. In order to account for the horizon, one would have to start from the scalar potential which diverges to $-\infty$ there. The Keres-Israel potential (10), instead, reaches $-\infty$ only at the very singularity $\Sigma=0$. The potential proposed by Artemova et al. (1996) is a better alternative in this respect, being the simplest generalization of the Paczyński-Wiita potential to the Kerr case which reproduces the horizon and approximates the important orbits. Also, the epicyclic frequency of small radial oscillation $\kappa(r)$, important in the theory of discoseismology, is not reproduced with good accuracy by potentials (10) and (19) although it does show a maximum, typical for a relativistic $\kappa$. Apparently, trying to incorporate different manifestations of the frame-dragging accurately, one may end with a long expression without practical sense.

To summarize, each particular relativistic effect can be well simulated within classical physics, but pseudo-Newtonian modelling of the complete relativistic situation has only a restricted validity. In order to clarify the value of our potential, we have carried out an extensive computation of trajectories and introduced a criterion which determines the accuracy of the pseudo-Newtonian model.

## 4. Testing approximative equations

As mentioned above, the pseudo-Newtonian potential has been used frequently in various situations in which general relativistic effects on the motion of material (either test particles or fluids) are essential but exact calculations are too difficult. It is however impossible to estimate, *a priori*, the error which is introduced by replacing the original system, described in the framework of general relativity, by a corresponding pseudo-Newtonian system. The plausibility of a particular form of the simulating potential can be verified by solving analogous situations within the exact theory. For example, several specific questions in the astrophysics of accretion discs had been first analysed using the pseudo-Newtonian theory, and the results were only later supported by more complicated calculations within the Schwarzschild spacetime (with identical local physics of the fluid). Indeed, it is quite trivial to recall that the above-mentioned standard model of thin discs was formulated within the Newtonian (Shakura & Sunyaev 1973), pseudo-Newtonian (Paczyński & Wiita 1980), and also relativistic frameworks (Novikov & Thorne 1973). A specific response of relativistic accretion discs to oscillations (Kato & Fukue 1980) was also explored with a modified Newtonian potential (Nowak & Wagoner 1991) before further steps were carried out (Perez et al. 1997). These examples indicate that the pseudo-Newtonian model is appropriate for investigating the motion of material around black holes.

It is quite straightforward to check whether the pseudo-Newtonian potential is suitable for treating the motion of test particles and fluids when one deals with a spherically symmetric system. What appears more difficult is to develop an analogous pseudo-Kerr theory, describing the relativistic effects near a rotating compact object. Now we will propose a systematic approach to estimate the quality of a particular model by computing a sufficiently large number of trajectories with different initial conditions and comparing results with exact calculations of geodesic motion. The test we suggest determines the rate of divergence of trajectories (as given by approximative versus exact equations). We propose that this type of check should be carried out for each particular set of approximative equations before they are applied to astrophysical situations. (Until now, only rather restricted tests on a relatively small number of trajectories have been applied. As a consequence, one cannot be sure about predictions based on approximative equations of such models because there is little or no control of their overall precision.) For illustration, we then submit the approximative equations of Sect. 3 to our test.

### 4.1. *Criterion of accuracy of approximative solutions*

Our criterion is motivated by the definition of the Lyapunov characteristic coefficients which have been introduced in order to characterize chaotic systems (see, e.g., Chapt. 5 of Lichtenberg & Lieberman (1983)). The most important features of the Lyapunov coefficients relevant for the present work are summarized in the Appendix.

We will now introduce a parameter which is analogous to the maximum Lyapunov characteristic exponent $\lambda$. One should however realize the basic difference between the standard formulation of the problem of chaotic system and our present situation. While $\lambda$ characterizes the rate of divergence of two neighbouring (initially close to each other) trajectories, in the present work one always starts with *exactly identical* initial conditions. Trajectories then separate because the motion of the first test particle is determined by the geodesic equation in the Kerr spacetime, while the other (fictitious) particle moves according to approximative equations. The geodesic equation could be integrated analytically but not the approximative equations, so we have to resort to numerical solution. Two points should be mentioned: (i) as the test particles move in a stationary system with preferred time coordinate ($t$), the separation of trajectories in the phase space is calculated in the $t=\mathrm{const}$ slice; (ii) periodic rescalings of the separation $w$ to zero length keep the two trajectories close to each other during their evolution.

In analogy with standard definition (A.2), we introduce a critical parameter

$$\Lambda=\lim_{n\to\infty}(n\bar{t})^{-1}\sum_{k=1}^{n}\ln\frac{d_k}{d_0}, \qquad (35)$$

with $d_0\equiv|w_0(\bar{t})|$.

Now:

(i) two particles start with identical initial conditions; the first one evolves according to geodesic equations (1)–(3), while the second one according to approximative equations (Eqs. (20)–(22), for example);

(ii) $\bar{t}$ is determined as a time interval during which the two trajectories remain close to each other and the initial separation $d_0$ is an arbitrary small number (typically, $\bar{t}$ is of the order of the Keplerian orbital period at the initial radius);

(iii) Eq. (35) is evaluated and it is determined, numerically, whether convergence has been reached for large $n$ with a pre-determined accuracy. The final value, $\Lambda_{\rm f}$, characterizes the rate of divergence of the two trajectories with the same initial conditions: $\Lambda_{\rm f} > 0$ corresponds to separation increasing exponentially while $\Lambda_{\rm f} \to -\infty$ corresponds to a polynomial (i.e. much slower) increase of the separation.

In order to estimate the accuracy of the approximative equations, one needs to follow a large number of trajectories with randomly chosen initial conditions. By averaging over $\Lambda_{\rm f}$ corresponding to different initial conditions, one obtains a value which shows whether the approximative equations describe, *on the whole*, the motion of test matter with good precision. Regions where the mean value is positive, $\langle\Lambda_{\rm f}\rangle > 0$, indicate that the approximation is not acceptable. The process is described in more detail below where we calculate $\langle\Lambda_{\rm f}\rangle$ for Eqs. (11)–(13) and (20)–(22) as examples.

### 4.2. An example: test of the pseudo-Kerr equations

We submitted our approximative equations to the above-described test on $\langle\Lambda_{\rm f}\rangle$. Relative position of the two test particles is determined by difference equations:

$$\delta\dot{x}^i \equiv \dot{x}^i_{\rm Kerr} - \dot{x}^i_{\rm approx}, \quad \delta\ddot{x}^i \equiv \ddot{x}^i_{\rm Kerr} - \ddot{x}^i_{\rm approx}, \qquad (36)$$

with $x^i = \{r(t), \theta(t), \phi(t)\}$. The suffix "Kerr" indicates that the trajectory is a geodesic in the Kerr spacetime while the suffix "approx" corresponds to pseudo-Newtonian equations. We should stress in this place that comparing processes in different spacetimes *is* a serious problem in general relativity. Here, however, we do not compare two relativistic spacetimes and the situation is quite different: though a proper physical justification gives an additional appeal to any pseudo-Newtonian model and we have therefore emphasized also a physical content of our approach (in Sect. 3), in the final stage one mainly asks whether the test particles in the model field in *some* coordinates move along trajectories that are sufficiently close to *certain* geodesics of a given relativistic field in some particular coordinates. It is only necessary to define the way of correspondence of the initial conditions. In our test, we consider as counterparts the particles which start from a given position ($r$,$\theta$) with given (specific) constants of motion ($e = E/m,\ l = \Phi/m$). Of course, it is possible that we would obtain a better fit with some other choice, e.g. if the "corresponding" particles had the same initial velocities with respect to some (corresponding) local frames.

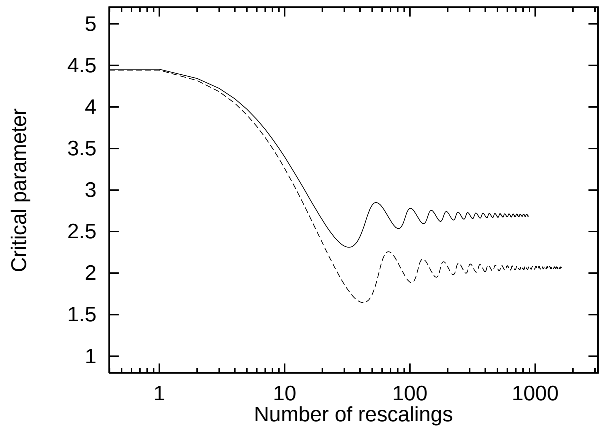


**Fig. 1.** This graph illustrates how the critical parameter $\Lambda$ oscillates at first and then settles to a positive value after $k \approx 10^3$ rescalings, indicating the chaotic-type increase of separation $d_k$ (cf. Eq. (35)). Two different examples of trajectories are shown.

We have integrated difference Eqs. (36) using the Bulirsch-Stoer scheme (Press et al. 1994). Initial conditions cover the parameter space of

$$r > r_+ \equiv M + \sqrt{M^2 - a^2},\ 0 < \theta \lesssim \tfrac{1}{2}\pi,\ e < e_{\rm max}. \qquad (37)$$

Typically, we set $e_{\rm max} \approx 1.1$. As expected, most of the particles with $e_{\rm max} \gg 1$ escape quickly to large distances $(r \gg r_+)$ where both the exact and approximative equations give identical results. It is thus relevant to investigate trajectories with lower energies which often make a number of revolutions around the black hole. Some of these trajectories tend to diverge in a chaotic-type manner, as we will see below.

Each run, characterized by the value of $a/M$, resulted in about $10^7$ values of $\Lambda_{\rm f}$. For each set of initial conditions (37) within the run, the separation of the two corresponding trajectories was periodically rescaled down to a pre-determined value after a fixed interval $\bar{t}$. We have typically chosen $\bar{t} = 2M$ which is comparable to the Keplerian orbital period near the horizon. The integration was terminated when one of the following conditions had been satisfied: (i) $\Lambda$ converged to a finite positive value $\Lambda_{\rm f}$ (numerically, we checked that the relative change of $\Lambda$ during the last 30 rescalings did not exceed 0.5 %); (ii) the upper limit of $10^4$ on the number of rescalings was exceeded, while $\Lambda$ was oscillating or decreasing monotonically to negative values; (iii) the trajectory was captured by a black hole, $r \leq r_+$. The case (iii) was excluded from further consideration.

Fig. 1 illustrates the evolution of $\Lambda$ for two sets of initial conditions which both fall under item (i) above. Apparently, both cases shown there converge to positive $\Lambda_{\rm f}$, although the numbers of rescalings were different. In this respect, it is interesting to note that the separation of nearby geodesics in the Kerr spacetime never increases exponentially. This fact is a consequence of the existence of the fourth (Carter's) constant of motion, as discussed in Karas & Vokrouhlický (1992).

Next we illustrate the mean value $\langle\Lambda_{\rm f}\rangle$ as a function of the constants of motion, $e$ and $l$. Graphs corresponding to Eqs. (20)–

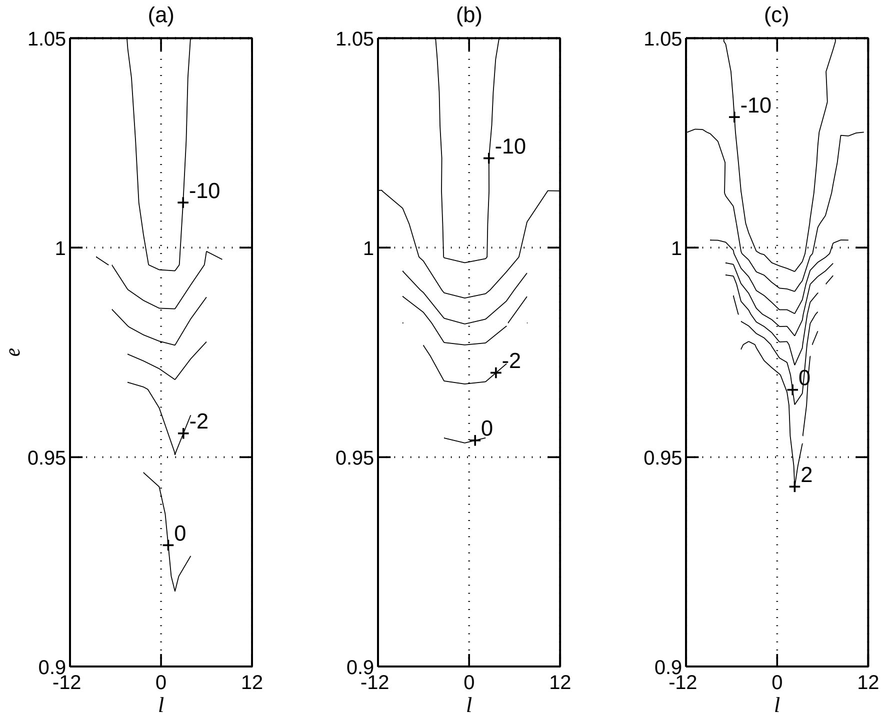


**Fig. 2a–c.** Isocurves of the mean terminal value of the critical parameter, $\langle\Lambda_{\rm f}\rangle$, as a function of the particles' specific energy $e$ and their specific axial angular-momentum $l$. Three panels correspond to **a** Eqs. (20)–(22) with $a = M$, **b** Eqs. (20)–(22) with $a = 0$, and, for comparison, **c** Eqs. (11)–(13) with $a = M$ (scalar, Keres-Israel model). Positive $\langle\Lambda_{\rm f}\rangle$ indicates chaotic-type behaviour (see the text for details).

(22) are shown for an extremely rotating black hole (Fig. 2a) and a non-rotating black hole (Fig. 2b). For comparison, the Keres-Israel model (11)–(13) with $a = M$ was also included (Fig. 2c). One can locate easily the regions of positive $\langle\Lambda_{\rm f}\rangle$. In general these regions are bound to small $e$ and $l$ which correspond to trajectories that plunge close to the horizon. In the case of a non-rotating hole the graph is symmetrical about $l = 0$ because both the Schwarzschild spacetime and the adopted pseudo-Newtonian field with $a = 0$ are spherically symmetric. Distortion introduced by rotation is visible in the graphs (a) and (c). Comparing these two graphs one can verify that model (19) is superior to (10) (cf. the positive values of $\Lambda_{\rm f}$ in Fig. 2c, indicating that most of the corresponding trajectories separate rapidly from each other).

## 5. Conclusions

We have described a method to test the overall accuracy of the pseudo-Kerr models of the relativistic gravitational field and we applied our approach to two particular potentials, given by Eqs. (10) and (19). One can conclude that the corresponding equations of motion, (11)–(13) and (20)–(22) respectively, do not provide an acceptable approximation to exact Kerr geodesic equations in the part of the $(e, l)$-plane where $\langle\Lambda_{\rm f}\rangle > 0$, and vice-versa. This statement does not apply strictly to *all* trajectories (with different initial conditions) because our graphs are given in terms of mean values. We suggest that a check of this type should always be employed when some particular form of approximative equations is formulated. For example, calculation of the spectra of accretion discs requires integration of a large number of photon trajectories; the overall accuracy in the description of the gravitational field is then important.

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## Appendix

The Lyapunov characteristic exponent $\lambda$ of two nearby trajectories, $\ell$ and $\ell + w$, is defined by

$$\lambda(\ell, w) = \lim_{\substack{t\to\infty \\ d(0)\to 0}} t^{-1} \ln\left[\frac{d(t)}{d(0)}\right], \tag{A.1}$$

where $\ell(t)$ is a solution of the equations of motion in the phase space, $w(t)$ is a connecting vector, $d(t) = |w|$ is its length in the phase space and $d(0)$ is an initial separation (the value of $d(0)$ must be small; otherwise it is arbitrary and one usually scales $d(0)$ to unity). The Kerr spacetime being stationary, we can measure the separations on the $t = \mathrm{const}$ surfaces. One defines $\lambda_{\max}$ as the maximum value of $\lambda$ with respect to variations of $w$ and characterizes the chaoticity of the system by the value of $\lambda_{\max}$. Positive values of $\lambda_{\max}$ indicate that neighbouring trajectories diverge exponentially in the course of time while negative $\lambda_{\max}$ corresponds only to polynomial divergence. The above definitions have been originally introduced within the framework of non-relativistic systems but they are directly applicable also to stationary systems in general relativity.

The maximum Lyapunov characteristic exponent is frequently determined numerically. This approach requires a careful choice of the integration scheme. In order to keep computational errors under control, one lets the trajectories evolve for a short interval of time, $\bar{t}$, after which $w$ is rescaled back to unity (one denotes $d_k \equiv |w_{k-1}(\bar{t})|$ the norm of the connecting vector at the moment of the $k$-th rescaling). One can show (Benettin

1984) that the Lyapunov characteristic exponent corresponding to the original $w_0$ is given by

$$\lambda = \lim_{n\to\infty} (n\bar{t})^{-1} \sum_{k=1}^{n} \ln \frac{d_k}{d_0}, \tag{A.2}$$

independently of the value of $\bar{t}$. In addition, $\lambda = \lambda_{\max}$ for almost all $w_0$. Again, one can conveniently set $d_0 = 1$.

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**Note added in proof:** Very recently, Løvås (1998) compared the pseudo-potentials of Paczyński & Wiita (1980), of Nowak & Wagoner (1991) and of Artemova et al. (1996) with the exact black hole fields. For each of the cases, she computed several quantities crucial in the theory of (optically thin) accretion discs and proposed to use $V_{\rm PW}$ for non-rotating holes while the potentials by Artemova et al. for rotating holes.