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Theoretical aspects of the inverse Tully-Fisher relation as a distance indicator: incompleteness in $\log V_{\rm Max}$, the relevant slope, and the calibrator sample bias

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Abstract. We study the influence of the assumption behind the use of the inverse Tully-Fisher relation: that there should be no observational cutoffs in the TF parameter $\log V_{\rm M}$. It is noted how lower and upper cutoffs would be seen in a $\log H_0$ vs. "normalized distance" diagram. Analytical expressions, under the simplifying assumption of a normal distribution and the use of the correct TF slope, are derived for the resulting biases, especially the average bias which $\log V_{\rm M}$ cutoffs produce in the derived value of H_0. This bias is shown to be relatively weak, and as such cannot explain the large differences in the reported values of H_0 derived from direct and inverse TF relations.

Some problems of slope and calibration are shown to be more serious. In particular, one consequence of fitting through the calibrators either the slope relevant for field galaxies or the steeper slope followed by calibrators is that the derived value of the Hubble constant comes to depend on the nature of the calibrator sample. If the calibrator sample is not representative of the cosmic distribution of $\log V_{\rm M}$, large errors in the derived value of H_0 are possible. Analytical expressions are given for this error that we term the calibrator sample bias.

Key words: galaxies: distances and redshifts – galaxies: spiral – cosmology: distance scale

1. Introduction

There have been sincere hopes that the inverse Tully-Fisher (TF) relation could overcome the distance dependent selection bias, the so-called Malmquist bias of the 2nd kind (Teerikorpi 1997), influencing the determination of the value of the Hubble constant. First, Schechter (1980) pointed out that the inverse relation of the form

$$p = a'M + b' \quad (1)$$

where M is absolute magnitude (or linear diameter) and p is an observable parameter not affected by selection effects (e.g. the maximum rotation velocity $\log V_{\rm M}$ in the TF relation), can be derived even from magnitude limited samples in an unbiased manner. Secondly, Teerikorpi (1984, or T84) showed explicitly that the inverse relation gives unbiased average distances for a sample of galaxies (if p is not affected by selection), a result later confirmed by Tully (1988) via simulations (see also Hendry & Simmons 1994). Thirdly, Ekholm & Teerikorpi (1997, or ET97) showed that with the inverse relation, one does not necessarily need a volume-limited calibrator sample, which on the contrary is quite critical for the use of the direct relation:

$$M = ap + b \quad (2)$$

The direct TF relation has been succesfully applied for derivation of H_0 by the method of normalized distances that was first used by Bottinelli et al. (1986). Theureau et al. (1997b) have used an improved version of this method in an analysis of a sample of 5171 spiral galaxies that leads to the value of $H_0 \approx 55 \pm 5\,{\rm km\,s^{-1}\,Mpc^{-1}}$. In this method, which is rather similar to the Spaenhauer diagram method of Sandage (1994), one identifies the so-called unbiased plateau in the $\langle \log H \rangle$ vs. normalized distance diagram for a magnitude or angular diameter limited sample, from which H_0 is determined without the need to make further model dependent correction due to selection. Normally, the unbiased plateau contains only 10–20 percent of the total sample. That one can use the whole sample without bias correction is a remarkable advantage of the inverse relation approach.

However, several problems balance the advantages of the inverse relation (Fouqué et al. 1990, Teerikorpi 1990, Sandage et al. 1995, ET97), and one has not yet been able to use it reliably for an independent determination of H_0 (see Theureau 1997, Ekholm et al. 1999). In the present paper, we study two important points which must be clarified before a successful use of the inverse TF relation:

1. How would possible cutoffs in $\log V_{\rm M}$ be seen in the data?
2. How large an influence do upper and lower cutoffs in $\log V_{\rm M}$ have on derived average distances (and hence on H_0)?
3. How does the nature of the calibrator sample influence the derived average distances if the inverse TF slopes for calibrators and field galaxies differ?

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As regards the first point, we emphasize that our approach follows the lines which we have adopted and found useful with the direct TF relation: to detect and overcome the bias without detailed modeling, by studying how the cutoffs would appear in kinematical applications (cf. the increase of $\langle H \rangle$ as a function of true (or normalized) distance, when the direct TF relation is used). Another, more technical approach to the direct TF relation is that of Willick (1994), based on corrections to individual galaxies within an iterative scheme. In principle, a good knowledge of the $\log V_{\rm M}$ selection function could also permit Willick's method to work for the inverse TF relation, in order to derive the slope as it would be without the selection.

Our approach necessarily cuts away a part of the sample. However, as shown by Ekholm et al. (1999) for the KLUN sample, the remaining subsample suitable for the determination of H_0 with the inverse TF method is still much larger than the unbiased subsample for the direct TF relation.

The observational motivation for studying question 2) comes from Theureau et al. (1998), who discuss the detection rate of 21cm line profiles at the Nançay radio telescope, in connection with the angular size limited KLUN galaxy sample. The detection rate is relatively high, 86 percent for galaxies with a previous redshift measurement and 61 percent for those with unknown z. However, more important than the average detection rate, is *how detection depends on* p*, i.e. the selection function* $S(p)$. Unfortunately this is difficult to derive from the raw data. Theureau (1997) notes that one expects loss of galaxies with either a narrow or broad 21 cm line width. Narrow profiles are difficult to detect among the noise, while broad profiles, being low, also tend to be missed.

Hence, as a first approximation we suppose that the distribution of $\log V_{\rm M}$ is affected by sharp lower and upper cutoffs, $p_{\rm l}$ and $p_{\rm u}$, although the selection $S(p)$ more probably depends on various galactic parameters in a complicated manner. How such cutoffs influence, e.g. the value of H_0 as determined by the inverse relation method, has not been previously *quantitatively* discussed.

We study the above questions using normal distributions, which allow analytic expressions. This forms a natural sequel to our previous discussions of the TF relations, both direct and inverse, where the assumption of Gaussianity has been adopted (Teerikorpi 1984, 1987, 1990, 1993). It is worth noting that though we occasionally refer to the KLUN sample (see e.g. Theureau et al. 1997a), the theoretical ideas concerning the above mentioned points 1–3 have general validity. As an extensive sequel to the present work, Ekholm et al. (1999) have investigated whether the large value of H_0, derived with a straightforward application of the inverse TF relation for the KLUN sample, could be due to the errors arising from cutoff and/or calibrator sample biases. They show that the calibrator sample bias is more important.

In this paper we also aim at a better general understanding of the inverse relation as a distance indicator, and present some arguments in a heuristic manner, since the method has been justly criticised as difficult to visualize, in comparison with the "more natural" direct relation approach.

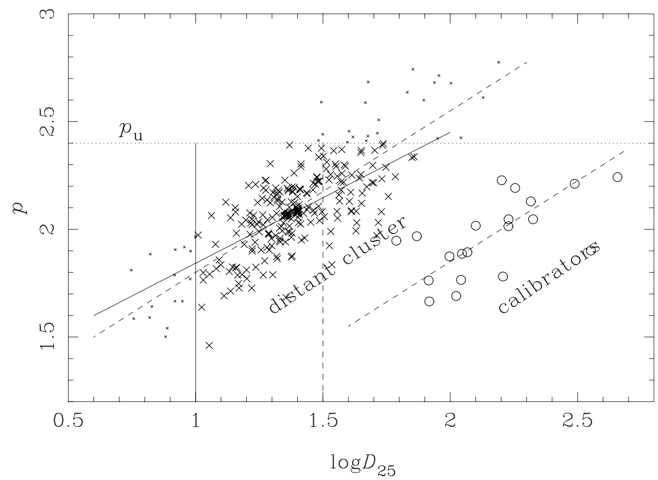

Fig. 1. p vs. log of apparent diameter, $\log D_{25}$ diagram for distant cluster and calibrator galaxies. Upper cutoff at p (dotted line) makes the inverse TF distance to the cluster too small when we force the correct slope, followed by both the calibrators and the cluster (dashed lines), through the cluster. The solid vertical line shows the angular diameter limit, the dashed vertical line is the 'unbiased plateau' limit for the distant cluster. Cluster members that are not in the observed sample are marked with small symbols.

This paper is structured as follows: In Sect. 2 we show qualitatively how cutoffs in $\log V_{\rm M}$ influence the distance determination. In Sect. 3 the method of normalized distance $d_{\rm n}$ is introduced for the inverse TF relation. In Sect. 4, the behaviour in the $\log H$ vs. $d_{\rm n}$ diagram is calculated analytically. In Sect. 5, we derive an analytical expression for the average bias in $\log H$ when there is an upper or lower cutoff in $\log V_{\rm M}$, and give some examples in Sect. 6. The fundamental problems of slope and calibration are discussed in Sects. 7 and 8. Sect. 9 contains concluding remarks, emphasizing the essential points and consequences of the present study.

A note on the nomenclature: We use capital D's for the diameter of a galaxy, D_{25} is the usual apparent 25th-mag isophotal diameter ($[D_{25}] = 0.1$ arcmin), D is the linear diameter ($[D] =$ kpc). In Sect. 4, to make the formulae more readable, we use $\mathcal{D}$ for the $\log D$'s. Small d's are used for inferred distances, $d_{\rm n}$ is the normalized and $d_{\rm kin}$ the kinematical distance. It is assumed that kinematical distances have been calculated from a realistic velocity field model, giving reliable relative distances (in practice from a Virgo-centric infall model as e.g. in Theureau et al. 1997b). The true distance is denoted by r.

2. Influence of cutoffs in $\log V_{\rm M}$: schematic treatment

Let us assume that the selection may be described as cutoffs on the wings of the symmetric p distribution. To illustrate, let us consider the simple case when there is only an upper cutoff $p_{\rm u}$, so that all $p > p_{\rm u}$ are missed, while all $p \le p_{\rm u}$ are measured. The situation is sketched in Fig. 1. The calibrator sample (at a fixed distance) is on the right, while a distant cluster of galaxies is on the left. The $p_{\rm u}$-limit is indicated by a dotted line. The correct regression lines, with the same slopes, are shown. Note that unaware of the p selection, one would derive from the cluster galaxies (or more generally from field galaxies) a too shallow slope. For the moment we simply assume that the correct slope is known. We return to the problem of slope in Sect. 7, and this issue will be discussed elsewhere (Ekholm et al. 1999) in connection with real data.

In inverse TF method (iTF) for deriving distances, one basically shifts one of the regression lines over the other, and the required shift of $\log D_{25}$ gives the (logarithmic) ratio of cluster to calibrator distance.

In practice, one calculates for each cluster galaxy the iTF "distance" $d_{\rm cl}$ using the calibrating relation $p = a' \log D + b'$, so that $\log(d_{\rm cl}/d_{\rm cal}) = \log D(p) - \log D_{25}$. Averaging over all cluster galaxies gives the unbiased distance. This is equivalent to "shifting the regression lines". The iTF "distance" has a systematic error depending on p, though one does not know how large the error is (c.f. Sect. 5 in T84).

In the depicted situation, even if one knows the correct slope, the attempt to force-fit it on the cluster data will shift the line towards larger $\log D_{25}$, because the upper cutoff has eliminated more galaxies from above the "correct" (unbiased) regression line than from below it. Hence, the inevitable result of an upper cutoff is that the iTF distance to the cluster will be underestimated.

Now, because of the limiting angular size, not all galaxies of the cluster are included. So one should add to the diagram a vertical line, only galaxies to the right of which are large enough to enter the sample. Note that this cutoff does not bias the distance indicator if there is no p-cutoff.

If one moves the cluster closer (to the right), the angular size limit cuts away less of the cluster galaxies. Then the influence of the $p_{\rm u}$-cutoff on the derived distances becomes weaker. In terms of the calculated Hubble constant (actually up to an unknown factor, because b'/a' is still unknown), one should see in a $\log H$ vs. $d_{\rm kin}$ diagram first at small kinematical distances (proportional to the true ones) a constant "plateau" when the angular size limit keeps out of the cluster. Then there is an increase due to the larger influence of the $p_{\rm u}$-cutoff.

3. "Unbiased plateau" for the inverse relation

The plateau in the $\log H$ vs. $d_{\rm kin}$ diagram is not an unbiased plateau; it still gives too large a value of $\langle \log H \rangle$, because of a residual effect of the $p_{\rm u}$-limit. The bias can be reduced by applying an upper cutoff to the cluster sample in $\log D_{25}$, as illustrated by the dashed vertical line in Fig. 1. The galaxies between the upper and lower $\log D_{25}$ cutoffs now provide the best iTF distance estimate.

The position of the critical righthand side vertical line in Fig. 1, which cuts away the "bad" $\log D_{25}$ interval, depends on the distance of the cluster (compare the cluster and the calibrators). However, its position is practically constant on the *linear size scale*, independently of the cluster distance. This is clearly seen if one shifts in Fig. 1 the cluster galaxies to the right, i.e. decreases their distance until it coincides with the distance of the calibrator cluster. Because the TF relations are essentially the same, the critical vertical lines will almost coincide. Hence, in the general case of a field galaxy sample, one may shift all galaxies independent of their distances, to one diagram of $\log V_{\rm M}$ vs. $\log D_{25} + \log d_{\rm kin}$, where the abscissa is up to an unknown constant (as far as H_0 is unknown) the same as linear diameter, $\log D$. If one now accepts galaxies left of the (common) critical line only, an unbiased average distance is derived.

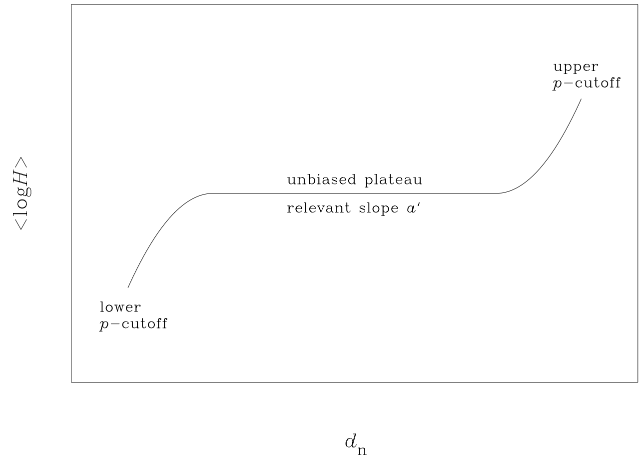

Fig. 2. Effect of the p-cutoffs on the value of $\langle \log H \rangle$ at different $d_{\rm n}$. The normalized distance $d_{\rm n}$ is defined as $D_{25} d_{\rm kin}$.

One might term $\log D_{25} + \log d_{\rm kin}$ (or $d_{\rm n} = D_{25} \cdot d_{\rm kin}$) the normalized distance, analogously to the case of the direct TF normalized distance introduced in T84. Then in the $\log H$ vs. $\log d_{\rm n}$ diagram, using the correct slope a', there is at small normalized distances an unbiased plateau, after which $\langle \log H \rangle$ starts to grow.

The resulting unbiased plateau should be horizontal, if the slope a' is correct. ET97 pointed out a simple device of deciding which is the correct inverse slope: in the $\log H$ vs. $d_{\rm kin}$ diagram the run of points should be horizontal. This was, however, made under the premise of no p-selection. Now we understand that instead of $d_{\rm kin}$ one should use $d_{\rm n} = D_{25} d_{\rm kin}$ on the abscissa, in order to identify the cutoff $d_{\rm n}$. Only galaxies below this cutoff are expected to show a horizontal run on the $\log H$ vs. $d_{\rm kin}$ diagram, if a' is correct. Fig. 2 shows how in the more general case, the lower cutoff causes too small $\langle \log H \rangle$ below the unbiased plateau. Ekholm et al. (1999) will discuss the evidence for upper and lower cutoffs in $\log V_{\rm M}$ in the KLUN sample and how the described method can in practice be implemented.

Though in this manner one may detect sharp cutoffs in $\log V_{\rm M}$ and derive an unbiased value of H_0 without detailed modelling, it is important to know how large an influence the cutoffs would have on H_0, if their presence were ignored. We discuss this in Sects. 4–6.

4. Analytic calculation of $\langle \log H \rangle$ vs. $d_{\rm n}$

Assuming Gaussian distributions and sharp cutoffs at $p_{\rm l}$ and $p_{\rm u}$, it is easy to calculate the expected run of $\langle \log H \rangle$ vs. normalized distance $d_{\rm n}$. For simplicity, we write the inverse relation as

$$p = a' \mathcal{D} + b' \tag{3}$$

We use $\mathcal{D} \equiv \log D_{\rm lin}$ as shorthand notation for the logarithm of the linear diameter. At any fixed $\mathcal{D}$ the dispersion of p is σ_p.

Now the task is to calculate the bias in $\log H$ at each $\mathcal{D}$. This is done starting from the same initial formula (28) of T84 which was used to show the absence of bias. However, in this case one must add the selection function $S(p)$:

$$\langle {\rm bias} \rangle_{\mathcal{D}} = \frac{\int_{-\infty}^{\infty} (\mathcal{D} - \mathcal{D}(p)) S(p) \Phi(p) dp}{\int_{-\infty}^{\infty} S(p) \Phi(p) dp} \tag{4}$$

where $\Phi(p) = {\rm const} \cdot \exp\left[-(p - p(\mathcal{D}))^2 / (2\sigma_p^2)\right]$ and $\mathcal{D}(p) = (p - b')/a'$.

This formula gives for the sample galaxies with true (log) diameter $\mathcal{D}$, the (log) error in the diameter deduced by the inverse relation (using $\mathcal{D}(p)$). It is directly reflected to the derived value of $\langle \log H \rangle$ at constant $\mathcal{D}$ (i.e., at fixed normalized distance $d_{\rm n}$).

With the present choice of the selection function (sharp upper and lower cutoffs), and after a change of variable,

$$x = \frac{p - a'\mathcal{D} - b'}{\sqrt{2}\sigma_p}, \tag{5}$$

Eq. (4) becomes

$$\langle {\rm bias} \rangle_{\mathcal{D}} = -\sqrt{2/\pi} (\sigma_p / a') B(x_{\rm l}, x_{\rm u}) \tag{6}$$

Here the term $B(x_{\rm l}, x_{\rm u})$ can be evaluated in terms of the error function ${\rm erf}(x) = (2/\sqrt{\pi}) \int_0^x \exp(-t^2) dt$:

$$B(x_{\rm l}, x_{\rm u}) = \frac{\exp(-x_{\rm l}^2) - \exp(-x_{\rm u}^2)}{{\rm erf}(x_{\rm u}) - {\rm erf}(x_{\rm l})} \tag{7}$$

where $x_{\rm l}$ and $x_{\rm u}$ correspond to the cutoffs $p_{\rm l}$ and $p_{\rm u}$ via Eq. (5).

Inspection of the formula confirms the qualitative conclusions of Sect. 2. Putting $x_{\rm l} = -\infty$, so that only the upper cutoff remains (and recalling that ${\rm erf}(-\infty) = -1$), the bias is seen to be positive: the diameters are underestimated, leading to underestimated distances, hence to overestimated $\log H$. Note that this calculation does not tell us the average bias in the sample.

5. Calculation of the average bias

In order to have an idea of how large an influence the p-cutoffs have on the value of $\langle \log H \rangle$ calculated from the whole sample using a (correct) inverse relation, we derive the average bias for a special case of radial space density behaviour:

$$n(r) = kr^{-\alpha}, \tag{8}$$

where r is the radial distance from our Galaxy and $\alpha = 0$ corresponds to homogeneity. It is easy to understand that the average bias depends on how the observed galaxies are distributed along the $d_{\rm n}$-axis, which must depend on how galaxies are distributed in space.

Rather than trying to start with averaging the derived $\langle {\rm bias} \rangle_{d_{\rm n}}$ over $d_{\rm n}$ (Eq. 6), we use another route, based on the fact that the bivariate distribution $F(d_{\rm n}, p)$ is known for an angular diameter limited sample, when space density behaves as Eq. (8). We assume for simplicity that the cosmic distribution of p (as well as of $\mathcal{D}$) is normal, which is a rough approximation for fixed Hubble types, see e.g. Fig. 1 in Theureau et al. (1997a).

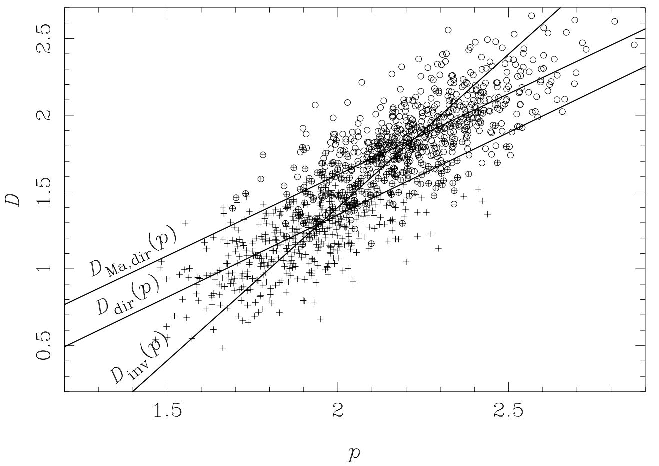

Fig. 3. A volume limited subsample of a synthetic data-set is shown as crosses, and the diameter limited subsample of the same parent data-set is marked with circles. Direct TF relations of these subsamples, $\mathcal{D}_{\rm dir}(p)$ and $\mathcal{D}_{\rm Ma,dir}(p)$, are parallel and separated vertically by $(3 - \alpha) \cdot \ln 10 \cdot \sigma_{\mathcal{D}_p}^2$ (Eq. 11). The inverse relation $\mathcal{D}_{\rm inv}(p)$ (Eq. 12), that under ideal conditions is the same for both subsamples, intersects the direct regression lines at $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D} = \mathcal{D}_0 + (3 - \alpha) \cdot \ln 10 \cdot \sigma_{\mathcal{D}}^2$.

As illustrated by Fig. 1 in T84 for the case of magnitude TF relation, in magnitude and volume limited samples the direct regression lines are parallel and separated by $1.382\sigma_M^2$ (when $\alpha = 0$). The separation happens so that the volume-limited line (carrying with it the bivariate distribution) glides along the unchanged inverse line which goes through the average M of both the volume ($= M_0$) and magnitude limited ($= M_0 - 1.382\sigma_M^2$) bivariate distributions. Fig. 3 gives a similar diagram for the diameter TF relation, showing the three kinds of regression lines for a synthetic sample of field galaxies.

In the present calculation we need to know how the diameter direct relation is constructed from the parameters defining the iTF (c.f. T84, Teerikorpi 1993, where this was given for the magnitude TF relation):

$$\mathcal{D}_{\rm dir}(p) = (a'/\sigma_p^2)\sigma_{\mathcal{D}_p}^2 \cdot p + (\mathcal{D}_0/\sigma_{\mathcal{D}}^2 - a'b'/\sigma_p^2)\sigma_{\mathcal{D}_p}^2 \tag{9}$$

where

$$1/\sigma_{\mathcal{D}_p}^2 = 1/\sigma_{\mathcal{D}}^2 + 1/(\sigma_p/a')^2. \tag{10}$$

The Malmquist shifted (angular size limited) direct relation is

$$\mathcal{D}_{\rm Ma,dir}(p) = \mathcal{D}_{\rm dir}(p) + (3 - \alpha) \cdot \ln 10 \cdot \sigma_{\mathcal{D}_p}^2 \tag{11}$$

and the inverse relation is

$$\mathcal{D}_{\rm inv}(p) = (1/a')p - b'/a'. \tag{12}$$

One can express the average bias at a fixed p using the inverse and (Malmquist-shifted) direct relations:

$$\langle {\rm bias} \rangle_p = \frac{\int_{-\infty}^{\infty} (\mathcal{D} - \mathcal{D}(p)) \, \Phi_{{\rm Ma},p}(\mathcal{D}) d\mathcal{D}}{\int_{-\infty}^{\infty} \Phi_{{\rm Ma},p}(\mathcal{D}) d\mathcal{D}} \tag{14a}$$

where

$$\Phi_{\mathrm{Ma},p}(\mathcal{D}) = \mathrm{const} \cdot \exp\left(-(\mathcal{D} - \mathcal{D}_{\mathrm{Ma,dir}}(p))^2/2\sigma^2_{\mathcal{D}_p}\right) \quad (15)$$

so that

$$\langle \mathrm{bias} \rangle_p = \mathcal{D}_{\mathrm{Ma,dir}}(p) - \mathcal{D}_{\mathrm{inv}}(p) \quad (14b)$$

This simple expression is then easy to integrate over all p in the sample, when one notes that the distribution of p is

$$\mathrm{const} \cdot \exp\left(-(p - p'_0)^2/(2a'^2\sigma^2_{\mathcal{D}})\right) \quad (16)$$

where p'_0 gives the Malmquist-shifted average value of p (shifted by a' times the shift of $\mathcal{D}_0$):

$$p'_0 = a'\mathcal{D} + b' + (3 - \alpha) \cdot \ln 10 \cdot \sigma^2_{\mathcal{D}_p} \quad (17)$$

Averaging $\langle \mathrm{bias} \rangle_p$ over all p from p_{l} to p_{u} gives:

$$\langle \mathrm{bias} \rangle_{\mathrm{ave}} = -\left(\sigma_{\mathcal{D}_p}/\sigma_{\mathcal{D}}\right)^2 \sigma_{\mathcal{D}} \sqrt{2/\pi} B(x_{\mathrm{l}}, x_{\mathrm{u}}) \quad (18)$$

where x_{l} and x_{u} are

$$x_{\mathrm{l,u}} = \frac{p_{\mathrm{l,u}} - p'_0}{\sqrt{2}a'\sigma_{\mathcal{D}}} \quad (19)$$

Note that the information on the space density (via α and Eq. 16) is contained in x_{l} and x_{u}, which show how far away from the Malmquist-shifted average p'_0 the cutoffs are, in terms of the dispersion $\sigma_P = a'\sigma_{\mathcal{D}}$ of the cosmic (Gaussian) distribution function of p. The factor before $B(x_{\mathrm{l}}, x_{\mathrm{u}})$ can also be written in terms of σ_p and σ_P. Then

$$\langle \mathrm{bias} \rangle_{\mathrm{ave}} = -(\sigma^2_p/(\sigma^2_p + \sigma^2_P))(\sigma_P/a')\sqrt{2/\pi} B(x_{\mathrm{l}}, x_{\mathrm{u}}) \quad (20)$$

Such an analytic calculation is possible only in the given special cases of space density. Generally, the distribution of points in the p vs. $\log d_{\mathrm{n}}$ diagram will differ from the Gaussian bivariate case: at each p, the shift of $\langle \mathcal{D}_{\mathrm{dir}}(p) \rangle$ will be due to different Malmquist biases of the 1st kind. However, even in this case the inverse relation is not affected and the calculation and prediction of Sect. 3 is valid.

6. Application of the average bias formula

We are interested in seeing how large an effect on $\log H$ cutoffs of different severities have. Table 1 shows the values of $B(-\infty, x_{\mathrm{u}})$ ($B(x_{\mathrm{l}}, \infty)$) and the bias in $\langle \log H \rangle$ for different values of upper (lower) cutoff $x_{\mathrm{u(,l)}}$, and the corresponding completeness percentage of the sample.

In particular, if half of the galaxies are lost because of the upper cutoff (while there is no lower cutoff), formula (18) adopts a simple form, which gives a rough estimate of the bias for such distorted samples.

$$\begin{aligned} \langle \mathrm{bias} \rangle_{50\%} &= -(\sigma_{\mathcal{D}_p}/\sigma_{\mathcal{D}})^2 \sigma_{\mathcal{D}} \sqrt{2/\pi} B(-\infty, 0) \\ &= -0.8(\sigma_{\mathcal{D}_p}/\sigma_{\mathcal{D}})^2 \sigma_{\mathcal{D}} \end{aligned} \quad (21)$$

With representative values $\sigma_{\mathcal{D}} = 0.28$ and $\sigma_{\mathcal{D}_p} = 0.14$ this gives the bias -0.056, or distances underestimated by about 14 percent, in the average. Intuitively, one might have expected a more substantial influence of the p cutoff, knowing that the assumption of no p-selection is so fundamental for the inverse relation method.

Table 1. Bias in $\langle \log H \rangle$ due to p cutoffs. The upper sign in the $\pm$ symbols refers to p_{u} while the lower sign refers to p_{l}.

$x_{\mathrm{u(,l)}}$	%	$-\sqrt{2/\pi}B(-\infty, x_{\mathrm{u}})$ $(-\sqrt{2/\pi}B(x_{\mathrm{l}}, \infty))$	bias*
$\pm\infty$	100	0.00	0.000
±0.84	80	±0.22	±0.015
0.00	50	±0.80	±0.056
∓0.84	20	±1.68	±0.118

$*$ for $\sigma_{\mathcal{D}} = 0.28$, $\sigma_{\mathcal{D}_p} = 0.14$.

7. Problems of slope and calibration

In the discussion thus far, we have assumed that the normalized distance has been accurately calculated, so that there are no observational errors in $\log D_{25}$ and the kinematical distance scale (d_{kin}) is accurate as well. However, this cannot be the case, and as a result, field galaxies collected from a fixed true distance will show in the p vs. $\log d_{\mathrm{n}}$ diagram a different slope than without such errors. This slope will differ from the intrinsic slope followed by calibrators.

Two questions arise: first, in such a situation, what happens to the slope-criterion requiring that $\langle \log H \rangle$ for the unbiased plateau galaxies goes horizontally in the $\log H$ vs. d_{kin} diagram? Secondly, how do we calculate the actual value of $\log H_0$ with calibrators known to have a different iTF slope? Here we expand somewhat the compact treatment in Teerikorpi (1990).

Consider two galaxy clusters in the p vs. $\log D_{25}$ diagram, both having (in their unbiased d_{n}-part), the same iTF slope a', which however differs from the calibrator slope. It is evident that the slope a' gives the correct relative distance of these clusters, when one shifts the regression lines one over the other (c.f. Fig. 1). Hence, as an answer to the first question, this is also the slope which corresponds to the horizontal distribution of $\langle \log H \rangle$; correct distance ratios reveal the linear Hubble law.

Assume now that the calibrators follow the slope $a'_{\mathrm{c}} > a'$. One may describe in a picturesque manner how to do the comparison with calibrators, leading to an unbiased value of $\log H_0$. As the shallower slope for the field galaxies is assumed to be due to errors in d_{n}, one may think of “adding” random errors to calibrator linear log diameters $\mathcal{D}$, until the slope falls to a' and then use the obtained relation for calibration (“shifting of regression lines”). Fortunately it is not necessary to actually “shuffle” $\mathcal{D}$’s - this is equivalent to forcing the slope a' through the calibrators, at least in an ideal situation.

So, let us assume first that the calibrator sample is volume-limited. Then adding random errors to $\mathcal{D}$ means that $\langle \mathcal{D} \rangle = \mathcal{D}_0$ remains the same, and one arrives at the relation

$$\langle p \rangle = a'\langle \mathcal{D} \rangle + b' \quad (22)$$

On the other hand, the actual calibrators follow

$$\langle p \rangle = a'_{\mathrm{c}}\langle \mathcal{D} \rangle + b'_{\mathrm{c}} \quad (23)$$

Hence, after this trick, the new zero-point which must be applied for the sample galaxies is:

$$b' = b'_{\rm c} + (a'_{\rm c} - a')\langle\mathcal{D}\rangle \quad (24)$$

There is also a longer route to this result, using Eq. (7) giving the direct relation in terms of the inverse parameters and $\mathcal{D}_0$. If one increases $\sigma_{\mathcal{D}_p}$ so that $a'_{\rm c} \to a'$ and requires that the direct relation does not change, one also arrives at Eq. (24).

On the other hand, the formula for b' is exactly what is obtained if one simply forces the slope a' through the calibrators. From this follows that a correct calibration is achieved by deriving the zero-point by a force-fit of the slope a' through the calibrators.

8. The calibrator sample bias when one uses relevant field slope a'

In the above discussion it is noteworthy that one does not have to know the intrinsic slope $a'_{\rm c}$ for the calibrators. However, the calibrator sample must fulfill a special condition. It must be equivalent to a volume-limited sample, or reflect the cosmic Gaussian distribution, as was already noted in Teerikorpi (1990). From Eq. (23), it is clear that two calibrator samples with different $\langle\mathcal{D}\rangle$ yield different zero-points b' for the same a' of the field galaxies, while there can exist only one such correct b', the one corresponding to $\langle\mathcal{D}\rangle = \mathcal{D}_0$ or $\langle p\rangle = a'\mathcal{D}_0 + b' = p_0$.

If $\langle\mathcal{D}\rangle$ of the calibrator sample deviates from the cosmic $\mathcal{D}_0$, there is a systematical error in b', resulting in an average error in the derived $\langle\log H\rangle$:

$$\Delta\langle\log H\rangle = (a'_{\rm c}/a' - 1)\Delta\mathcal{D} \quad (25)$$

or in terms of the average p:

$$\Delta\langle\log H\rangle = (a'_{\rm c}/a' - 1)\Delta p/a' \quad (26)$$

This source of error is very significant if the calibrator sample has not been constructed with the intention of reaching the cosmic distribution of p, if for example the selection of galaxies has aimed at detection of Cepheids. In this way, one may have a volume-limited sample for each p (c.f. Theureau et al. 1997a), but *this does not guarantee that the distribution of p reflects the cosmic distribution* from which the field galaxy sample has been drawn.

That this could be a genuine problem, consider the calibrator sample of KLUN with Cepheid distances. An indication of the calibrator sample bias is seen by comparing the distribution of calibrator p's with that of the whole sample. Because the latter is diameter limited, it suffers from a Malmquist bias, hence its median should be displaced to a larger value of p in comparison with the calibrators. Ekholm et al. (1999) show that there is no such shift, which implies that calibrators' $\langle p\rangle$ is larger than the cosmic value. Such a situation results in an overestimated $\log H$ when one is compelled to use $a' < a'_{\rm c}$.

Application of Eqs. (25–26) requires a knowledge of the average $\mathcal{D}_0$ or p_0 of the cosmic distribution functions to be compared with those of the calibrator sample. It is clear that any calculation of $\mathcal{D}_0$ from the field sample requires, besides a suitable method, the value of H_0 itself, hence an iterative approach. Below we show that there is another much more convenient route which does not require an explicit knowledge of the difference $\Delta\mathcal{D}$ (or the value of H_0). However, it assumes a radial space distribution law of galaxy number density.

Table 2. H_0 as derived using $a' = 0.5$ when $H_0(\rm true) = 55$ and the true calibrator slope is $a'_{\rm c}$.

$a'_{\rm c}$	$H_0(\alpha=0)$	$H_0(\alpha=0.8)$
0.60	67	64
0.65	74	68
0.70	82	74
0.75	90	79

9. Another route: using calibrator slope $a'_{\rm c}$ and making $a'_{\rm c}/a'$ and Malmquist corrections

In principle, the problem of the p-distribution of the calibrators does not affect the determination of the zero-point $b'_{\rm c}$ when one forces the correct slope $a'_{\rm c}$ on the calibrators. This makes one ask what happens if $a'_{\rm c}$ and $b'_{\rm c}$ are then used for the field sample. First of all one expects a distance-dependent bias, according to Eq. 15 from Teerikorpi (1990), here adopted to diameters:

$$\Delta\langle\log H\rangle(r) = (\langle\mathcal{D}\rangle(r) - \mathcal{D}_0)(a'_{\rm c}/a' - 1) \quad (27)$$

Here $\langle\mathcal{D}\rangle(r)$ is the average (log) diameter for the galaxies at true distance r. Now one may ask what is the average bias for the sample? The average bias clearly depends on the average of $\langle\mathcal{D}\rangle(r) - \mathcal{D}_0$. This is actually the Malmquist bias of the 1st kind for Gaussian distribution of diameters with dispersion $\sigma_{\mathcal{D}}$, and in the case of a space density distribution $\propto r^{-\alpha}$ one gets:

$$\langle\Delta\log H\rangle = (3-\alpha)\cdot\ln 10\cdot\sigma^2_{\mathcal{D}}(a'_{\rm c}/a' - 1) \quad (28)$$

When $\alpha = 0$ the distribution is homogeneous. In order to use this as a correction, one must know both $a'_{\rm c}$ and a'. Clearly, if the field and calibrator slopes are identical, then $\langle\log H\rangle$ is unbiased, and this is so irrespective of the nature of the calibrator sample. Note also that this formula contains the width $\sigma_{\mathcal{D}}$ of the total diameter function, not the smaller dispersion $\sigma_{\mathcal{D}_p}$. Hence the effect can be formidable even when the slopes $a'_{\rm c}$ and a' do not differ very much.

As a concrete example, consider the KLUN field galaxies following the inverse slope $a' \approx 0.5$ which is in agreement with the simple disk model of Theureau et al. (1997a). This slope, when forced through the calibrators, gives $H_0 \approx 75\,{\rm km\,s^{-1}\,Mpc^{-1}}$. Above it was noted that the calibrator sample cannot represent the cosmic distribution of $\log V_{\rm M}$, and this is the situation when a too high H_0 is derived if the actual slope is steeper than the slope from the field galaxies. It is natural to ask, how large a slope $a'_{\rm c}$ would in this manner explain the difference between $H_0(\rm dir) \approx 55$ and $H_0(\rm inv) \approx 75$. Taking $\sigma_{\mathcal{D}} \approx 0.25$, we get predicted $H_0(\rm inv)$ for different slopes $a'_{\rm c}$ as given in Table 2.

The last column corresponds to a decrease in the average space density, corresponding to density $\propto r^{-0.8}$ as derived by Teerikorpi et al. (1998). It may be concluded that in order to explain the difference between $H_0 = 75$ and 55 solely in this manner, it is required that actually $a'_{\rm c} \approx$ 0.65–0.75, instead of ≈ 0.5. Ekholm et al. (1999) show that such a steep calibrator slope really is consistent with the data.

10. Concluding remarks

In the present study we have searched for answers to a few basic questions concerning the inverse TF relation as a distance indicator. *First*, how to see the signature of observational incompleteness in the TF parameter $\log V_{\rm M}$, analogously to the well-known increase in $\log H$ at increasing (normalized) distance, in the case of the direct TF relation? *Secondly*, in which sense and how *significant* is the influence of the $\log V_{\rm M}$ incompleteness alone on the derived value of $\log H$ (supposing one knows the correct inverse TF slope)? *Thirdly*, how to work with the situation when the inverse slope obeyed by the calibration sample differs from the relevant slope for the field sample (or for a cluster), and how important is in this regard the nature of the calibration sample?

We have generalized the "fine-tuning" scheme discussed by ET97, and have pointed out that there is a useful $\log H$ vs. $d_{\rm n}$ representation for the inverse relation, revealing both the relevant inverse slope and cutoffs in $\log V_{\rm M}$. Here $d_{\rm n}$ is the normalized distance applicable to the inverse TF relation, actually the (log) linear diameter within an unknown constant.

We have derived simple analytical formulae which give the influence of the $\log V_{\rm M}$ cutoffs on derived average distance (hence, on $\log H$). The treatment is based on Gaussian distributions, and, in the case of the average bias, on the assumption that the space density is proportional to $r^{-\alpha}$. The probable case that there is an upper observational cutoff in $\log V_{\rm M}$, would cause an overestimated value of H_0. However, this effect is not at all dramatic – even if the upper cutoff excludes half of the galaxies, H_0 would be increased typically by only 14 percent. It may be concluded that a cutoff in $\log V_{\rm M}$ cannot alone explain derived large values of H_0.

The problems of the relevant inverse TF slope and the nature of the calibrator sample lead to serious consequences on the use of the inverse TF relation as a distance indicator. It has been previously realized that the inverse slope a' applicable to the field sample is not necessarily the same as the slope $a'_{\rm c}$ obeyed by the calibrators, which may result in a biased value of H_0 (Teerikorpi 1990). In the present paper, such situations were systematically studied.

If the two slopes are the same (in practice, when accuracies of calibrator distances are high and the photometric measurements for the field sample are not less accurate than those for the calibrators), then there is no bias, and it is also true that the nature of the calibrator sample (volume-limited, magnitude-limited, etc.) is not important (ET97).

If the sample inverse slope a' is shallower than the calibrator slope (as normally expected), then matters come to depend critically on the nature of the calibrator sample. First, there is the possibility that the calibrator sample is a true, volume-limited representation of the Gaussian cosmic distribution function of $\log V_{\rm M}$ from which the field sample has been taken (the latter subject only to Malmquist bias). In this ideal case, it is permitted to use the field sample slope, with the zero-point obtained by a force-fit through the calibrators. This was the solution suggested by Teerikorpi (1990), and in this case it is not necessary to know the value of the calibrator slope.

However, generally the calibrator sample's $\langle p \rangle$ is shifted from the cosmic p_0. If one knows the shift Δp (or $\Delta \mathcal{D}$) *and* the slopes $a'_{\rm c}$ and a', then one may calculate the resulting error in $\log H$, when the field sample's a' is used. The systematic error is given by Eq. (25).

Another, more convenient route, which bypasses the explicit need for Δp, is to use the calibrators' slope $a'_{\rm c}$ (and corresponding zero-point) and to calculate analytically the resulting systematic error in terms of the Malmquist bias of the 1st kind. This assumes that the average space number density around us is proportional to $r^{-\alpha}$. The systematic error is given by Eq. (27).

We repeat that in the general case both $a'_{\rm c}$ and a' (with corresponding zero-points from force-fits through calibrators) result in erroneous average distance estimates. Fortunately, we have now some quantitative control on this error.

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