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Avoided crossings in radial pulsations of neutron and strange stars

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Abstract. Radial pulsations of neutron stars and strange quark stars with nuclear crust are studied. The avoided crossing phenomenon occurring for the *radial* modes is found and discussed. Neutron star models are constructed using a realistic equation of state of dense matter and strange star models using a phenomenological bag model of quark matter. The eigenfrequencies of the three lowest modes of linear, adiabatic pulsations are calculated, using the relativistic equations for the radial oscillations.

Key words: dense matter – stars: neutron – stars: oscillations

1. Introduction

Radial oscillations of dense stars (neutron and strange stars) were studied by many authors (see eg. Glass & Lindblom (1983), Väth & Chanmugam (1992) and references therein).

In the present paper we study the radial pulsations of neutron stars and strange quark stars with nuclear crust in the framework of general relativity. Our aim is to present an interesting phenomenon – avoided crossing of this radial modes. Avoided crossing manifest itself as a rapid change of the frequency of the certain oscillatory mode when stellar configuration gradually changes. At this point the frequencies of two consecutive modes are very close and star "switches" the oscillatory properties from one to another. This effect was previously noticed in the analysis of the *nonradial* oscillations in "ordinary" stars (Aizenman et al. (1977), Christensen-Dalsgaard (1980)) and has been recently discussed by Buchler et al. (1997) in the case of radial pulsations of the classical variable stars. For neutron stars the avoided crossing phenomenon was considered by Lee & Strohmayer (1996) for *nonradial* oscillations of rotating neutron stars.

The avoided crossings of radial modes in neutron stars was mentioned in our recent paper (Gondek, Haensel and Zdunik (1997, hereafter Paper I) also in the case of hot protoneutron star.

In this paper we study the avoided crossing phenomenon for two models of dense matter: the low temperature limit of the Lattimer & Swesty (1991) equation of state of the neutron matter and the strange quark matter described by the simple bag model.

The plan of the paper is as follows. In Sect. 2 we describe equations of state for which we found the avoided crossing phenomenon. We calculate here also the adiabatic index of the matter which is crucial for the existence of the avoided crossings. In Sect. 3 we present the formulation of the problem of linear, adiabatic, radial pulsations of stars. We discuss here also the method of determination separately the oscillatory properties of the inner part of a star and the outer regions (envelope). Numerical results for the eigenfrequencies of the lowest modes of radial pulsations of stars are presented in Sect. 4. Finally, Sect. 5 contains a discussion of our results and conclusions.

2. Equation of state and adiabatic indices

The starting point for the construction of our equation of state (EOS) for the neutron star models was the model of hot dense matter of Lattimer and Swesty (1991), hereafter referred to as LS. Actually, we used one specific LS model, corresponding to the incompressibility modulus at the saturation density of symmetric nuclear matter $K = 220$ MeV. We use LS model of matter in the low temperature limit, which can be treated as a zero-temperature $T = 0$ model.

The detailed discussion of the EOS and equilibrium conditions is presented in Paper I.

As a second example we consider the strange stars with nuclear crust. The interior of this star is build of strange quark matter, containing nearly equal number of u, d and s quarks. This matter is described in the framework of the bag model with the value of bag constant $B = 60\ \mathrm{MeV\ fm^{-3}}$ (Witten (1984), Farhi & Jaffe (1984), Haensel et al. (1986)). The crust of a quark star, at densities below neutron drip point ($\rho_{\rm ND} = 4.3\cdot 10^{11}\ \mathrm{g\ cm^{-3}}$), is represented by the BPS equation of state (Baym et al. 1972).

In Fig. 1 we show our equations of state for nuclear and quark matter.

2.1. Adiabatic indices

The adiabatic index within the star, Γ, plays a crucial role for oscillatory properties of the star.

The oscillation frequency depends sensitively on the value of the adiabatic index and on the shape of the $\Gamma(\rho)$ function. The

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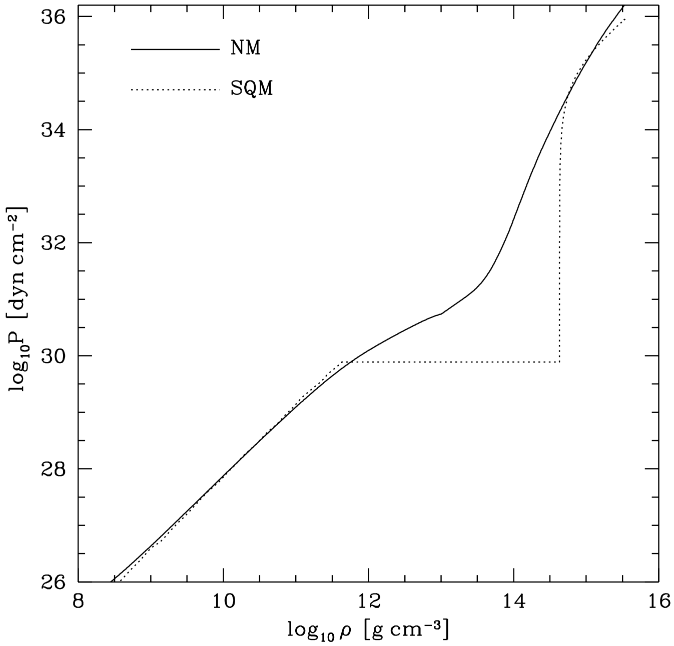

Fig. 1. Pressure versus baryon density for our models of dense matter. The solid curve corresponds to cold catalyzed nuclear matter. The dotted curve presents the EOS for strange quark star with nuclear crust.

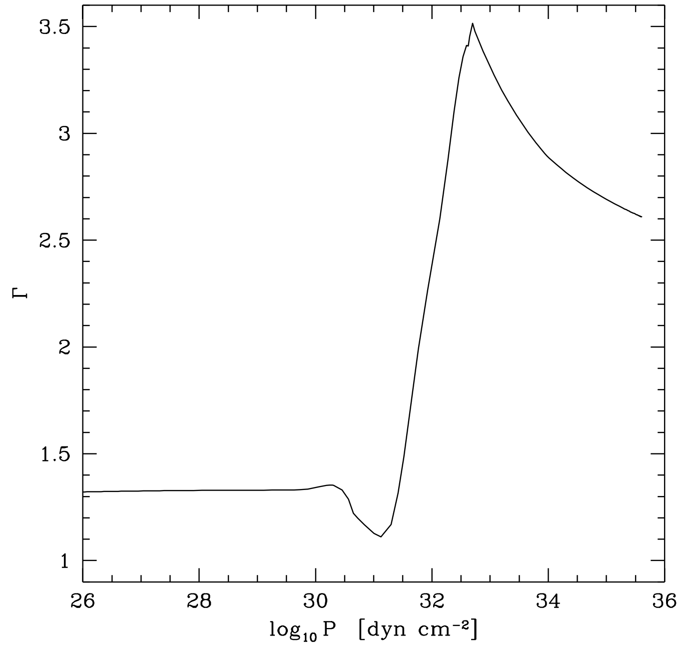

Fig. 2. Parameter $\Gamma_{\rm osc}$ versus pressure, for the nucleon matter.

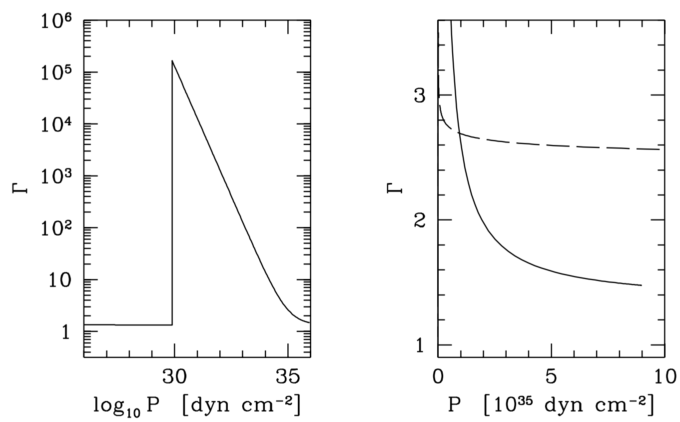

Fig. 3. Parameter $\Gamma_{\rm osc}$ versus pressure, for the strange quark matter. The left figure in logarithmic scale visualizes the huge jump in Γ at the crust–strange core boundary. In the right figure we presented Γ in linear scale in the region of core oscillations. For comparison we show Γ for nuclear matter (dashed line)

increase of Γ means the growing stiffness of the matter leading to the effect of the "wall", setting the bounds for the regions of oscillatory motion.

To calculate properly the oscillations of the star one has to know the relevant index, governing linear perturbation of the pressure due to the density perturbation. This index will be denoted by $\Gamma_{\rm osc}$.

In the case of a sufficiently cold matter, the reactions between matter constituents are so slow, that the matter composition remains fixed (frozen) during perturbations in the dynamical timescale, because $\tau_{\rm react} \gg \tau_{\rm dyn}$. In such a case, the appropriate adiabatic index is $\Gamma_{\rm frozen}$ i.e.

$$\Gamma_{\rm osc} = \Gamma_{\rm frozen} \equiv \frac{n}{P}\left(\frac{{\rm d}P}{{\rm d}n}\right)_{s,Y_e}, \qquad (1)$$

where s, Y_e correspond to the equilibrium model. Fixed value of Y_e during oscillatory motion results in freezing of the weak interaction processes (β-reactions).

The detailed discussion of the relevant adiabatic indices $\Gamma_{\rm osc}$ at different physical conditions (hot protoneutron star with or without trapped neutrinos) was presented in Paper I.

The dependence of Γ's on the pressure in the stellar interior, for our models is displayed in Figs. 2, 3. In this figures (and also some next figures) we choose pressure as an independent variable because pressure is continuous through the star and describes very well some stellar parameters (e.g. mass above given radius). This is not the case of density ρ, which exhibits large density jump at the crust–core boundary of quark stars.

3. Linear adiabatic radial pulsations of stars

The stellar configurations in hydrostatic equilibrium are determined by the integration of Tolman-Oppenheimer-Volkoff (TOV) equations (Tolman 1939, Oppenheimer & Volkoff 1939).

The mass-radius relation for the neutron and quark star models is shown in Fig. 4. In Fig. 5 we show mass vs. central pressure for our stellar configurations. The points on the M(R) curve for neutron stars correspond to the stellar models for which oscillatory properties were studied in details and presented in Fig. 8 and discussed in Sect. 4.

To find eigenfrequencies ω we solve the equations governing infinitesimal radial adiabatic stellar pulsations in general relativity derived by Chandrasekhar (1964), and rewritten by

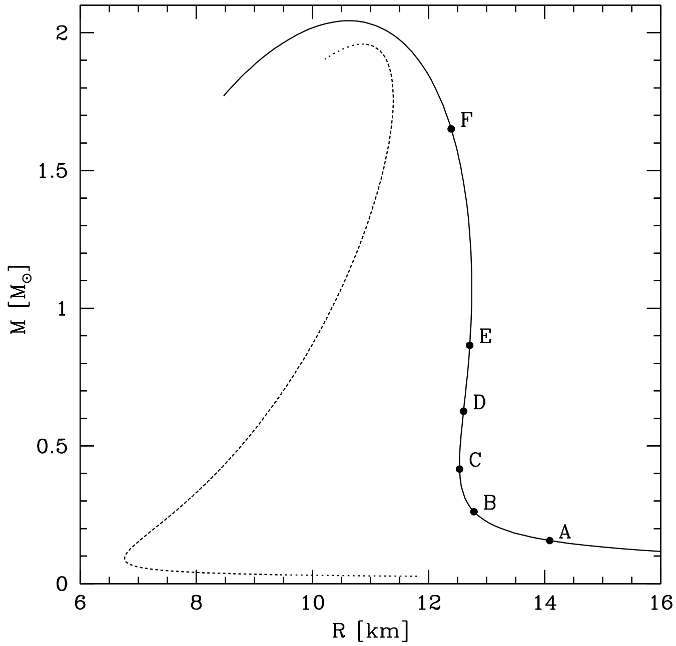

Fig. 4. The gravitational mass versus stellar radius for static models of neutron stars (solid curve) and quark stars (dotted curve). The points on the M(R) curve for neutron stars correspond to the stellar models for which oscillatory properties were studied in details and presented in Fig. 8.

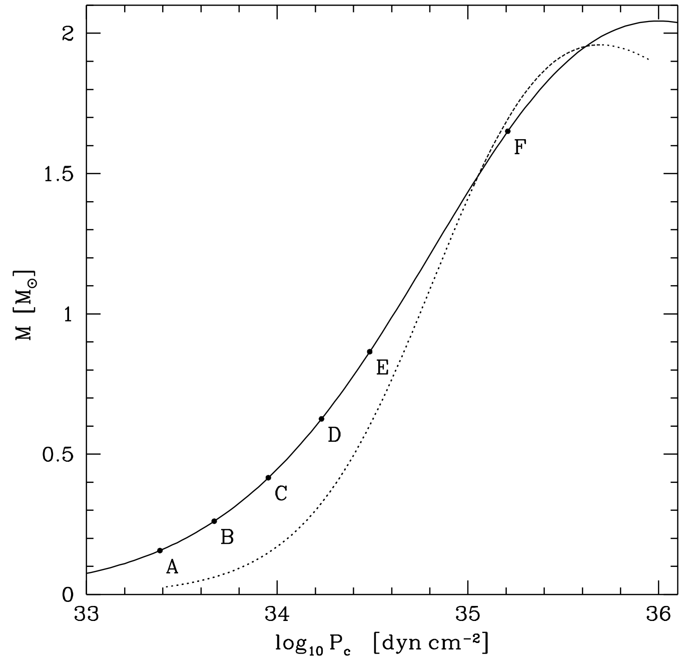

Fig. 5. The gravitational mass versus central pressure for static models of neutron stars (solid curve) and quark stars (dotted curve). The points on the M(R) curve for neutron stars correspond to the stellar models for which oscillatory properties were studied in details and presented in Fig. 8.

Chanmugan (1977) in a form, which turns out to be particularly suitable for numerical applications. Two important quantities, describing pulsations, are: the relative radial displacement, $\xi = \Delta r/r$, where Δr is the radial displacement of a matter element, and ΔP – the corresponding Lagrangian perturbation of the pressure. These two quantities are determined from a system of two ordinary differential equations which can be reduced to the one second order (in ξ), linear radial wave equation, of the Sturm-Liouville type with ω^2 as the eigenvalue of the Sturm-Liouville problem. As a result for a given stellar configuration, we get a set of the eigenvalues $\omega_0^2 < \omega_1^2 < ... < \omega_{\rm n}^2 < ...$, with corresponding eigenfunctions $\xi_0, \xi_1, ..., \xi_{\rm n}, ...$, where the eigenfunction $\xi_{\rm n}$ has n nodes within the star, $0 \le r \le R$ (see, e.g., Cox 1980). The detailed form of the equations and solving method is presented in Paper I.

To solve linear radial wave equation one needs two boundary conditions. The condition of regularity at $r = 0$ requires

$$(\Delta P)_{\rm center} = -3\,(\xi \Gamma P)_{\rm center} \quad . \tag{2}$$

The surface of the star is determined by the condition that for $r \to R$, one has $P \to 0$. This implies

$$(\Delta P)_{\rm surface} = 0 \tag{3}$$

To study the oscillations of the core and envelope separately we divide the star into two parts: core from the center down to the boundary density $\rho_{\rm b}$ and envelope with density smaller than $\rho_{\rm b}$. In the case of neutron matter the density $\rho_{\rm b}$ is defined by the steep change in Γ (Fig. 2) and is equal to $\simeq 10^{14}\ {\rm g\,cm^{-3}}$. The corresponding pressure is equal to $P_{\rm b} \simeq 10^{32}\ {\rm dyn\,cm^{-2}}$. For quark stars the crust–core boundary is very well defined and connected with huge density jump at the pressure $P_{\rm b} = 10^{30}\ {\rm dyn\,cm^{-2}}$. At this boundary the density of the crust is equal to $\rho_{\rm ND} = 4.3 \cdot 10^{11}\ {\rm g\,cm^{-3}}$ and the density of quark core is slightly above $4\,B/c^2 = 4.28 \cdot 10^{14}\ {\rm g\,cm^{-3}}$.

We have to apply the boundary conditions appropriate for core or envelope pulsations. In the investigation of the core oscillations we stop the integrations of the linear radial wave equation at the $P_{\rm b}$ with the boundary conditions corresponding to the "free" envelope oscillation with the frequency defined by the central core – we treat the envelope as a mass $M_{\rm env}$ laying on the central core and simply moving in a way governed by this central region. This assumptions leads to the boundary condition in the form:

$$\frac{\Delta P}{P} = -\left(4 + \left(\frac{GM}{Rc^2} + \frac{\omega^2 R^3}{GM}\right)\left(1 - \frac{2GM}{Rc^2}\right)^{-1}\right)\xi \tag{4}$$

The oscillation of the envelope we can define by the boundary condition:

$$\xi(\rho_{\rm b}) = 0 \tag{5}$$

which means that central core do not move (in other words $\xi(\rho \ge \rho_{\rm b}) = 0$).

4. Eigenfrequencies

As a result of the numerical integration of linear wave equation with boundary conditions appropriate for three situations

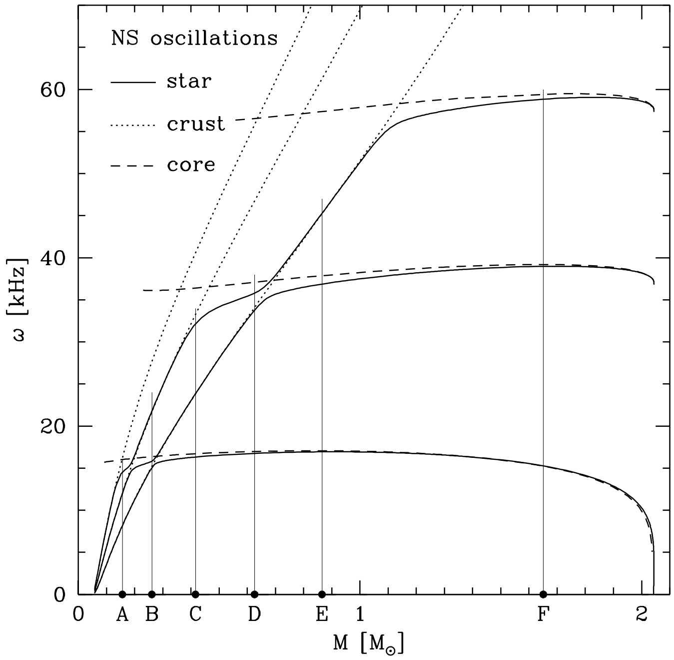

Fig. 6. The eigenfrequencies of the three lowest modes, versus stellar mass for neutron star's radial oscillations. The solid curve corresponds to the oscillations of the star as a whole, the dotted curve describes oscillations of the crust of a star and dashed curve – oscillations of the core. The points on the M(R) curve for neutron stars correspond to the stellar models for which oscillatory properties were studied in details and presented in Fig. 8

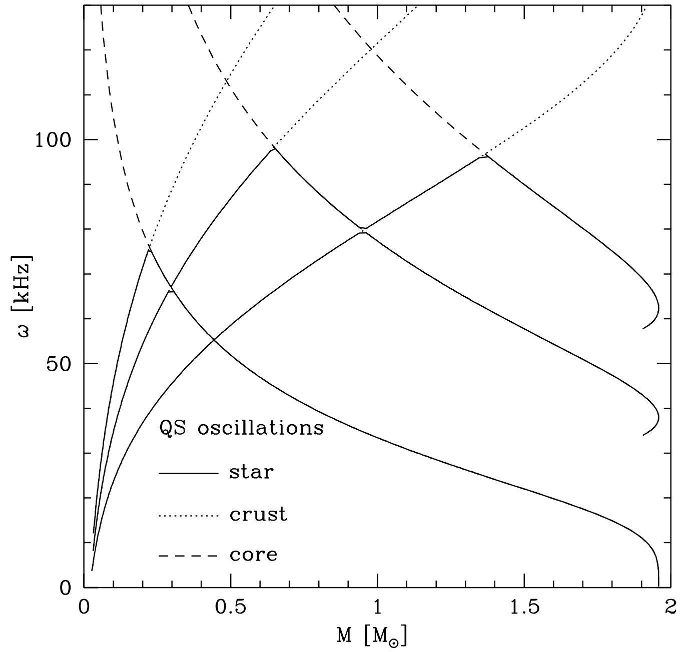

Fig. 7. The eigenfrequencies of three lowest modes, versus stellar mass for quark star's radial oscillations. The solid curve corresponds to the oscillations of the star as a whole, the dotted curve describes oscillations of the crust of a star and dashed curve – oscillations of the core.

(Eqs. (2, 3, 4, 5)) we obtain eigenfrequencies of oscillations for each stellar configuration.

In Figs. 6, 7 we plotted the eigenfrequencies of the three lowest $n = 0,\ 1,\ 2$ radial modes versus stellar mass for neutron and quark star oscillations respectively. The solid line corresponds to a whole star pulsations with boundary conditions at the center and stellar surface (Eqs. (2, 3)). The avoided crossing phenomenon is clearly visible and manifest itself as abrupt, nearly stepwise changes in frequencies of the consecutive (neighbouring) oscillation modes. This effect occurs also in the case of hot, isothermal models of protoneutron stars (Paper I).

This stepwise changes of $\omega_{\rm n}$ are due to the change of the character of the standing-wave solution for the eigenproblem. Namely, at the avoided crossing point the solution changes from the standing wave localized mainly in the outer layer of the star, to that localized predominantly in the central core. This is very clearly visible from the comparison with solutions of oscillatory equation for core and crust pulsations separately. The solid line of the frequency of stellar pulsations nearly coincides with either "core" or "crust" solutions. With increasing mass of the star the subsequent modes $\omega_{\rm n}(M)$ choose between the crust or core solution to fulfill the condition $\omega_0^2 < \omega_1^2 < ... < \omega_{\rm n}^2 < ...$,. Because of the different dependence $\omega_{\rm n}(M)$ for crust and core oscillatory modes this leads to the avoided crossing effect.

The crust region in the stars under consideration is relatively small. Thus the oscillations of the crust could be very well described neglecting the mass of the crust in the gravitational force. In the Newtonian limit and for constant $\Gamma = \Gamma_{\rm crust}$ throughout the stellar crust the wave equation for crust oscillations can be reduced to the form of Bessel equation of the order of ν defined by the formula:

$$\Gamma = 1 + \frac{1}{\nu} \tag{6}$$

The resulting oscillatory frequency should then be proportional to the gravity at the stellar surface

$$\omega_{\rm n} = \frac{1}{2}\,\frac{j_{\nu,\rm n}}{\nu\sqrt{(1+\frac{1}{\nu})}}\,\sqrt{\frac{\rho_{\rm b}}{P_{\rm b}}}\,g \tag{7}$$

where $j_{\nu,\rm n}$ denotes the n-th zero of the Bessel function $J_\nu(z)$ of the order of ν and $g = \frac{GM}{R^2}$ is gravitational acceleration at the stellar surface. This approximation agrees very well with our exact numerical results for neutron and quark stars. Especially for strange quark stars where Γ is nearly constant and equal to $4/3$ in the large part of the crust (Fig. 3) the difference between true, numerical results and approximation by the formula (7) with $\nu = 3$ is smaller than 5%. In the case of neutron stars due to the changes of Γ through the outer layers the identification of subsequent oscillatory modes with Bessel functions is less accurate, but the linear dependence of $\omega_{\rm n}$ on g in formula (7) holds very well.

To confirm our conclusions we have studied in details properties of the three lowest radial modes for some stellar configurations. The parameters of this configurations and eigenfrequencies $\omega_{\rm n}$ are presented in Figs. 4–5 by dots A–F. In Fig. 8 we plotted the energy density of oscillations calculated per pressure gradient in the star i.e. $e_{\rm osc}(P)\,{\rm d}P$ is equal to the energy due to the

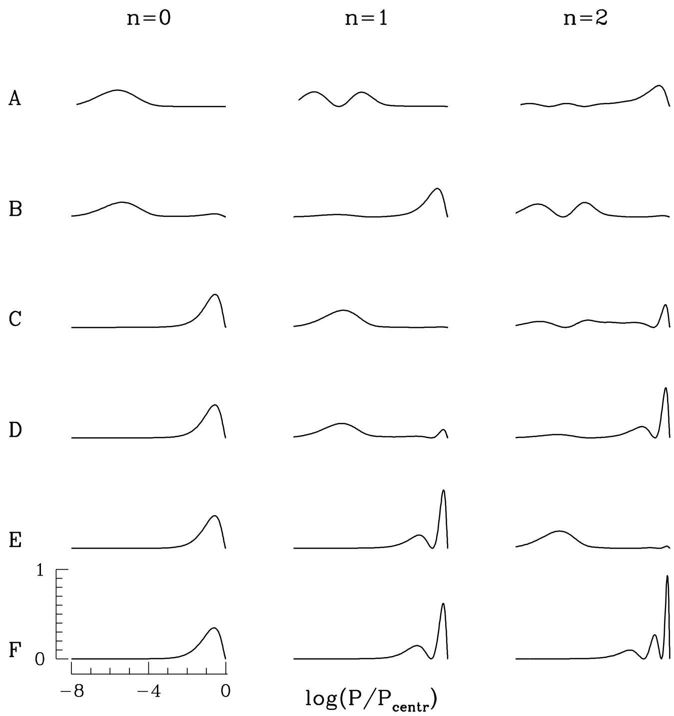

Fig. 8. The oscillatory energy ($e_{\rm osc}$) vs relative pressure ($P(r)/P_{\rm centr}$) for three lowest radial modes, $n = 0,\ 1,\ 2$ for neutron star. The subsequent rows (marked by letters A–F) correspond to stellar models with growing central density (from A to F) marked by dots in Figs. 4, 5, 6. The energy is normalized to unity i.e. $\int e_{\rm osc}\,{\rm d}(P/P_{\rm centr}) = 1$. The function $P(r)/P_{\rm centr}$ approximates very well the relative mass from the stellar surface $(M - m(r))/M$.

oscillatory motion of the matter in the sphere $r(P) - r(P + {\rm d}P)$. This function is proportional to $\xi_{\rm n}^2(r)$. As an independent variable we choose in Fig. 8 $P(r)/P_{\rm centr}$. For the outer layers of the star this function is proportional to the relative mass above the radius r i.e.:

$$P(r)/P_{\rm centr} \sim (M - m(r))/M \tag{8}$$

with proportionality constant of the order of unity. The integral of the energy in Fig. 8 is normalized to unity i.e. $\int e_{\rm osc}\,{\rm d}(P/P_{\rm centr}) = 1$.

The Fig. 8 very well visualizes where the mode is localized. We can also identify modes by the shape of ξ or energy of oscillation. The three lowest radial "core modes" are characterized by the functions presented in row F in Fig. 8. The stellar configuration F is the most massive and crust plays very little role in oscillatory properties (see also Fig. 6). The two lowest "crust modes" correspond to the columns $n = 0$ and $n = 1$ in row A of Fig. 8. For the low massive model A the crust is relatively large and determines frequencies for fundamental mode and first overtone. Second overtone $n = 2$ is strongly contaminated by the core oscillations (Figs. 8 and 6). Fig. 8 allows for very clear division of oscillatory modes by the real pulsational properties and not the number of nodes. This is presented in Table 1, where for example A1 denotes configuration A, number of nodes $n = 1$.

Table 1. Identification of radial oscillatory modes of neutron star (see Fig. 8)

propagation region	fundamental mode	first overtone	second overtone
crust	A0 B0 C1 E3	A1 B2	
core	B1 C0 D0 E0 F0	D2 E1 F1	F2
mixed	A2 C2 D1		

The identification of the radial modes in Table 1 is consistent with the Fig. 6. The specific oscillatory mode for stellar configurations A–F corresponds to the crossection points of the vertical lines at points A–F and the curves $\omega_{\rm n}(M)$ for given stellar configuration. The location of these points very well define type of oscillatory motion. The nearly constant parts of the $\omega_{\rm n}(M)$ functions correspond to the core oscillations while regions of increasing $\omega_{\rm n}(M)$ describes crust oscillations. The region close to avoided crossing points is characteristic to mixed modes: star pulsates in crust as well as in the core.

5. Discussion and conclusions

The avoided crossing phenomenon in the *radial* pulsations of dense stellar objects was presented. Both strange quark stars with crust and neutron stars could be subject to this effect.

The main reason for the abrupt changes of the oscillatory properties of the star as one change a little stellar configuration is the very well defined division of the star into two regions: inner core and outer crust. This two parts have different equation of state and oscillatory properties. The latter have been studied in this paper by the separation of the outer and inner regions by appropriate boundary conditions. Resulting oscillatory frequencies of crust and core pulsations depend on the stellar configurations (e.g. mass of the star) in completely different manner. The functions $\omega_{\rm n}(M)_{\rm crust}$ are rather steeply increasing while $\omega_{\rm n}(M)_{\rm core}$ are nearly constant (neutron stars) or decreasing (quark stars).

The oscillatory properties of the star could be characteristic either to the crust or core pulsations. The frequency spectrum depends on the mass of the star. The core pulsations determine the oscillatory properties of the lower order radial modes for relatively massive neutron or strange stars. The star could be subject to the crust pulsations for sufficiently high frequency or low stellar mass. For given mass the fundamental mode of crust oscillations $\omega_0(M)_{\rm crust}$ define the frequency below which the star pulsates in the core (Figs. 6, 7).

The avoided crossing phenomenon is strongly related to the changes of compressibility of the matter throughout the star, which is described by the shape of the $\Gamma_{\rm osc}(\rho)$ function. The increase of the stiffness of the matter outward leads to the maximum of $\Gamma_{\rm osc}(\rho)$ close to the boundary between core and crust. In the case of strange quark matter this is the consequence of the the fact that it is self-bound at the very high density and $\Gamma \to \infty$ as $P \to 0$. In neutron matter described by LS equation of state the symmetry energy leads to the increase of Y_e inward and in this case more symmetric matter is softer. The same effect could be seen for the field model of dense matter considered by Diaz Alonso & Ibáñez (1985). The function $\Gamma(\rho)$ for their EOSs (Fig. 9 of their paper) resembles LS result at low temperature limit.

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