On the minimum period of uniformly rotating neutron stars

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Abstract. We show that the neutron star minimum period of uniform rotation is determined, for causal equations of state, by the maximum value of the relativistic (compactness) parameter $2GM/Rc^2$, allowed by causality for static neutron stars, 0.7081, and by the largest measured mass of a neutron star. The relation between these three quantities, resulting from the extrapolation of the empirical formulae of Lasota et al. (1996), yields the minimum period of 0.288 ms, only 2% higher than an absolute lower bound obtained in extensive exact numerical calculations of Koranda et al. (1997).

Key words: dense matter – stars: neutron – stars: pulsars: general

1. Introduction

The lower limit on the period of a uniformly rotating neutron star is sensitive to the equation of state (EOS) of dense matter above the nuclear density. Therefore, an uncertainty in the high density EOS implies a large uncertainty in the minimum period of uniform rotation, $P_{\min}$ (see, e.g. Friedman & Ipser 1987; Friedman et al. 1989; Salgado et al. 1994a,b; Cook et al. 1994). Hence, it is, therefore, of interest to find a lower limit on $P_{\min}$, that is independent of the EOS. This limit results from the condition of causality, combined with the requirement that EOS yields neutron stars with masses compatible with observed ones [currently the highest accurately measured neutron star mass is $M_{\text{obs}}^{\max} = 1.442 \, \text{M}_\odot$ (Taylor & Weisberg 1989)]. It will be hereafter referred to as $P_{\min}^{\text{CL}}$.

The first calculation of $P_{\min}^{\text{CL}}$ was done by Glendenning (1992), who found the value of 0.33 ms. Glendenning (1992), however, used a rather imprecise empirical formula, to calculate a lowest $P_{\min}$ by using the parameters (mass and radius) of the maximum mass configurations of a family of non-rotating neutron star models. His result, therefore, should be considered only as an estimate of $P_{\min}^{\text{CL}}$. Recently, Koranda et al. (1997) extracted the value of $P_{\min}^{\text{CL}}$ from extensive *exact* calculations of

uniformly rotating neutron star models. They have shown, that the method of Glendenning (1992) overestimated the value of $P_{\min}^{\text{CL}}$ by 6%. The result of Koranda et al. (1997) calculations can be summarized in a formula

$$P_{\min}^{\text{CL}} = 0.196 \, \frac{M_{\text{obs}}^{\max}}{\text{M}_\odot} \, \text{ms} , \qquad (1)$$

which combined with measured mass of PSR B1913+16 yields today's lower bound for $P_{\min}^{\text{CL}} = 0.282$ ms. This absolute bound on the minimum period was obtained for the "causality limit (CL) EOS" $p = (\rho - \rho_0)c^2$, which yields neutron star models of the surface density ρ_0 and is maximally stiff $(\mathrm{d}p/\mathrm{d}\rho = c^2)$ everywhere within the star; it does not depend on the value of ρ_0. In the present letter we show that Eq. (1) can be reproduced using an empirical formula for $P_{\min}$ derived for *realistic* causal EOS by Lasota et al. (1996), combined with an upper bound on the relativistic (compactness) parameter $2GM/Rc^2$ for static neutron stars with causal EOS.

2. Relation between x_{s} and $P_{\min}$

As shown by Lasota et al. (1996), numerical results of Salgado et al. (1994a,b) for the maximum frequency of uniform stable rotation can be reproduced (within better than 2%), for a broad set of realistic causal EOS of dense matter, by an empirical formula

$$(\Omega_{\max})_{\text{e.f.}} = \mathcal{C}(x_{\text{s}}) \left(\frac{GM_{\text{s}}}{R_{\text{s}}^3} \right)^{\frac{1}{2}} , \qquad (2)$$

where M_{s} is the maximum mass of a spherical (nonrotating) neutron star and R_{s} is the corresponding radius, and $\mathcal{C}(x_{\text{s}})$ is a universal (i.e. independent of the EOS) function of the compactness parameter $x_{\text{s}} \equiv 2GM_{\text{s}}/R_{\text{s}}c^2$ for the static maximum mass configuration,

$$\mathcal{C}(x_{\text{s}}) = 0.468 + 0.378 x_{\text{s}} . \qquad (3)$$

Combining Eq. (2) and Eq. (3) we get

$$(P_{\min})_{\text{e.f.}} = \frac{8.754 \times 10^{-2}}{\mathcal{C}(x_{\text{s}}) x_{\text{s}}^{\frac{3}{2}}} \, \frac{M_{\text{s}}}{\text{M}_\odot} \, \text{ms}. \qquad (4)$$

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At given maximum mass of a spherical configuration, the maximum rotation frequency (minimum rotation period) is obtained for the maximum value of x_s. At fixed x_s, the value of P_{min} is proportional to M_s. Neutron stars for which masses have been measured, rotate so slowly that their structure can be very well approximated by that of a spherical star. Observations impose thus a condition $M_s \geq M_{obs}^{max}$.

3. Lower bound on P_{min}

Our empirical relation, Eq. (4), indicates, that to minimize P_{min} for given M_{obs}^{max} we have to look for an EOS which yields maximum x_s at $M_s = M_{obs}^{max}$. It is well known, that if one relaxes the condition of causality, the absolute upper bound on x_s for stable neutron star models is reached for an incompressible fluid (i.e., $\rho = const.$) EOS; the value of x_s is then independent of M_s and equal $8/9$ (see, e.g., Shapiro & Teukolsky 1983). It is therefore rather natural to expect that in order to maximize x_s under the condition of causality, one has to maximize sound velocity throughout the star. Together with condition of density continuity in the stellar interior this points out at the CL EOS, $p = (\rho - \rho_0)c^2$, as to that which yields "maximally compact neutron stars"; introducing density discontinuities does not increase the value of x_s, see Gondek & Zdunik (1995). [The conjecture that the CL EOS minimizes P_{min} was already proposed and then confirmed numerically in extensive exact calculations by Koranda et al. (1997)]. Note, that the value of x_s for CL EOS does not depend on ρ_0 (and therefore is M_s-independent). It represents an absolute upper bound on x_s for causal EOS, $x_{s,max}$. Our numerical calculation gives $x_s(\text{CL EOS}) = x_{s,max} = 0.7081$. This corresponds to an absolute upper bound on the surface redshift of neutron star models with causal EOS, $z_{max} = (1 - x_{s,max})^{-1/2} - 1 = 0.8509$.

Let us consider the effect of the presence of a crust (more generally, of an envelope of normal neutron star matter). For a given EOS of the normal envelope, the relevant (small) parameter is the ratio $p_b/\rho_b c^2$, where p_b and ρ_b are, respectively, pressure and mass density at the bottom of the crust (Lindblom 1984). The case of $p_b = 0$ corresponds to stellar models with no normal crust. Numerical calculations show, that adding a crust onto a CL EOS core implies an increase of R_s, which is linear in $p_b/\rho_b c^2$; for a solid crust we have typically $p_b/\rho_b c^2 \sim 10^{-2}$. The change (increase) in M_s is negligibly small; it turns out to be quadratic in $p_b/\rho_b c^2$. This implies, that the decrease of $x_{s,max}$, and of the maximum surface redshift $z_{s,max}$, due to the presence of a crust, is proportional to $p_b/\rho_b c^2$. This is consistent with Table 1 of Lindblom (1984). However, the extrapolation of his results to $p_b = 0$ yields $z_{max} = 0.891$, which is nearly 5% higher than our value of z_{max}! This might reflect a lack of precision of the variational method used by Lindblom (1984), which led to an overestimate of the value of z_{max}. It should be stressed that while a precise determination of $M_{max} \equiv M_s$ for static neutron star models is rather easy, determination of the precise value of the radius of the maximum mass configuration, R_s, (with the same relative precision as M_s) and consequently of the value of x_s (with, say, four significant digits), is much

more difficult and requires a rather high precision of numerical integration of the TOV equations.

In what follows, we restrict ourselves to the case of the absolute upper bound on x_s, obtained for neutron star models with no crust. Inserting the value of $x_{s,max}$ into Eq. (4) we get

$$\left(P_{min}^{CL}\right)_{e.f.} = 0.1997 \frac{M_{obs}^{max}}{M_\odot} \text{ ms} . \tag{5}$$

Current lower bound on P, resulting from the above equation, is thus 0.288 ms, which is only 2% higher than the result of extensive exact numerical calculations of Koranda et al. (1997).

The formula (5) deserves an additional comment. In numerical calculations, of a family of stable uniformly rotating stellar models, for a given EOS of dense matter, one has to distinguish between the rotating configuration of maximum mass, which corresponds to the rotation frequency $\Omega_{M_{max}}(\text{EOS})$, and the maximally rotating one, which rotates at $\Omega_{max}(\text{EOS})$ (Cook et al. 1994, Stergioulas & Friedman 1995). Notice, that determination of a maximum mass rotating configuration (and therefore of $\Omega_{M_{max}}$) is a much simpler task than the calculation of exact value of Ω_{max}, which is time consuming and very demanding as far as the precision of numerical calculations is concerned. Usually, both configurations are very close to each other, and Ω_{max} is typically only 1–2% higher than $\Omega_{M_{max}}$; such a small difference is within the typical precision of the empirical formulae for Ω_{max}. Actually, the formula for $\mathcal{C}(x_s)$, Eq. (3), was fitted to the values of $\Omega_{M_{max}}(\text{EOS})$ calculated in (Salgado et al. 1994a,b). Therefore, Eq. (5) should in principle be used to evaluate the causal lower bound to $P_{min,M_{max}}$; it actually reproduces, within 0.2%, the exact formula for this quantity, obtained by Koranda et al. (1997) [see their Eq. (8)].

It should be stressed that Eq. (5) results from an *extrapolation* of the empirical formula of Lasota et al. (1996). General experience shows that - in contrast to interpolation - extrapolation is a risky procedure. The fact that in our case extrapolation of an empirical formula yields - within 2% - the value of P_{min} of Koranda et al. (1997) (and reproduces their value of $P_{min,M_{max}}$), proves the usefulness of compact "empirical expressions" which might summarize, in a quantitative way, a relevant content of extensive numerical calculations of uniformly rotating neutron star models.

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