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The vertical eddy-heat flux as a stabilizer of cold accretion disks

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Abstract. The time-dependent vertical structure of cold accretion disks and their thermal stability have been studied with turbulent heat transfer being included. A strong turbulent heat transport enforces a nearly adiabatic stratification and the disk evolves quasi-homologously, although homology is broken by the outer boundary condition. The radiative energy loss scales with the disk's optical depth τ like $Q^- \propto T_{\rm c}^4/\tau^m$ with $T_{\rm c}$ the midplane temperature. The disk is only stable if the resulting m fulfills the stability criterion $3 + m\,(n/2 - q) > 0$, with n and q taken from the opacity law $\kappa \propto \rho^n T^q$. For rather cool disks with $n = 1/3$ and $q = 10$ the vertical structure proves to be thermally unstable unless the turbulent Prandtl number (the ratio between the eddy viscosity and turbulent heat conductivity) is less than, say, 0.1. For weaker temperature power-laws of the opacity (smaller q) the disks become more and more stable even without the stabilizing support of the eddy-heat flux. Numerical simulations confirm the quasi-analytically derived stability criterion.

Key words: accretion, accretion disks – hydrodynamics – instabilities

1. Introduction

The standard accretion disk model (cf. Shakura & Sunyaev 1973) must be complemented by stability considerations. Here, in particular, short term flickering is concerned as well as the limit cycle behaviour of cataclysmic variables. Even more surprising is the lack of the theory to explain the existence of 'cold disks' such as T Tauri disks or those on the cold branch of the S-curve concept of dwarf nova systems.

Piran (1978) investigated the role of various cooling mechanisms on the viscous and thermal stability of thin, non-self-gravitating accretion disks, particularly for the standard viscosity scaling

$$\nu_{\rm T} \propto H^2 \Omega_{\rm K}, \tag{1}$$

with H as the disk half thickness and a Keplerian angular velocity $\Omega_{\rm K} = \sqrt{GM/R^3}$. Provided the vertically integrated cooling rate goes like

$$Q^- \propto H^k \Sigma^l, \tag{2}$$

with Σ as the surface density, a disk proves stable only if

$$k > 2 \cdot {\rm Max}(1, l). \tag{3}$$

This predicts stability for a hot disk and instability for a cool one. The reason is the following. In order to determine the exponents k and l one adopts as usual for the case of radiative cooling $Q^- \propto T_{\rm c}^4/\tau \propto H^8/(\kappa\Sigma)$, with midplane temperature $T_{\rm c}$ and optical depth τ. The disk thickness scales like $H \propto \sqrt{T_{\rm c}}$. A Kramers opacity ($\kappa \propto \rho T^{-3.5}$) yields then $Q^- \propto H^{16}\Sigma^{-2}$, i.e. $k = 16$ and $l = -2$. Hence, after (3) the solution is stable. For cool gas with

$$\kappa \propto \rho^{1/3} T^{10} \tag{4}$$

(Faulkner et al. 1983) the result is

$$Q^- \propto H^{-35/3}\Sigma^{-4/3}, \tag{5}$$

i.e. $k = -35/3$ and $l = -4/3$, and the stability criterion (3) cannot be fulfilled. As cool disks nevertheless exist, the question arises how the theory must be modified in order to yield cool, but yet stable solutions.

In Paper I (Fröhlich 1997) the thermal stability for a radiative disk has been discussed using homology considerations. Homologous height contraction of the cooling primordial solar nebula was first studied by Lin & Papaloizou (1980), assuming that the opacity depends on some power of temperature. But, homology solutions exist even if there are internal heat sources. In Paper I diffusion of radiative energy has been modelled with power-law dependencies of the opacity on temperature *and* density, i.e.

$$\kappa \propto \rho^n T^q. \tag{6}$$

As long as $q < 0.87$–$1.55\,n$ the temperature gradient in the disk is everywhere less steep than the adiabatic one, ensuring a purely radiative transport. The $q = 10$ of the cool opacity law, however, does never fulfill the condition so that a particular

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stabilizer must exist, e.g. the vertical heat transport of the disk turbulence.

For a thermally relaxed accretion disk the effect of a turbulent heat transfer has been considered by Liang (1977), Shakura et al. (1978), Urpin (1983), Goldman & Wandel (1995), and in two papers by Rüdiger et al. (1988, 1990). Now, owing to homology considerations, the vertical disk structure computations by Rüdiger et al. shall be extended to the non-equilibrium case in order to check stabilizing effects of the vertical eddy-heat fluxes.

2. Basic equations

The quantities ρ, p, T, and u_z have their usual meanings. The specific entropy, S, is that of an ideal gas. The acceleration is for a thin disk proportional to the height z above the midplane. The viscous heating is $(9/4)\,\alpha_{\rm SS}\,p\,\Omega_{\rm K}$, with the viscosity parameter $\alpha_{\rm SS}$. The flux, F, in the vertical direction is the sum of both the radiative and the turbulence-driven part,

$$F = -\frac{16\,\sigma\,T^3}{3\,\kappa\,\rho}\frac{\partial T}{\partial z} - \rho\,\chi_{\rm T}\,T\frac{\partial S}{\partial z}\,. \tag{7}$$

In what follows the turbulent heat conductivity $\chi_{\rm T}$ is considered in terms of the Prandtl number ${\rm Pr} = \nu_{\rm T}/\chi_{\rm T}$ with the eddy viscosity after $\nu_{\rm T} = \alpha_{\rm SS}\,p/(\rho\Omega_{\rm K})$. Hence, the basic equations are

$$\frac{\partial p}{\partial z} = -\rho\,\Omega_{\rm K}^2\,z, \tag{8}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_z)}{\partial z} = 0, \tag{9}$$

$$\frac{\partial S}{\partial t} + u_z\frac{\partial S}{\partial z} = \frac{1}{\rho T}\left(\frac{9}{4}\,\alpha_{\rm SS}\,p\,\Omega_{\rm K} - \frac{\partial F}{\partial z}\right), \tag{10}$$

$$\frac{\partial T}{\partial z} = -\frac{F + \frac{\nu_{\rm T}}{\rm Pr}\,\rho\,\Omega_{\rm K}^2\,z}{\frac{\gamma}{\gamma-1}\frac{p}{T}\frac{\nu_{\rm T}}{\rm Pr} + \frac{16\sigma T^3}{3\kappa\rho}}\,. \tag{11}$$

In (9) the radial term is ignored. Replacing in (10) the entropy by $C_v\,\ln(p/\rho^\gamma)$ with $\gamma = C_p/C_v$ one obtains the energy equation in the form

$$\frac{\partial p}{\partial t} + u_z\frac{\partial p}{\partial z} + \gamma\,p\,\frac{\partial u_z}{\partial z} = (\gamma-1)\left(\frac{9}{4}\,\alpha_{\rm SS}\,p\,\Omega_{\rm K} - \frac{\partial F}{\partial z}\right). \tag{12}$$

Heights may now be expressed by $\xi = z/z_0(t)$ with

$$z_0(t) = \frac{1}{\Omega_{\rm K}}\sqrt{\frac{p_{\rm c}(t)}{\rho_{\rm c}(t)}}. \tag{13}$$

Then self-similarity is assumed with the relations

$$p(z,t) = p_{\rm c}(t)\cdot f_p(\xi), \tag{14}$$

$$\rho(z,t) = \rho_{\rm c}(t)\cdot f_\rho(\xi), \tag{15}$$

$$u_z(z,t) = u_0(t)\cdot \xi, \tag{16}$$

$$F(z,t) = p_{\rm c}(t)\,z_0(t)\,\Omega_{\rm diss}(t)\cdot f_F(\xi), \tag{17}$$

$$T(z,t) = T_{\rm c}(t)\cdot f_T(\xi). \tag{18}$$

The label ‘c’ refers to midplane values. Hence,

$$\frac{d\,f_p}{d\,\xi} = -f_\rho\,\xi, \tag{19}$$

$$\frac{d\,f_F}{d\,\xi} = f_p, \tag{20}$$

$$\frac{d\,f_T}{d\,\xi} = -\frac{f_F + K_2\,f_p\,\xi}{\frac{\gamma}{\gamma-1}K_2\,f_p + K_1^{-1}f_T^{3-q}f_\rho^{-n-1}} \tag{21}$$

with

$$K_1 = \frac{3\,\Omega_{\rm diss}\,p_{\rm c}z_0^2\kappa_{\rm c}\rho_{\rm c}}{16\sigma T_{\rm c}^4}, \qquad K_2 = \frac{\alpha_{\rm SS}\,\Omega_{\rm K}}{{\rm Pr}\,\Omega_{\rm diss}}, \tag{22}$$

and with the dissipation rate

$$\Omega_{\rm diss} = \frac{9}{4}\,\alpha_{\rm SS}\,\Omega_{\rm K} - \frac{\gamma+1}{\gamma-1}\,\frac{\dot z_0}{z_0}\,. \tag{23}$$

Strictly speaking, homology requires the constancy of K_1 and K_2. With regard to K_2 this demands the scale-height to evolve with a time-scale exceeding the thermal one. Homology would be also recovered for ${\rm Pr} \to 0$. Paper I considers the limiting case ${\rm Pr} \to \infty$.

Eqs. (19)–(21) are integrated numerically, subject to the symmetry conditions $f_p(0) = f_T(0) = 1$ and $f_F(0) = 0$. The only free parameter is K_1. With the integrals

$$a_2 = \frac{1}{2}\int_{-\infty}^{\infty} f_\rho\,d\,\xi, \qquad a_3 = \frac{\int_{-\infty}^{\infty} f_\rho f_\kappa^{\rm rad}\,d\,\xi}{\int_{-\infty}^{\infty} f_\rho\,d\,\xi} \tag{24}$$

we can write

$$\Sigma = \int_{-\infty}^{\infty}\rho\,dz = 2\,a_2\,\rho_{\rm c}\,z_0, \tag{25}$$

$$\tau = \int_{-\infty}^{\infty}\rho\,\kappa\,dz = a_3\,\kappa_{\rm c}\,\Sigma. \tag{26}$$

At last, for the cooling flux, $Q^- = F(\infty) = \sigma T_{\rm eff}^4$, the expression

$$Q^- = \frac{32}{3}\,a_2 a_3\,K_1 f_F(\infty)\frac{\sigma T_{\rm c}^4}{\tau} \tag{27}$$

results. The view is adopted that somewhere in the disk atmosphere the real temperature matches the effective one. Because there the viscous heating is unimportant and in the absence of an eddy-heat flux the Eddington approximation for a grey atmosphere can be used, i.e.

$$T(\infty)^4 = \frac{17}{32}\,T_{\rm eff}^4. \tag{28}$$

Replacing in (27) the effective temperature by the asymptotic temperature, one obtains the conditions

$$K_1 = \frac{3}{17}\,\frac{f_T^4(\infty)}{f_F(\infty)}\,\frac{\tau}{a_2 a_3}. \tag{29}$$

Hence, K_1 depends in a complicated manner on the optical depth as well as the Prandtl number.

3. Numerical results

In Paper I for purely radiative disks it has been found that *provided K_1 converges for large τ to a finite value* a disk should be thermally stable only for $q < 3 + n/2$. Now we ask, how the eddy-heat transport affects the stability problem, especially for cold accretion disks with opacity laws like (4) which simply do not fulfill the given stability criterion.

3.1. Hot disks

As an example for a thermally stable disk the Kramers' case ($n = 1, q = -3.5$) is reconsidered. As in the case $\mathrm{Pr} \to \infty$ there is a limiting value for K_1, whose dependence on the Prandtl number ($\mathrm{Pr} = 4/(9\,K_2)$) is indicated in Table 1. We find that an eddy-heat flux steepens the temperature gradient, enhancing somewhat the mid-plane temperature as compared with the purely radiative case, but the dependence of K_1 on the Prandtl number proves to be weak. An Eddy-heat flux does not alter the stability criterion, viz. $3 + n/2 - q > 0$.

3.2. Cold disks

While for a hot disk the atmospheric constraint (28) hardly affects the outcome, the contrary holds for a cold one with an opacity like (4): It is now the radiative atmosphere that restricts the flux. There is no longer a unique limiting value for K_1 if the optical depth grows beyond any limit. For small Prandtl number, however, numerical integration of Eqs. (19)–(21) provide the scaling relation

$$K_1 = K\tau^{1-m} \tag{30}$$

or equivalently,

$$Q^- \propto \frac{\sigma T_\mathrm{c}^4}{\tau^m}, \tag{31}$$

with K slightly depending on K_2. The question is how strong a scaling like (31) influences the stability problem. One should be aware that the approximation $Q^- \simeq (8/3)\,T_\mathrm{c}^4/\tau$ works well if the radiative part of the flux does not strongly depend on optical depth, otherwise it provides a lower limit only.

Fig. 1 shows how for a given Prandtl number m converges for large τ. The opacity law (4) is used refering to H^- as the main source of the opacity. The resulting exponent m is near 0.3, slightly decreasing for decreasing Prandtl number[1].

4. Thermal instability

4.1. Theory

With the surface pressure

$$\Pi = \int_{-\infty}^{\infty} p\, dz \tag{32}$$

[1] We have also worked with other opacity laws. For $q < 3 + n/2$ stability results like for hot disks. The location of the dividing line in the opacity n-q-plane is still unknown.

Table 1. How K_1 depends, for a hot and optically very thick disk, on K_2, or Prandtl number Pr, respectively

K_2	Pr	K_1
0	∞	0.2687
$\sqrt{0.1}$	1.405	0.2491
1	0.444	0.2176
$\sqrt{10}$	0.140	0.1614
10	0.044	0.0962

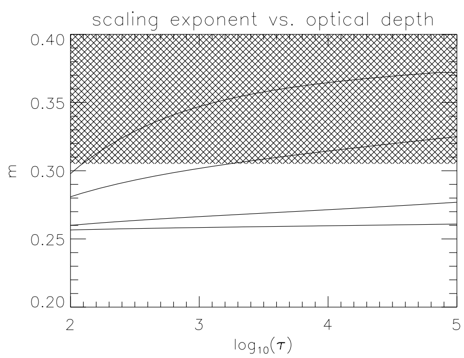

Fig. 1. Disks are stabilized by the eddy-heat flux for sufficiently small Prandtl numbers. The slope $m = 1 - d\,\log K_1/d\,\log\tau$, converges towards a limiting value. From *top* to *bottom*: Pr = 1, 0.1, 0.01, 0.001. Cross-hatched: The unstable region with $m > 0.305$.

the energy equation (12) can be written as

$$\dot{\Pi} = 2\,\frac{\gamma-1}{\gamma+1}\left(\frac{9}{4}\,\alpha_\mathrm{SS}\,\Pi\,\Omega_\mathrm{K} - 2\,Q^-\right). \tag{33}$$

Expressing the cooling term with the help of Eqs. (27)–(29) in terms of Σ and Π yields

$$\dot{\Pi} = \frac{9}{2}\,\frac{\gamma-1}{\gamma+1}\,\alpha_\mathrm{SS}\,\Pi\,\Omega_\mathrm{K}\,\left[1 - (\Pi/\Pi_\mathrm{eq})^{3+m\left(\frac{n}{2}-q\right)}\right]. \tag{34}$$

Π_eq denotes the equilibrium value

$$\Pi_\mathrm{eq} = a_2\,\frac{\mathcal{R}}{\mu}\,\Sigma\,\left[A\,\alpha_\mathrm{SS}\,\Sigma^{1+m(1+n)}\Omega_\mathrm{K}^{1+mn}\right]^{\frac{1}{3+m(n/2-q)}} \tag{35}$$

with

$$A = \frac{3^3\kappa_0^m(\mathcal{R}/\mu)^{1-\frac{mn}{2}}\,a_3^{m-1}}{2^{8+mn}a_2^{2+mn}\sigma K}. \tag{36}$$

In order to avoid a thermal run-away (for Σ fixed) the exponent in (34) must be positive, i.e.

$$3 + m\left(\frac{n}{2} - q\right) > 0. \tag{37}$$

This criterion holds for homologous behaviour. Using equilibrium series, as outlined by Lin & Papaloizou (1996), one is lead to a more general criterion for thermal stability, viz. our criterion (42) for viscous (in)stability, given below.

In order to stabilize the cold disk, the m in (31) must be small enough. Its critical upper limit,

$$m_\mathrm{crit} = \frac{3}{q - n/2}\,, \tag{38}$$

Table 2. Theoretical and numerical growth rates (in units of $1/(\alpha_{\rm SS}\Omega_{\rm K})$) for various cold disk models. In all simulations $K_1 = 100$

$\alpha_{\rm SS}$	Pr/$\alpha_{\rm SS}$	τ	$t_{\rm growth}$	simulation
0.05	20	8000	$+1.1$	$+1.6$
0.01	1	12800	-2.7	-2.1
0.05	0.2	12800	-2.9	-2.1
0.001	1	25500	-2.0	-1.8
0.05	0.02	25500	-2.0	-1.5

is 0.305 for $n = 1/3$ and $q = 10$, which is very close to the minimum value in Fig. 1. Even with a strong eddy-heat transport a cold disk with $q = 10$ proves to be only marginally stable.

The growth rate for small deviations from the equilibrium (and $\gamma = 5/3$) becomes

$$t_{\rm growth} = -\frac{8}{9\alpha_{\rm SS}\Omega_{\rm K}(3 + m(n/2 - q))}. \quad (39)$$

In the equilibrium we have

$$Q^- = \frac{9}{8}\alpha_{\rm SS}\Omega_{\rm K}\Pi, \quad (40)$$

so that with (35) and $\Omega_{\rm K}$ fixed

$$Q^- \propto \Sigma^{\frac{4+m(1+3n/2-q)}{3+m(n/2-q)}} \quad (41)$$

results. Viscous instability, therefore, always happens for

$$\left(4 + m(1 + \frac{3n}{2} - q)\right)\left(3 + m(\frac{n}{2} - q)\right) < 0. \quad (42)$$

For thermal stability with (37), however, the second factor in (42) is positive. Then for $n > -1$ also the first one is positive. Non-negative n provided (cf. Bell et al. 1997) the thermally stable disks do never fulfill (42) so that they are also viscously stable. They are always located on the stable branches of the S-type diagram (cf. Meyer & Meyer-Hofmeister, 1981; Smak 1982).

4.2. Numerical simulations

The validity of our analytical approach has been examined by numerical simulations, using Lagrangian coordinates, and with the in Eq. (8) neglected inertial terms being included. The energy equation (12) has been used with a fully implicit time-stepping method. The code were run for at least a few thermal time-scales. The numerically found damping or rising times are roughly the theoretical ones (Table 2). They exceed the thermal time-scale, which justifies our assumption of a quasi-homologous evolution.

5. Conclusions

For cold disks the well-known scaling relation, $T_{\rm eff}^4 \simeq T_{\rm c}^4/\tau$ must be substituted by the more general relation $T_{\rm eff}^4 \simeq T_{\rm c}^4/\tau^m$. For small m otherwise unstable vertical structures are thermally stabilized. But, with $n = 1/3$ and $q = 10$ the numerically found minimal value, $m \approx 0.26$, is only a little smaller than the threshold value $m_{\rm crit} \approx 0.305$. In order to stay below this limit, the value for the Prandtl number must be less than, say, 0.1. A Prandtl number less than $\alpha_{\rm SS}$, however, seems not to be realistic. Our stability consideration therefore indicates for a cold disk an upper limit, of order of 0.1, for $\alpha_{\rm SS}$, too.

One should be aware that all of the above rests solely on the atmospheric boundary condition, viz. that the surface temperature goes with the effective one, a point stressed already by Faulkner et al. (1983). Thus, it is the radiative atmosphere that – being a kind of bottle-neck – suppresses the radiative cooling.

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