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Dynamical scaling of matter density correlations in the Universe

An application of the dynamical renormalization group

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Abstract. We show how the interplay of non–linear dynamics, self–gravity, and fluctuations leads to self–affine behavior of matter density correlations quite generically, i.e., with a power law exponent whose value does not depend in a very direct way on the specific details of the dynamics. This we do by means of the Renormalization Group, a powerful analytical tool for extracting asymptotic behavior of many-body systems.

Key words: gravitation – hydrodynamics – methods: analytical – cosmology: theory – cosmology: large-scale structure of Universe

1. Introduction

The Universe at short and moderate distance scales is inhomogeneous, being filled by numerous structures. In fact the clumpy structure of matter in the Universe extends for over 15 orders of magnitude in linear size, from stars to the largest clusters of galaxies, and for about 18 orders of magnitude in mass (Ostriker 1991). At scales from galaxies to clusters of galaxies and even superclusters, the Universe exhibits self-similar, fractal behavior (Mandelbrot 1983; Peebles 1993). At the largest scales, those probed by COBE observations (Smoot et al. 1991), for example, the Friedmann-Robertson-Walker (FRW) cosmology which is based on the hypotheses of homogeneity and isotropy, provides adequate description of many uncorrelated observations. Isotropy is broken by the primordial fluctuations in the matter density which subsequently become amplified by the gravitational instabilities.

In this paper we explore the observed self-affinity and scaling in the density correlation function and tackle them with the renormalization group, a tool originating in quantum field theory (Gell-Mann & Low 1954), then extended to condensed matter physics (Wilson & Kogut 1974), later on extended to problems in hydrodynamics (Forster et al. 1977) and applied to surface growth (Kardar et al. 1986) and in the last few years applied to gravitation and cosmology (Pérez-Mercader et al. 1996, 1997).

Our starting point will be the non-relativistic hydrodynamic equations governing the dynamical evolution of an ideal self-gravitating Newtonian fluid in a FRW background. Through a series of steps involving physically justified approximations, these equations reduce to a single evolution equation for the cosmic fluid's velocity potential. At this point one has a single deterministic hydrodynamic equation. To this we add a noise source which represents the influence of fluctuations and dissipative processes on the evolution of the fluid, arising from, but not limited to, viscosity, turbulence, explosions, late Universe phase transitions, gravitational waves, etc. The resulting equation is recognized as a cosmological variant of the Kardar-Parisi-Zhang (KPZ) equation (Kardar et al. 1986) which is the simplest archetype of nonlinear structure evolution and has been studied extensively in recent years in the context of surface growth phenomena (Barabàsi & Stanley 1995). While the essential steps involved in arriving at this KPZ equation are reviewed in Sect. 2, we refer the reader to the concise details of its complete cosmological-hydrodynamical derivation in Buchert et al. 1997. The relevance of stochastic fluctuations in structure formation in the Universe has been addressed in Berera & Fang 1994, where the rôle of the KPZ equation was emphasized. For FRW cosmologies with curved spatial sections, this equation contains a time-dependent mass-term. In Sect. 3 we discuss briefly the adiabatic approximation which allows one to treat this mass term as a constant during most of the expansion of the background cosmology.

Having thus reduced the essential hydrodynamics to a single dynamical stochastic equation, we turn to the results of a dynamical renormalization group analysis to calculate the precise form of the density-density correlation function using the KPZ equation. To do so we proceed first by extracting the renormalization group equations governing the evolution of couplings appearing in the cosmological KPZ equation with respect to changes in scale and integrating out the short distance degrees of freedom. As we are interested in the behavior of the system at ever larger scales, we will run the renormalization group (RG) equations into the large scale (infrared) limit, a process known as coarse-graining. The fixed points of these RG equations determine the long–time, long–distance behavior of the system and their subsequent analysis yields both the power law form of the correlation functions as well as their explicit exponents. In particular, the value of the galaxy-galaxy correlation func-

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tion exponent $\gamma \sim 1.8$ is calculated, and its value understood in terms of spatio-temporal correlated noise. A summary of our results and discussion are presented in the closing section.

2. From hydrodynamics to the KPZ equation

Structure formation in the Universe in the range of a few Mpc up to several hundred Mpc can be modelled as the dynamical evolution of an ideal self-gravitating fluid and studied within the framework of General Relativity. However, we will restrict our treatment to the case of Newtonian gravity for a number of physically justifiable reasons. First and foremost, the length scales involved in large scale structure formation by matter after decoupling from radiation are smaller than the Hubble radius (of order ≈ 3000Mpc at the present epoch), so that general relativistic effects are negligible. Secondly, it is known observationally that matter flow velocities are much smaller than the speed of light and that this non-relativistic matter plays a dominant rôle over radiation in the epoch of structure formation. Thus, the evolution in space and time of large scale structure in the Universe will be described adequately by the Newtonian hydrodynamic equations for a fluid whose components are in gravitational interaction in an expanding background metric[1]. The connection with observations and the subsequent phenomenology is made by writing these equations in terms of the comoving coordinates $\boldsymbol{x}$ and coordinate time t, and are (Peebles 1993)

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \boldsymbol{\nabla} \cdot [(1+\delta)\boldsymbol{v}] = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + H\boldsymbol{v} + \frac{1}{a}(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{w} - \frac{1}{a\rho}\boldsymbol{\nabla} p, \tag{2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{w} = -4\pi G a \bar{\rho}(t)\delta, \;\; \boldsymbol{\nabla} \times \boldsymbol{w} = 0, \tag{3}$$

where $\boldsymbol{v}(\boldsymbol{x}, t)$ and $\boldsymbol{w}(\boldsymbol{x}, t)$ are the peculiar velocity and acceleration fields of the fluid, respectively. In the above equations, $H = \dot{a}(t)/a(t)$ is the Hubble parameter, $a(t)$ is the scale factor for the underlying cosmological background and $\bar{\rho}(t)$ is the (time–dependent) average cosmological density. Thus, the local density is $\rho = \bar{\rho}(1+\delta)$ and the dimensionless density contrast is $\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$.

In the linear regime and for dust ($p = 0$), the vorticity of the velocity field $\boldsymbol{v}$ damps out rapidly by virtue of the background expansion, and therefore $\boldsymbol{v}$ can be derived from a velocity potential ψ, $\boldsymbol{v} = -\nabla\psi$, in the long-time limit. Furthermore, parallelism between $\boldsymbol{w}$ and $\boldsymbol{v}$ holds, i. e.,

$$\boldsymbol{w} = F(t)\boldsymbol{v} \tag{4}$$

where $F(t)$ is the long-time limit solution of the following Ricatti equation

$$\dot{F} = 4\pi G \bar{\rho}(t) - HF - F^2. \tag{5}$$

[1] The resolution length must be large enough so as to legitimize the use of continuum equations (i.e., to validate the long-wavelength, or hydrodynamic limit) for the description of the matter density, peculiar velocity and acceleration fields. This requirement implicitly singles out a lower length scale which we may take to be greater than roughly the mean galaxy-galaxy separation length (~ 5 Mpc).

Following the Zel'dovich approximation, we make the assumption that in the weakly non-linear regime, the parallelism condition (4) continues to hold, with $F(t)$ given as the solution of (5) but we now employ the fully non-linear Euler equation (2) with pressure term included. That is, we assume that the non-linearities and pressure have not yet had enough time to destroy the alignment in the acceleration and velocity fields of the cosmic fluid. Assuming also that the pressure is a function of the density, $p = p(\rho)$, one arrives at the following equation for the velocity potential (Buchert et al. 1997)

$$\frac{\partial \psi}{\partial t} = \nu f_1(t)\nabla^2\psi + \frac{\lambda}{2} f_2(t)(\nabla\psi)^2 + \frac{f_3(t)}{T}\psi + \eta(\boldsymbol{x}, t). \tag{6}$$

where we have also added a stochastic source or noise η (see a few lines below for a discussion). The three dimensionless functions of time, f_1, f_2 and f_3 are

$$f_1(t) = \frac{p'(\bar{\rho})\, F(t)}{4\pi G \bar{\rho}(t) a^2(t)} \frac{1}{\nu}, \tag{7}$$

$$f_2(t) = \frac{1}{a(t)\lambda}, \tag{8}$$

$$f_3(t) = (F(t) - H(t))T. \tag{9}$$

The positive constants ν, λ and T are introduced to carry dimensions. One can think of them as typical values of the corresponding time-dependent coefficients during the epoch we are interested in. The reason why we introduce them here explicitly (instead of setting them equal to unity) will become clear in Sect. 4. The f_1-term arises from the pressure term in the Euler equation which we have expanded to lowest order about the zero-pressure limit. Note that higher-order terms in this Taylor expansion will yield higher-derivative terms ($O(\nabla^2\psi)^2$) involving quadratic and higher powers of the field ψ. The f_2-term is simply the convective term from the original Euler equation (2) written in the new variables, while the f_3-term entails the competition between the damping of perturbations due to the expansion ($-H\psi$) and the enhancement due to self-gravity ($+F\psi$).

The Zel'dovich approximation (4) together with the Poisson equation yield an important relationship between the density contrast and $\nabla^2\psi$, the Laplacian of the velocity potential, namely

$$\delta(\boldsymbol{x}, t) = \frac{F}{4\pi G a \bar{\rho}} \nabla^2 \psi(\boldsymbol{x}, t), \tag{10}$$

which shows that the density contrast tracks the divergence of the peculiar velocity. Because of this identity, we are able to calculate density correlation functions in terms of (derivatives of) velocity potential correlations. This is why it is worthwhile investigating the scaling properties of the KPZ equation in detail.

The noise $\eta(\boldsymbol{x}, t)$ appearing in (6) represents the effects of random forces acting on the fluid particles, including the presence of dynamical friction, and also of degrees of freedom whose size is smaller than the *coarse–graining length* (indeed, the very fact that we are using continuous fields to describe the dynamics of a system of discrete particles means that we are implicitly introducing a coarse–graining length, such that the details below this resolution length are not resolvable. (The relevance of this length-scale will become clear in Sect. 4.) There exist a number of physical processes on various length and time scales that contribute to an effective stochastic source in the Euler equation. Indeed, any dissipative or frictional process leads to a stochastic force, by virtue of the fluctuation-dissipation theorem. So, fluid viscosity and turbulence should be accountable to some degree by adding a noise term in the dynamical equations. Early and late Universe phase transitions, the formation of cosmic defects such as strings, domain walls, textures, are sources for a noisy fluctuating background, as are also the primordial gravitational waves and gravitational waves produced during supernovae explosions and collapse of binary systems. Another way to visualize the noise is to imagine a flow of water in a river; there are stones and boulders in the river-bed which obviously perturb the flow and have an impact on its nature. The noise can be thought of as a means for modelling the distribution of obstacles in the river bed. In cosmology the obstacles in the river-bed represent, e.g., density fluctuations and the fluid is the matter flowing in this "river". The noise is phenomenologically characterized by its average value and two–point correlation function as follows,

$$\langle \eta(\boldsymbol{x}, t) \rangle = 0 \tag{11}$$

$$\langle \eta(\boldsymbol{x}, t)\eta(\boldsymbol{x}', t') \rangle = 2D(\boldsymbol{x}, \boldsymbol{x}'; t, t'), \tag{12}$$

with $D(\boldsymbol{x}, \boldsymbol{x}'; t, t')$ a given function of its arguments (see below). All higher cumulants vanish. The noise is thus Gaussian. However, the velocity field need not be (and in general, will not be) Gaussian as a consequence of coarse-graining of the dynamics.

The stochastic hydrodynamic equation (6) can be further cast into

$$\frac{\partial \Psi}{\partial \tau} = \nu \nabla^2 \Psi + \frac{1}{2}\lambda (\nabla \Psi)^2 - m^2(\tau)\Psi + \tilde{\eta}(\boldsymbol{x}, \tau), \tag{13}$$

by means of the following simple change of variables and redefinition of the "physical–time" t into a "conformal–time" τ

$$\Psi(\boldsymbol{x}, \tau) \equiv \frac{f_2(t(\tau))}{f_1(t(\tau))}\psi(\boldsymbol{x}, t(\tau)), \tag{14}$$

$$\tau = \int_{t_0}^{t} dt' f_1(t'). \tag{15}$$

The linear (mass) term in Eq. (13) is given by

$$m^2(\tau) = -\frac{d}{d\tau} \ln \left[\frac{f_2(\tau)}{f_1(\tau)} \exp\left(\frac{1}{T} \int_{t_0(\tau_0)}^{t(\tau)} dt' \; f_3(t') \right) \right], \tag{16}$$

and the noise transforms into

$$\tilde{\eta}(\boldsymbol{x}, t) = \frac{f_2(t)}{f_1(t)^2}\eta(\boldsymbol{x}, t). \tag{17}$$

Finally, Eq. (13) is a generalization to a cosmological setting of the Kardar–Parisi–Zhang (KPZ) equation for surface growth (Kardar et al. 1986), and differs from the standard KPZ equation in the linear (mass) term which, as seen from Eq. (16), originates in the expansion present in the background cosmology. Here, ν plays the rôle of a diffusion constant while λ is proportional to the average "speed of growth" of ψ.

3. Adiabatic approximation

In this section we briefly comment on some results about the time–dependent mass term (16) of Eq. (13). The detailed analysis that justifies our assertions is carried out in Buchert et al. 1997.

For equations of state of the form $p = \kappa\rho$, the time–dependent mass term (16) is identically zero for flat FRW backgrounds, but evolves with time in closed and open FRW backgrounds, keeping a constant sign during the epoch of interest: $m^2 > 0$ in open backgrounds and $m^2 < 0$ when the background is closed.

There are two time scales in our problem: an intrinsic time scale associated with the dynamical evolution prescribed by Eq. (13), which is determined by the values of the coefficients in it, and an extrinsic time scale associated with the background expansion, which is determined by the parameters defining the background cosmology. A numerical study of the function $m^2(t)$ for the physically interesting range of parameters, reveals that its relative variation over these two time scales is small, which justifies an "adiabatic" approximation: we can assume $m^2(t)$ to be a constant, rather than a function of time, when performing the Renormalization Group analysis in the next section. This assumption greatly simplifies the analysis, and the presence of a non-zero mass term in the KPZ equation results in a much richer renormalization group trajectory flow and fixed point structure than that corresponding to the massless case, as we will see below.

4. The renormalization group equations

The starting point for the subsequent analysis is the KPZ equation for the velocity potential, eq. (13). Because this is a nonlinear equation, and because there are fluctuations, as represented by the stochastic noise term, it is to be expected on general grounds that renormalization effects will modify the coefficients appearing in this equation. Due to the existence of fluctuations, the coefficients in Eq. (13) (ν, λ, m^2) do not remain constant with scale: as is well known (Ma 1976; Amit 1984; Binney 1992) there are renormalization effects that take place and modify these coefficients (or "coupling" constants); the modifications are calculable and can be computed using dynamical renormalization group (DynRG) techniques. The renormalization group is a standard tool developed for revealing in a systematic way, how couplings change with scale in any physical system under the action of (coarse)-graining. Moreover, it predicts that all correlation functions go as power laws when the system is near a fixed, or critical point, i.e., they exhibit self–similar behavior. However, the value of the scaling exponent changes from one fixed point to another. *In fact, the values for the couplings* *at some reference scale establish the fixed point to which the system will be attracted to or repelled from, because these reference values will belong to a specific "basin of attraction".* We will look for the IR stable fixed points since these reflect the system behavior characteristic of long times and large distances and ceases to change as we look at the system at ever larger scales.

Indeed, even before fluctuations are accounted for, one can easily obtain the so-called canonical scaling laws for the parameters appearing in (13); these follow by performing the simultaneous scaling transformation $\mathbf{x} \to s\mathbf{x}$, $\tau \to s^z \tau$ and $\psi \to s^\chi \psi$ and requiring the resultant equation to be *form-invariant*, that is, that the KPZ equation transforms to a KPZ equation at the new, larger ($s > 1$) length and time scale. This simple requirement leads to scale dependence in the parameters appearing in the original equation of motion as well as in the correlation functions built up from the KPZ field $\psi(\boldsymbol{x}, t)$. When fluctuations and noise are "turned on", renormalization effects change the scaling behavior of the couplings and correlation functions away from their canonical forms, in a way which depends on the basins of attraction for the fixed points of the DynRG equations of the couplings.

4.1. Calculation of the RGEs

The RG transformation consists of an averaging over modes with momenta k in the range $\Lambda/s \leq k \leq \Lambda$ where $s > 1$ is the scale factor for the transformation, followed by a dilatation of the length scale $\mathbf{x} \to s\mathbf{x}$ in order to bring the system back to its original size[2]. Here, Λ plays the rôle of an UV-momentum, or short distance, cut-off characterizing the smallest resolvable detail whose physics is to be described by the dynamical equations. In classical hydrodynamics, this scale would be identified with the scale in the fluid at and below which the molecular granularity of the medium becomes manifest and the hydrodynamic limit breaks down. The renormalization group transformation has the important property that it becomes an *exact* symmetry of the physical system under study whenever that system approaches or is near a critical point, because it is near the critical point (or points) where the system exhibits scale-invariance or, equivalently, where the system can be described in terms of a conformal field theory. The hallmark for a system near criticality is that its correlation functions display power law behavior. This means that the statistical properties of the system remain the same, except possibly up to a global dilatation or change of unit of length. We can calculate the power law exponents by requiring that the dynamical equations remain invariant under the above RG transformation and under the further change of scale

$$x \to sx \quad t \to s^z t, \text{ and } \Psi \to s^\chi \Psi, \tag{18}$$

where z and χ are numbers which account for the response to the re-scaling. By eliminating s from the above, we arrive at the fact that the two-point correlation function for the velocity potential scales as

$$\langle \Psi(\boldsymbol{x}, t)\Psi(\boldsymbol{x}', t')\rangle \propto \mid \boldsymbol{x} - \boldsymbol{x}' \mid^{2\chi} f\left(\frac{\mid t - t' \mid}{\mid \boldsymbol{x} - \boldsymbol{x}' \mid^z}\right), \tag{19}$$

where χ is the roughness exponent, z the dynamical exponent, and the scaling function $f(u)$ has the following asymptotic behavior (see, e.g., Barabàsi & Stanley 1995):

$$\lim_{u\to\infty} f(u) \to u^{2\chi/z}, \tag{20}$$

$$\lim_{u\to 0} f(u) \to \text{constant}. \tag{21}$$

Notice that because of eq. (18), large s means going to the large distance or infrared (IR) and (for $z > 0$) long time limit, while small s corresponds to the short distance or ultraviolet (UV) and short time limit.

Making use of the relations Eq. (14) and Eq. (10), one can immediately obtain the scaling behavior of the 2–point correlation function for the density contrast, and thus study the asymptotic behavior of this function in different regimes. In the following we will implement this procedure. First, we obtain the scaling or RG equations for the couplings by imposing form-invariance on the Eq. (6), then by using the property of constancy of the couplings near fixed points, we obtain and calculate the fixed points themselves and the corresponding values of the exponents.

We now characterize the Gaussian noise, $\eta(\boldsymbol{x}, t)$. This is done by choosing the noise correlation function. Here we will use colored or correlated noise, whose Fourier transform satisfies

$$\begin{aligned} \langle \tilde{\eta}(\boldsymbol{k}, \omega)\rangle &= 0, \\ \langle \tilde{\eta}(\boldsymbol{k}, \omega)\tilde{\eta}(\boldsymbol{k}', \omega')\rangle &= 2\tilde{D}(k, \omega)(2\pi)^{d+1}\delta(k + k')\delta(\omega + \omega'), \\ \tilde{D}(k, \omega) &= D_0 + D_\theta k^{-2\rho}\omega^{-2\theta}, \end{aligned} \tag{22}$$

where D_0 and D_θ are two couplings describing the noise strength, and the ρ, θ exponents characterize the noise power spectrum in the momentum and frequency domains. White noise corresponds to $D_\theta = 0$. The explicit functional form of the noise amplitude $\tilde{D}(k, \omega)$ reflects the fact that correlated noise has power law singularities of the form written above (Medina el al. 1989).

The solution of (13) can be carried out in Fourier space where iterative and diagrammatic techniques may be developed (Medina et al. 1989). A standard perturbative expansion of the solution to Eq. (13) coupled with the requirement of form–invariance and the property of renormalizability, lead to the following RG equations for the couplings:

$$\begin{aligned} \frac{\mathrm{d}m^2}{\mathrm{d}\log s} &= zm^2, \\ \frac{\mathrm{d}\nu}{\mathrm{d}\log s} &= \nu[z - 2 - \frac{\lambda^2}{\nu^3}\frac{K_d}{4d}\Lambda^{d-2}V^{-2}\{(d - 2V^{-1})D_0 \end{aligned}$$

[2] The term "renormalization group" is actually a misnomer, since due to the process of averaging over small distances and the subsequent loss of information, the RG transformations do not have an inverse. Consequently the technically correct name would have to be the "renormalization semi-group".

$$
\begin{aligned}
& \quad +(d-2V^{-1}-2\rho)D_\theta V^{-2\theta}\sec(\theta\pi)(1+2\theta)\}], \\
\frac{\mathrm{d}D_0}{\mathrm{d}\log s} &= D_0(z-2\chi-d)+\frac{\lambda^2}{\nu^3}\frac{K_d}{4}\Lambda^{d-2}V^{-3}\times \\
& \quad [D_0^2+2D_0D_\theta V^{-2\theta}\sec(\theta\pi)(1+2\theta) \\
& \quad +D_\theta^2V^{-4\theta}\sec(2\theta\pi)(1+4\theta)], \\
\frac{\mathrm{d}D_\theta}{\mathrm{d}\log s} &= D_\theta[z(1+2\theta)-2\chi-d+2\rho], \\
\frac{\mathrm{d}\lambda}{\mathrm{d}\log s} &= \lambda[\chi+z-2 \\
& \quad -\frac{\lambda^2D_\theta}{\nu^3}\frac{K_d}{d}\Lambda^{d-2}V^{-3-2\theta}\theta(1+2\theta)\sec(\pi\theta)].
\end{aligned} \tag{23}
$$

The full details of the calculation of these equations will be presented elsewhere (Martín–García & Pérez-Mercader 1998). The calculation of the RGE's for the *massless* KPZ equation in the presence of colored noise is given in (Medina et al. 1989). Here, s is the scale factor of Eq. (18), and $K_d = \frac{S_d}{(2\pi)^d}$ is a dimensionless geometric factor proportional to the surface area of the d-sphere S_d, where d is the spatial dimension. This set of equations describes how the coupling constants $\nu, \lambda, m^2, D_0, D_\theta$ evolve as one varies the scale at which the system is studied, a process commonly referred to in the condensed matter literature as "graining". In the present context we will be interested in the coarse-graining behavior, since we seek to uncover the behavior of our cosmological fluid as one goes to larger and larger scales. In actual practice we will investigate the coarse-graining flow in a two-dimensional (white noise) or three-dimensional (correlated noise) parameter space. This is because one may cast the above set of RG equations in terms of a smaller, yet equivalent set, by employing the dimensionless couplings defined as $V = 1 + \frac{m^2}{\nu\Lambda^2}$, $U_0 = \lambda^2 D_0 K_d \Lambda^{d-2}/\nu^3$, and $U_\theta = \lambda^2 D_\theta K_d \Lambda^{d-2-2\rho-4\theta}/\nu^{3+2\rho}$. Doing so, one arrives at the following reduced set of renormalization group equations:

$$
\begin{aligned}
\frac{\mathrm{d}U_0}{\mathrm{d}\log s} &= (2-d)U_0+\frac{U_0^2}{4dV^3}[d+3(dV-2)] \\
& \quad +\frac{U_\theta^2}{4V^{3+4\theta}}\sec(2\pi\theta)(1+4\theta) \\
& \quad +\frac{U_0U_\theta}{4dV^{3+2\theta}}(1+2\theta)\sec(\pi\theta)\times \\
& \qquad [2d-8\theta+3((d-2\rho)V-2)] \\
\frac{\mathrm{d}U_\theta}{\mathrm{d}\log s} &= (2-d+2\rho+4\theta)U_\theta+\frac{U_0U_\theta}{4dV^3}(dV-2)(3+2\theta) \\
& \quad +\frac{U_\theta^2}{4dV^{3+2\theta}}(1+2\theta)\sec(\pi\theta)\times \\
& \qquad [-8\theta+((d-2\rho)V-2)(3+2\theta)] \\
\frac{\mathrm{d}V}{\mathrm{d}\log s} &= (V-1)[2+\frac{1}{4dV^3}[(dV-2)U_0 \\
& \quad +((d-2\rho)V-2)V^{-2\theta}(1+2\theta)\sec(\pi\theta)U_\theta]]
\end{aligned} \tag{24}
$$

The dimensionless couplings U_0, U_θ measure the strength of the "roughening" effect due to the combined action of noise and the non-linearity against the "smoothing" tendency of the diffusive term in Eq. (13): these couplings grow when either λ or the noise intensity (as given by the parameters D_0 and D_θ) grow, and they become smaller when the diffusive action (measured by ν) grows. The dimensionless coupling V measures the competition between the diffusion and mass terms. Indeed, neglecting for the moment the noise and nonlinear terms, one can write (13) in Fourier space as

$$
\frac{\partial\tilde{\Psi}}{\partial\tau} = -(\nu k^2+m^2)\tilde{\Psi}. \tag{25}
$$

Now, if $m^2 \geq 0$ (i.e., $V \geq 1$), the perturbations in the field Ψ are damped. But if on the other hand, $m^2 < 0$ ($V < 1$), there exists a length scale $L = \sqrt{-\frac{\nu}{m^2}}$ below which the perturbations are damped, but above which they grow. The case $V < 0$ corresponds to a length scale $L < \Lambda^{-1}$: this scale is *smaller* than the minimum resolvable length scale in the problem at hand, and therefore the diffusive term becomes unimportant. We remark in passing that the limiting behavior in the RG flow as $V \to 0^+$ leads to a technical requirement: one must assume that the diffusive term is non-negligible (i.e., $\nu \neq 0$) in order to compute the RG equations.

From the above equations one calculates their fixed points to which the graining flow drives the couplings. The corresponding fixed point exponents z and χ follow from substituting the fixed point solutions so obtained back into the previous set of RG equations (23). These exponents control and determine the *calculated* asymptotic behavior of the correlation functions. The system will be attracted to fixed points in the IR or UV regimes depending on whether they are IR-attractive or UV-attractive although, as with any autonomous set of differential equations, other possibilities exist.

5. Fixed points

The fixed points, collectively denoted by g^*, are the solutions to the system of algebraic equations obtained by setting the right hand sides of equations (24) equal to zero. They control the stability properties of the system of DynRG equations and therefore they characterize the asymptotic behavior of correlation functions. When the couplings attain their critical values, the system becomes critical and the correlation functions enter into power law regimes, characteristic of physical systems in the critical state where they are scale invariant. *Which fixed point the system is attracted to or repelled from depends on the basin of attraction where the initial conditions for the DynRG equations are located.* As the couplings evolve under graining into their critical values, the n–point correlation functions approach power law (or scaling) forms because they are the solution to first order quasi–linear partial differential equations (the Callan-Symanzik (CS) equations (Amit 1984; Binney et al. 1992)) whose characteristics are the solutions to Eqns. (23). This CS equation simply expresses the fact that the correlation functions are invariant under the renormalization group transformation. In other words, the system is scale-invariant at its fixed points, hence, correlation functions must take the form of algebraic power laws, since these are the only mathematical functions that are scale-invariant. The $physics$ of each of these

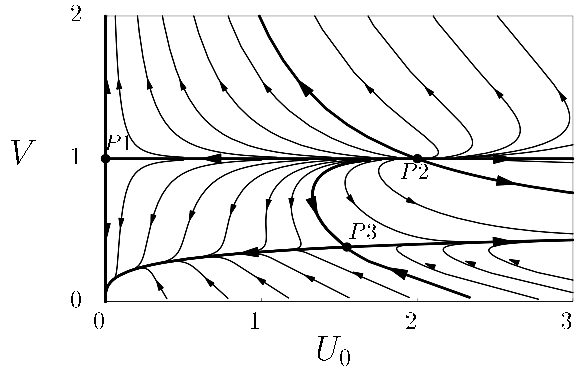

Fig. 1. Flow diagram in (U_0,V) space. Arrows indicate IR ($s \to \infty$) flow.

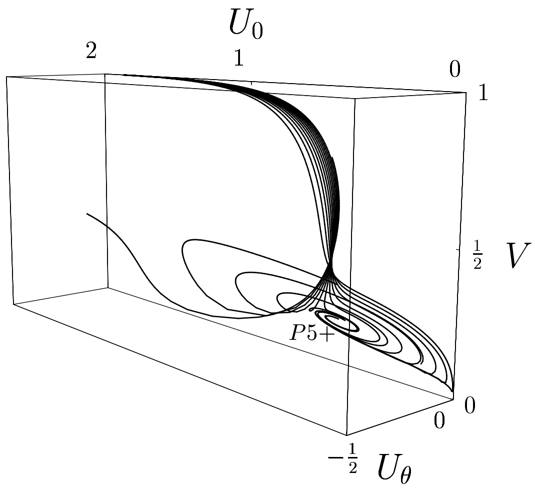

Fig. 2. Flow diagram in (U_θ, U_0, V) space around P5+.

fixed points is different, because it depends to a large extent on the magnitude and sign of the scaling exponent as well as on the attractive or repulsive nature of the fixed point.

The flows and the fixed points of our set of RG equations are represented in Fig. 1, for white noise and Fig. 2 for colored noise, while the features of our fixed point analysis are conveniently summarized in the adjoining Table 1.

5.1. The value of the critical exponents at each critical point. White noise

In the case of white noise ($D_\theta = 0$) we solve for the fixed points of the (reduced set) of RG equations. Since $U_\theta^* = 0$ in this case, these fixed points will all lie in a two-dimensional coupling space spanned by (U_0, V). There are three fixed points, labelled as $P1, P2$ and $P3$, whose coordinates are listed in Table 1. Two of them, $P1, P2$ lie in the line $V^* = 1$, corresponding to ${m^*}^2 = 0$ in the RG-evolved KPZ equation, while $P3$ lies in the region bounded between the $V = 1$ and $V = 0$ lines. By substituting these fixed point values back into the original set of RG equations (23), we solve for the corresponding fixed point exponents χ and z. These values are listed in the third column of Table 1. Linearizing the RG equations about each one of their fixed points allows one to calculate the infrared (IR) stability properties of the fixed points and thus characterize their behavior with respect to the coarse-graining. This entails expanding the RG equations in $U_0 \to U_0^* + \delta U_0$ and $V_0 \to V_0^* + \delta V_0$ to first order in the perturbations and solving for δU_0 and δV_0. In general, the eigenvectors of this linearized system will involve linear combinations of δU_0 and δV_0. The associated eigenvalues, which determine the stability properties of the fixed point, are also listed in Table 1. A positive eigenvalue indicates that the coarse-graining induces a flow away from the point along the eigen-direction, while a negative eigenvalue indicates the point is stable in the infrared, since the flow will be into the fixed point. The type, or class, of fixed point, whether it be IR-attractive (all eigenvalues negative), IR-repulsive (all eigenvalues positive) or a saddle point (mixed sign eigenvalues) is listed in Table 1. The coarse-graining flow of the couplings in the white noise case is given in Fig. 1. These flow lines were calculated by integrating numerically the set of two coupled first order differential equations for U_0 and V, using general choices for the RG equation initial conditions.

5.2. The value of the critical exponents at each critical point. Colored noise

For colored or correlated noise ($D_\theta \neq 0$), we now deal with a three dimensional space of dimensionless couplings U_θ, U_0 and V. Using the same procedure as discussed above, we solve for the RG equation fixed points which lead to a total of seven fixed points, including the same three points $P1, P2$ and $P3$ obtained in the white noise limit. Thus, allowing for colored noise yields four additional fixed points which we will denote by $P4\pm$ and $P5\pm$, since they arise in pairs. Their coordinates are listed in the table, together with their associated exponents χ and z and IR eigenvalues and stability properties under coarse-graining. Both, the positions of these points and values of their exponents, depend in general upon the values of the noise exponents ρ and θ, which parametrize spatial and temporal correlations in the noise. The pair of fixed points labelled as $P4\pm$ always lie in the plane $V^* = 1$, (i.e., ${m^*}^2 = 0$) but their location within this plane varies with the noise exponents. Moreover, any RG flow which starts off in this plane will always remain in this plane (this plane acts as a separatrix). It is important to point out that there are values of the noise exponents ρ, θ which lead to *complex* values of the fixed point coordinates for $P4\pm$. This is because these fixed points arise as solutions of a quadratic algebraic equation whose discriminant can become negative for values of ρ and θ in certain domains in parameter space. We must rule out such values of the noise exponents because of physical reasons. For the allowed values of ρ and θ (i.e., those that lead to real fixed points), we find that $P4\pm$ have exponents z and χ that are complicated functions of these parameters, but we have checked that for all allowed values, either $\chi \leq 0$ or $\chi \geq \frac{4}{3}$. The IR eigenvalues and the nature of these fixed points depend on ρ, θ. For the choice shown, i.e., $\rho = 2.65$ and $\theta = 0.1$, both $P4\pm$ are saddle points. In fact, we have confirmed that the pair $P4\pm$ will always be saddle points whenever $\rho \in (2.60, 2.75)$

Table 1. Characteristics of the fixed points.

Point	Position (U_θ^*, U_0^*, V^*)	(z, χ)	$\gamma = 4 - 2\chi$	IR-eigenvalues	Class
P1	$(0, 0, 1)$	$(2, -1/2)$	5	$(4.7, -1, 2)$	Saddle point
P2	$(0, 2, 1)$	$(13/6, -1/6)$	$13/3$	$(5.2, 1, 2.2)$	IR-repulsive
P3	$(0, 1.55, 0.38)$	$(0, 2)$	0	$(-1.7, 2.0, -15.1)$	Saddle point
P4±	$V^* = 1,\ (U_0^*, U_\theta^*)$ depend on (ρ, θ) Some pairs are not allowed by a discriminant condition*	Depend on (ρ, θ) $\chi \leq 0$ or $\chi \geq 4/3$	Depends on (ρ, θ) $\gamma \geq 4$ or $\gamma \leq 4/3$	Sign depends on (ρ, θ)	
	For $\rho = 2.65,\ \theta = 0.1$				
	P4+ $(4.2, 6.4, 1)$	$(0.64, 1.53)$	0.94	$(-6.2, 1.8, 0.64)$	Saddle point
	P4– $(3.4, 1.7, 1)$	$(0.62, 1.52)$	0.96	$(-5.5, -1.6, 0.62)$	Saddle point
P5±	(U_θ^*, U_0^*, V^*) depend on (ρ, θ) Some pairs are not allowed by a discriminant condition*. Essentially the allowed region is $\rho \in [0, 7/2]$, $\theta \in [0, 1/4)$	$(0, \rho - 3/2) \Rightarrow \chi \in [-3/2, 2]$ which includes the range $(1.1, 1.25)$ for χ	$7 - 2\rho$	Sign depends on (ρ, θ)	
	For $\rho = 2.65,\ \theta = 0.1$				
	P5+ $(-0.1, 0.43, 0.19)$	$(0, 1.15)$	1.7	$(-0.4 \pm 3.5i, -27.3)$	IR-attractive
	P5– $(3.0, -1.96, -0.55)$	$(0, 1.15)$	1.7	$(-19.2, 8.5, -1.5)$	Saddle point

* These fixed points are obtained as real solutions of a second degree algebraic equation, and therefore the set of pairs that give real fixed points are restricted by a discriminant inequality.

Remarks:
– Only P5+ has a basin of attraction, which is approximately in the region $V \in (0, 1), U_0 \in (0, 1.5), U_\theta \in (-0.2, 0)$. P2 has a large "basin of repulsion", and the other points are saddle points.
– The trajectories outside the P5+ basin of attraction reach infinity or fall to the singular $V = 0$ plane at a finite "time".
– Only with P5+ can we adjust the observed values $\gamma \in (1.5, 1.8)$, because P5– has $V^* < 0$ in this region, which implies smaller length scales than the one corresponding to the UV cut-off Λ.

and $\theta \in (0, 0.23)$. The reason for choosing these particular exponent intervals will become clear in Sect. 6.

The next pair of fixed points $P5\pm$ also have coordinates and exponents depending on the noise exponents. As in the case of $P4\pm$, there are values of ρ, θ (same as for $P4\pm$) leading to complex couplings, which we rule out. The allowed values of the noise exponents lie roughly in the respective intervals $\rho \in [0, \frac{7}{2}]$ and $\theta \in [0, \frac{1}{4})$. The exponents for $P5\pm$ are given by the simple functions $z = 0$ and $\chi = \rho - \frac{3}{2}$, and the stability properties do not depend on the particular values of the noise exponents within the above mentioned intervals. For the choice $\rho = 2.65$ and $\theta = 0.1$, $P5+$ is IR attractive while $P5-$ is an unstable saddle point. We must nonetheless exclude $P5-$ from our consideration since it lies below the plane $V = 0$, i.e., it corresponds to a renormalized KPZ mass $m^{*2} < -\nu\Lambda^2$, which therefore exceeds the scale of the momentum cut-off imposed on our perturbative calculation.

6. Scaling properties of matter density correlations

We can now begin to put together the results we have obtained in relation with the scaling properties *predicted* by the DynRG. Using equations (10) and (14) we can write for the 2–point correlation function of the density contrast in comoving coordinates that

$$\xi(|\boldsymbol{x} - \boldsymbol{y}|, t) \equiv \langle \delta(\boldsymbol{x}, t)\, \delta(\boldsymbol{y}, t) \rangle = \left(\frac{F}{4\pi G a \bar{\rho}} \right)^2 \frac{f_1^2}{f_2^2} \nabla_x^2 \nabla_y^2 \langle \Psi(\boldsymbol{x}, t) \Psi(\boldsymbol{y}, t') \rangle \,|_{t'=t} \ . \tag{26}$$

We see that the scaling behavior of $\xi(r)$ is determined once we know the scaling behavior for the equal time two–point correlation function of the density contrast for the velocity potential $\Psi(\boldsymbol{x}, t)$,

$$C(|\boldsymbol{x} - \boldsymbol{y}|) \equiv \langle \Psi(\boldsymbol{x}, t) \Psi(\boldsymbol{y}, t) \rangle \ . \tag{27}$$

From Eqns. (19) and (21), we have

$$\langle \Psi(\boldsymbol{x}, t) \Psi(\boldsymbol{y}, t) \rangle \sim |\boldsymbol{x} - \boldsymbol{y}|^{2\chi} \,. \tag{28}$$

Introducing (for convenience and notational consistency with the literature in critical phenomena) $\eta = 2 - d - 2\chi$, we have

$$C(r) \sim r^{-(d-2+\eta)} \,;\ \tilde{C}(k) \sim k^{\eta-2} \tag{29}$$

which on using Eq. (26) above for equal times gives

$$\xi(r) \sim r^{-(d+2+\eta)}, \tag{30}$$

for the correlation function, and

$$P(k) \sim k^{2+\eta}, \tag{31}$$

for its Fourier transform, the power spectrum.

Eqs. (30) and (31) are now suitable for comparison with observations[3], since they are written in comoving coordinates, and therefore correspond to quantities directly inferred from observations and provided we assume that light traces mass. Otherwise we would need to correct for bias, a strategy we leave for a future publication. From (30) and the definition of the exponent η, we see that the exponent γ measured from large scale galaxy surveys, where $\xi_{OBS}(r) \sim r^{-\gamma}$, is calculable in term of the roughening exponent χ and is given by[4]

$$\gamma = 4 - 2\chi(\rho, \theta). \qquad (32)$$

Using our fixed point analysis we have computed all the exponents χ and z for all the fixed points (fourth column in Table 1). The predicted values for γ are tabulated in the table as shown. Thus, at each fixed point, we derive a power law for the density correlation function $\xi(r) \sim r^{-\gamma}$. In the case of white noise alone, we see that none of the three corresponding fixed points are capable of reproducing any of the inferred values of γ ($1.5 \lesssim \gamma \lesssim 1.8$) from observations. It is of interest to point out, however, that the point $P3$ yields the exponent $\gamma = 0$, which means that the correlation function is strictly constant at this point, i.e., the matter distribution is perfectly homogeneous. However, $P3$ is an IR unstable saddle point and a fine-tuning in the initial values of U_0 and V would be required in order to have the system flow into it under coarse-graining.

For colored noise, the RG behavior becomes significantly richer since we now must map out RG equation flow trajectories in a three dimensional space of couplings (see Fig. 2). A glance at the tabulated calculated values of the exponent γ shows that $P5+$ is the only fixed point capable of yielding values of γ within the currently accepted range of values inferred from observations, by choosing $\rho \in (2.60, 2.75)$ and any $\theta \in (0, 0.23)$, since $\chi = \chi(\rho) = 7 - 2\rho$ for $P5\pm$. Although γ is independent of θ, a non-zero value of θ in the above interval must be chosen in order that $P5\pm$ exist (this pair of fixed points vanish identically when $\theta = 0$). It is also the *only* IR-stable fixed point in the three–dimensional coupling space spanned by the couplings (V, U_0, U_θ). This implies the following behavior: any point initially in the basin of attraction of $P5+$ inevitably ends up at this fixed point as larger and larger scales are probed: *no fine tuning is required.*

[3] Observation has shown that large scale structure depends on cosmological parameters such as the age of the Universe t_0 and initial density Ω_0. While these parameters (and others) do not appear explicitly in our calculations, they are there implicitely in quantities such as r_0, the scale factor, whose calculation cannot be carried out without a deeper knowledge of the noise sources. Nevertheless, the renormalization group gives one a way to calculate the power law scaling law and its exponent as a function of which particular basin of attraction the systems happens to start out in. This explains exactly what we mean when we claim that no fine tuning of parameters is required in order to account for the value for the scaling exponent inferred from the observations. Irrespective of where the system starts out in a particular basin of attraction, in time it inevitably flows into that basin's fixed point: this is very much a kind of independence of initial conditions.

[4] Notice that when written in this way, the value of γ seems to be independent of the spatial dimensionality d. But this is fallacious, since χ itself depends on d as is seen from the RGE's.

7. Discussion and conclusions

In this paper we have presented an *analytical* calculation of the density-density correlation function for non-relativistic matter in a FRW background cosmology. The calculation hinges on blending two essential theoretical frameworks as input. First, we take and then reduce the complete set of non-relativistic hydrodynamic equations (Euler, Continuity and Poisson) for a self-gravitating fluid to a single *stochastic* equation for the peculiar-velocity potential ψ. The noise term $\eta(\boldsymbol{x}, t)$ models in a phenomenological but powerful way different stochastic processes on various length and time scales. Second, we apply the well-established techniques of the dynamical Renormalization Group to calculate the long–time, long–distance behavior of the correlation function of the velocity potential (which relates directly, as we have seen, and under the conditions we have specified, to the density-density correlation) as a function of the stochastic noise source in the cosmological KPZ equation.

When we carry out a detailed RG fixed point analysis for our KPZ equation, we find that simple white noise alone is not sufficient to account for the scaling exponent $(1.6 \lesssim \gamma \lesssim 1.8)$ inferred from observations of the galaxy-galaxy correlation function. However, we can get close to this range for the observed exponent if the cosmic hydrodynamics is driven by correlated noise. In fact, we need only adjust the degree of spatial correlation of the noise to achieve this, since the calculated value of the exponent $\gamma = 4 - 2\chi(\rho)$ associated with $P5+$ is independent of the degree of temporal correlations, as encoded in θ. Moreover, $P5+$ is the only fixed point with this property, and is simultaneously IR-attractive, both desirable properties from the phenomenological point of view. This last property is very important, since it means that the self–similar behavior of correlations is a *generic* outcome of the dynamical evolution, rather than an atypical property that the system exhibits only under very special conditions.

Thus we have learned that colored noise seems to play an important rôle in the statistics of large scale structure. Now, *why* it is that a non-zero θ and a $\rho \in (2.6, 2.75)$ are the relevant noise exponents is a question one still has to ask. To answer it, one would in principle have to derive the noise source itself starting from the relevant physics responsible for generating the fluctuations. Here, we content ourselves with the phenomenological approach. Nevertheless, at some future point, it may be possible to say more about the spectrum of noise fluctuations.

It must be pointed out that the Renormalization Group allows us to compute only *asymptotic* correlations, i.e., after sufficient time has elapsed so that the effect of noise completely washes out any traces of the initial conditions in the velocity field or initial density perturbations. Hence, there will be a time–dependent *maximum* length scale, above which noise has not become dominant yet and correlations therefore remember the initial conditions, as well as a transition region between these two regimes. The length scale at which this transition occurs depends on noise intensity and on the amplitude of initial perturbations. Therefore, a physical model of the origin of noise would enable us to predict this scale, but such a task is well beyond the scope of the present paper.

It is also interesting that the Renormalization Group allows us to compute the proportionality coefficient of the matter density correlations, i.e., the quantity r_0 in $\xi(r) = (r/r_0)^{-\gamma}$, by means of the so-called *improved* perturbation expansion (Weinberg 1996). This length scale r_0 marks the transition from inhomogeneity ($r \ll r_0$, $\xi(r) \gg 1$) to homogeneity ($r \gg r_0$, $\xi(r) \ll 1$). But r_0 is also dependent on the noise intensity and other free parameters of our model, so that a prediction is impossible without a deeper knowledge of the noise sources. At this point, we should mention the recent debate about the observational value of r_0: on the one hand there is the viewpoint that homogeneity has been reached well within the maximum scale reached by observations (Davis 1996) ($r_0 \lesssim 50 Mpc$); on the other hand there is an opposing viewpoint as expressed by Sylos Labini et al. 1998 who argue that $r_0 \gtrsim 200 Mpc$. Either viewpoint can be easily incorporated into our model, by suitably choosing the noise intensity. But if the latter value of r_0 turns out to be correct, this would indicate within our model that the noise intensity is really large and therefore, that noise and stochastic processes played a much more important rôle in the History of the Universe than has been thought to date.

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References

Amit D.J., 1984, Field Theory, the Renormalization Group and Critical Phenomena. World Scientific, Singapore
Barabàsi A.L., Stanley H.E., 1995, Fractal Concepts in Surface Growth. Cambridge University Press, Cambridge
Berera A., Fang Li-Zhi, 1994, Phys. Rev. Lett. 72, 458
Binney J.J., Dowrick N.J., Fisher A.J., et al., 1992, The Theory of Critical Phenomena. Oxford University Press, Oxford
Buchert T., Domínguez A., Pérez-Mercader J., submitted to Physical Review D., astro-ph/9709218
Davis M., astro-ph/9610149 and references therein.
Forster D., Nelson D.R., Stephen M.J., 1977, Phys. Rev. A16, 732
Gell-Mann M., Low F., 1954, Phys. Rev.95, 1300
Kardar M., Parisi G., Zhang Y.C., 1986, Phys. Rev. Lett. 56, 889
Ma S.K., 1976, Modern Theory of Critical Phenomena. Benjamin, Reading
Mandelbrot B., 1983, The Fractal Geometry of Nature. W.H. Freeman and Company, New York
Martín–García J.M., Pérez-Mercader J., 1999, in preparation.
Medina E., Hwa T., Kardar M., Zhang Y.C., 1989, Phys. Rev. A 39, 3053
Ostriker J.P., 1991, Development of Large-Scale Structure in the Universe. Cambridge University Press, Cambridge
Peebles P.J.E., 1993, The Large Scale Structure of the Universe, Princeton University Press, Princeton, 1980 and Principles of Physical Cosmology. Princeton University Press, Princeton, 1993
Pérez-Mercader J., Goldman T., Hochberg D., Laflamme R., 1996/1997, Phys. Lett. A222, 177; Hochberg D., Pérez-Mercader J., Gen. Rel. Grav. 28, 1427; Barbero J.F., Domínguez A., Goldman T., Pérez-Mercader J., Europhys. Lett. 38, 637
Smoot G.F., Bennett C.L., Kogut A., et al., 1991, ApJ 371, L1
Sylos Labini F., Montuori M., Pietronero L., 1998, Phys. Rep. 293, 61
Weinberg S., 1996, The Quantum Theory of Fields. Volume 2, Cambridge University Press
Wilson K.G., Kogut J., 1974, Phys. Rep. 12C, 75-200