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On modeling radiation-driven envelopes at arbitrary optical depths

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Abstract. We consider stationary outflowing stellar envelopes accelerated by radiation pressure. Making use of solutions of the transfer equation in the Eddington approximation, we derive relations for pressure, radiation energy density, and radiative energy flux at arbitrary optical depth τ. These relations are used in the equations of radiative hydrodynamics, which can be solved numerically. The solution proceeds through a singular point, where the velocity is equal to the isothermal sound speed, and satisfies zero temperature and pressure boundary conditions at infinity. A sample calculation is performed for a constant opacity and molecular weight. We argue that, with realistic opacity tables and thermodynamic functions which take into account variable ionization states and in conjunction with self-consistent evolutionary calculations, our method for constructing outflowing envelopes will provide unambiguous mass-loss rates for massive stars in yellow and red supergiant stages. Advantages of this approach are discussed in comparison with previously used methods.

Key words: hydrodynamics – radiative transfer – stars: carbon – stars: mass-loss

1. Introduction

The evolution of massive stars $(M \geq 20 M_{\odot})$ is accompanied by mass loss driven by a high luminosity and high radiation pressure. In blue supergiants situated near the main sequence, the mass-loss rate is moderate, $\dot{M} \sim 10^{-6} M_{\odot}/\,\mathrm{yr}$, and is associated with the outflow of layers having small optical depth; mass loss is due to radiation pressure on lines with large absorption coefficients. The theory of outflow from these stars is based on the Sobolev approximation, and the velocity gradient in the flow increases the wavelength range over which large absorption occurs (Castor, Abbott, Klein, 1975; hereinafter CAK). For recent developments of CAK theory, see Owocki & Puls (1996) and Pauldrach et al. (1994).

Evolved massive stars may lose mass at a much higher rate than blue supergiants. Single Wolf-Rayet (WR) stars are probably formed as a result of such intense mass loss. Even though the importance of mass loss from stars of different masses at different evolutionary stages is well understood from the observations, the theoretical understanding of the mass-loss process is rather undeveloped. This hampers progress in addressing theoretically problems such as supernova frequencies, birthrates of neutron stars and black holes, enrichment of the interstellar medium by heavy elements, and other critical problems of stellar and galactic evolution.

The mass-loss rate in evolutionary calculations is usually prescribed by phenomenological dependences valid over a restricted range of parameters (e.g., Stothers et al. 1979; Maeder 1981), or it is simply ignored. The main goal of the theory of mass loss from stars is to determine the mass-loss rate as an "eigenvalue" of the star, together with its luminosity and radius. CAK theory gives a solution of this problem only for blue supergiants, but for later evolutionary stages, where the mass-loss rate is expected to be much higher, a reliable theory does not exist.

For blue supergiants, comparison with the observations makes it possible to correct the theory to arrive at a self consistent description of the mass-loss rate as a function of position in the HR diagram. For more evolved stars, however, intensive mass loss can drastically change the position of the star in the HR diagram. But, without knowledge of the mass-loss rate, we do not know where the star is situated; it may occupy any region from the infrared to the yellow. In this situation, corrections to the theory which make use of the observations are not trivial, so the creation of a self-consistent theory of mass loss is vital.

An evolutionary scenario for the formation of WR stars as a result of intensive mass loss at the end of the core hydrogen-burning phase was first suggested by Bisnovatyi-Kogan & Nadyozhin (1972), who constructed somewhat crude, but self-consistent evolutionary models of mass-losing stars. They suggested that "a check of this hypothesis of the formation of stars of the Wolf-Rayet type would be the discovery around the stars of an extensive gaseous envelope, the mass of which would be of the same order as the mass of the star." It happened that, without their knowledge, ring nebulae around a single WR star had already been observed (Smith 1968). Subsequent results of observational and theoretical studies of WR winds, of their surrounding nebulae, and of evolutionary scenarios for their formation may be found in Lozinskaya (1986).

The main shortcoming of the Bisnovatyi-Kogan & Nadyozhin (1972) work was that it assumed radiation pressure and energy density to be everywhere in local thermodynamic equi-

Send offprint requests to: A.V. Dorodnitsyn librium (LTE), even in outer layers where the optical depth becomes small. Further, the knowledge of opacity in outer layers was rather poor, and many sources of opacity, in particular bound-bound transitions and dust, had not been taken into account. Finally, to overcome numerical problems, they used a very approximate treatment of the flow in the outer regions, resulting in a large overestimate of the self-consistently determined mass-loss rate.

In several papers, Zytkow (1972, 1973) used different simplifications, including the assumption that the outer surface is determined by $L = 8\pi\sigma T_{\rm surf}^4 R_{\rm surf}^2$ at optical depth $\tau = 0$. However, for a star with a strong wind, $\rho \propto 1/r^2$ outside the critical point, and there is no such thing as an "outer surface." Formally, $\tau \to 0$ only at infinity. The dependence of temperature on τ used by Zytkow in the region between the "stellar surface" and the stellar photosphere was taken from Paczynski's (1969) calculations of extended static atmospheres. In Zytkow's calculations, temperature and pressure are uniquely determined as functions of an outer radius and a luminosity by means of an approximate solution of a transfer equation. The "universal" character of the solution is based on the exponential drop in density with distance for all static atmospheres and on a small difference between photospheric and "infinite radius" temperatures in a plane atmosphere. However, for extended atmospheres, the plane approximation fails, and spherical symmetry must be taken into account, leading to considerable deviations from a universal law. Paczynski (1969) took these deviations only partially into account. More importantly, the manner in which ρ and P depend on the radius in an outflowing atmosphere is quite different from how they depend on radius in a static atmosphere. Further, the gas temperature in the flow goes to zero with increasing radius, and decreasing τ, while in the plane-parallel, static atmosphere it drops only $2^{1/4}$ times, from the photosphere to the "surface" at $\tau = 0$.

Another, but similar simplification was used by Kato (1985) and by Kato and Iben (1992), who adopted the boundary condition $L = 4\pi\sigma r_{\rm ph}^2 T_{\rm ph}^4$ at $\tilde\tau = \kappa\rho r = 8/3$ (instead of $\tau_{\rm ph} = \int_{r_{\rm ph}}^{\infty} \kappa\rho dr$), and terminated solutions at this point. Theirs was an attempt to avoid problems connected with the singular point at infinity, but, in fact, it creates another problem, especially when ionization is incomplete and the opacity gradient is steep. Far from the star, the velocity u of the mass flux is approximately constant, and it follows from the constancy of $\dot M = 4\pi\rho u r^2$ that $\rho \propto 1/r^2$. For a constant κ, we have $\tau \simeq \tilde\tau$. However, the opacity is not, in general, constant. Assuming that $\kappa = \rho^\alpha$, where α is constant, $\tau_{\rm ph} = (\kappa\rho r)_{\rm ph}/(1+2\alpha)$, which is $(1+2\alpha)$ times smaller than $\tilde\tau_{ph}$. For an even steeper (exponential) decrease of κ with r, the difference between τ_{ph} and $\tilde\tau_{ph}$ may be much larger. In real stars, the opacity may increase rather steeply. Even if the temperature is decreasing smoothly, partial recombination of ions and the formation of molecules and dust will cause a rapid increase in κ (Cox & Tabor 1975). Hence, regions will be created where the flow is accelerated; u can therefore not be taken constant and the dependence of ρ on r will be far from $\propto 1/r^2$. In this case, rather large uncertainities could be introduced by taking $\tilde\tau$ instead of τ. In a supergiant phase in massive stars there are regions of incomplete ionization of H and He with attendant large opacities where the main acceleration of the matter flow occurs.

Recently, another approximate approach for solution of the outflowing atmosphere was proposed by Schaerer et al. (1996). Expanding layers of the star are divided into a subsonicaly extended photosphere where the stationary momentum equation is adopted, and a supersonic wind region where matter is accelerated to the velocity v_∞. The velocity profile is approximately prescribed by a three parameter function, with parameters determined from matching conditions at the interface. It is not clear how this quasi-phenomenological approach can be used in the construction of stellar models with self-consistent mass loss rates.

The goal of the present paper is to derive equations which are approximately valid at all optical depths, giving exact limiting equations for the case of very large and very small τ. Solutions of these equations may be obtained by using correct boundary conditions at large r (infinity), where gas density ρ and gas temperature T tend to zero. After fitting the solution to the stellar core, this procedure gives self-consistent values of $\dot M$, as well as the parameters at the critical point and at $\tau_{\rm ph}$. Here, we concentrate on the first part of the problem: derivation of the equations for the gas and radiation and identification of the solution which satisfies the outer boundary conditions and passes through the critical point. Obtaining a self-consistent value for mass loss by fitting atmosphere solutions to stellar interiors is a problem we leave to the future.

To highlight the essence of the method, we use in this first paper exceedingly simple physics: a constant opacity, a perfect gas plus black body radiation for an equation of state, no ionization, no molecules, and no dust. The use of more realistic physics leads to more complex equations, but the complexities are only of a technical nature. In a subsequent paper, we will derive equations with variable $\kappa(\rho, T)$ and $P(\rho, T)$, after the manner of Bisnovatyi-Kogan and Nadyozhin (1969).

In Sect. 2, we derive relationships for pressure, energy density, and the energy flux of radiation which describe the transition between optically thick and optically thin regions. In the limiting cases, they reduce to corresponding solutions of the radiative transfer equations in the Eddington approximation. In Sect. 3, these relationships are used in the equations of radiation hydrodynamics with a constant total energy flow. The equations are written in a non-dimensional form with several non-dimensional parameters. Singular points of the equations are analysed, and an expansion about the isothermal sonic point, necessary for obtaining a numerical solution, is presented. In Sect. 4, a numerical solution which satisfies the boundary conditions at infinity and which uses parameters characterizing the underlying star is obtained. In Sect. 5, we analyse the validity of the adopted approximations, list the advantages of our method as compared with other methods, and discuss the prospect of its use for realistic self-consistent calculations of evolved massive stars with high mass-loss rates. The main results are summarized in Sect. 6. The coefficients of the expansion about the sonic point are given in an Appendix.

2. Thermodynamic relations in outflowing envelope at arbitrary optical depth

In deep layers, at large optical depth $\tau \gg 1$, the equation of state is taken as a mixture of an ideal gas and black body radiation.

$$P(\rho,T) = \frac{aT^4}{3} + \rho\mathcal{R}T. \tag{1}$$

When τ is small, radiation and matter are not in thermodynamic equilibrium. There are two ways to deal with this fact. The first focuses on the physics of the interaction between the outflowing matter and the propagating radiation, and the intensity of the radiation field is calculated by solving simultaneously the equations of hydrodynamics together with the equation of radiative transfer. The second, which we adopt in this paper, is heuristic. Instead of dealing directly with the intensity of the radiation field, we deal with the first two angular moments of this field (the Eddington approximation) and use several simplifying approximations. By introducing an Eddington factor that relates the radiation pressure to the radiation energy density, we can close the set of moments. Since there are different ways of averaging the intensity field over the angles, there exists a certain degree of freedom in choosing the factor. A variable Eddington factor (Sobolev, 1967) allows us to describe smoothly regions between $\tau >> 1$ and $\tau << 1$ and provides a correct description of the radiation field in the two limiting regimes.

In the limit $\tau >> 1$, the radiation field and matter are in LTE; the thermal energy flux $L_{\rm th}$ and the temperature gradient are related accurately by the diffusion approximation. When $\tau << 1$, a portion of the radiation field is not interacting with the outflowing matter, and the field acts primarily as a pushing force. Supposing that the isotropic component of the radiation field and the matter remain in LTE, we define an isotropic pressure

$$P^{\rm isotr.}_{\tau\to 0} = \frac{aT^4}{3}\,\tau + \rho\mathcal{R}T. \tag{2}$$

The multiplier τ in Eq. (2) ensures that the contribution of radiation pressure to the total isotropic pressure vanishes as $\tau \to 0$, and also takes into account that radiation pressure becomes increasingly anisotropic as τ decreases. We choose the following approximate representation of the thermodynamical relations for the radiation field:

$$P_r = \frac{aT^4}{3}\left(1-e^{-\tau}\right) + \frac{L(r)_{\rm th}}{4\pi r^2 c}, \tag{3}$$

$$\rho E_r = aT^4\left(1-e^{-\tau}\right) + \frac{L(r)_{th}}{4\pi r^2 c}. \tag{4}$$

The terms in Eqs. (3) and (4) which involve the total thermal radiative flux $L_{\rm th}$ are to be determined from a solution of a self-consistent set of equations. In general, these terms contribute importantly only when τ is small, so a further simplification is possible: $L_{\rm th}(r)$ in Eqs. (3) and (4) may be replaced by $L^\infty_{\rm th} = \lim_{r\to\infty} L_{\rm th}(r)$.

3. Basic equations

A system of equations of radiation hydrodynamics which provides a smooth transition between optically thick and optically thin regions is:

$$u\frac{du}{dr} = -\frac{1}{\rho}\frac{dP_{\rm g}}{dr} - \frac{GM(1-\tilde{L}_{\rm th})}{r^2}, \tag{5}$$

where $\tilde{L}_{\rm th} = \dfrac{L_{\rm th}(r)}{L_{\rm Edd}}$ and $L_{\rm Edd} = \dfrac{4\pi cGM}{\kappa}$,

$$L = 4\pi\mu\left(E + \frac{P}{\rho} - \frac{GM}{r} + \frac{u^2}{2}\right) + L_{\rm th}(r), \tag{6}$$

$$L_{\rm th} = -\frac{4\pi r^2 c}{\kappa\rho}\left(\frac{dP_r}{dr} - \frac{E_r\rho - 3P_r}{r}\right), \tag{7}$$

$$\frac{\dot{M}}{4\pi} \equiv \mu = \rho u r^2, \tag{8}$$

$$P = \frac{aT^4}{3}\left(1-e^{-\tau}\right) + \frac{L^\infty_{\rm th}}{4\pi r^2 c} + P_g, \tag{9}$$

$$E\rho = aT^4\left(1-e^{-\tau}\right) + \frac{L^\infty_{\rm th}}{4\pi r^2 c} + E_g\rho, \tag{10}$$

$$P_g = \rho\mathcal{R}T, \tag{11}$$

$$E_g = \frac{3}{2}\mathcal{R}T, \text{ and} \tag{12}$$

$$\tau = \int_r^\infty \kappa\rho dr. \tag{13}$$

In these equations, L is a constant total energy flux consisting of the radiative energy flux together with the energy flux of the matter flow, u is the speed of the outflow, κ is the opacity, assumed to be constant (for simplicity), a is the radiation constant, and $\mathcal{R}$ is the gas constant. We have assumed that M is constant and have neglected the self-gravity of the outflowing envelope beyond the critical point. This approximation is quite good in a realistic case when the mass $m_{\rm en}$ of the envelope beyond the critical point is much smaller than M (Zytkow 1973).

This system of equations provides a description of a stationary outflowing envelope accelerated by a radiative force at arbitrary optical depth, where continuum opacity prevails. A solution of these equations in the optically thick limit $\tau >> 1$ (when terms in Eqs. (3) and (4) with $L^\infty_{\rm th}$ are negligible so that $E_r\rho = 3P_r$) was obtained by Bisnovatyi–Kogan (1967). When τ is small (so that $E_r\rho \simeq P_r$), we recover the solution of the transfer equation in the Eddington approximation (Sobolev 1967).

At small optical depth, the partial decoupling of the radiation field from matter should be taken into consideration, for only a part of the radiation flow interacts with the outflowing gas. For this part, we assume LTE to be valid, which means that the mean energy of the quanta is characterized by the temperature of the outflowing gas (i.e., the temperature in Eqs. (3)-(4) is the same as in Eqs. (11)-(12)). For the rest of the radiation, another "temperature" (read mean energy of quanta) should be introduced. This part of the radiation field transfers momentum to the outflowing matter (pushes it) and thus produces only the anisotropic part of the pressure, determined by the term $L_{\rm th}$. The separation of the radiation field into two different parts occurs near the photosphere, and the mean energy of the freely propagating quanta are characterized by the effective temperature of the photosphere.

All previous papers dealing with this problem have not taken into account the bimodal behaviour of the radiation field in the transition zone between optically thick and optically thin regions, when it changes from an isotropic radiation field in thermodynamic equilibrium with matter to a completely anisotropic outflowing radiation field. For example, Bisnovatyi-Kogan and Nadyozhin (1969, 1972) consider radiation and matter to be in thermodynamic equilibrium everywhere. Zytkow (1972, 1973) takes the temperature behavior in optically thin regions to be that given by the static atmospheres of Paczynski (1969) and assumes the outer boundary condition at $\tau = 0$ to be the same as for a static case. Kato (1985) and Kato & Iben (1992) do not consider optically thin regions explicitly, assuming that all driving of the wind occurs in optically thick layers.

In the equation of motion, Eq. (5), the effect of the pressure gradient is explicitly written only for the gas component; all effects of the radiation field are included in the second term on the right through the quantity $L_{\rm th}/L_{\rm Edd}$. The radiative flux $L_{\rm th}$ is obtained from the radiation transfer equation, Eq. (7), taking into account relations (9) and (10). For large optical depth, this approach ensures that acceleration is provided by the total isotropic (radiation plus gas) pressure gradient, and, for small τ, it ensures that acceleration is due to the gas pressure gradient plus the pressure of the anisotropic radiation field. In the intermediate region, the last term in Eq. (5), taking into account Eqs. (3), (4), and (7), gives a description which smoothly joins the descriptions at large and small τ. Substituting Eqs. (7)-(10) into Eq. (6), we obtain

$$\frac{L}{4\pi} = \rho u r^2 \left(E + \frac{P}{\rho} - \frac{GM}{r} + \frac{\mu^2}{2\rho^2 r^4} \right) - \frac{r^2 c}{\kappa\rho} \left\{ \frac{d}{dr} \left[\frac{(1-e^{-\tau})aT^4}{3} + \frac{L_{th}^{\infty}}{4\pi r^2 c} \right] + 2\frac{L_{th}^{\infty}}{4\pi r^3 c} \right\}. \quad (14)$$

Differentiating in Eq. (14) and using

$$\frac{d\tau}{dr} = -\kappa\rho, \quad (15)$$

we obtain

$$\lambda r^2 (1-e^{-\tau}) \frac{dT}{dr} = \mu \left\{ \frac{5}{2}\mathcal{R}T - \frac{GM}{r} + \frac{\mu^2}{2\rho^2 r^4} \right\} - \frac{L}{4\pi} + \mu \left[2\frac{L_{th}^{\infty}}{4\pi r^2 c\rho} + \frac{4}{3\rho}(1-e^{-\tau})aT^4 \right] + \frac{1}{3} a r^2 c T^4 e^{-\tau}, \quad (16)$$

where we have introduced a coefficient of a heat conductivity

$$\lambda = \frac{4acT^3}{3\kappa\rho}. \quad (17)$$

For $\tau \to \infty$, Eq. (16) coincides with a corresponding equation from Bisnovatyi–Kogan (1967) when

$$\frac{L_{\rm th}}{4\pi r^2} \ll caT^4.$$

Using Eqs. (11) and (8) in Eq. (5) gives

$$\left(\frac{\mathcal{R}T}{\rho} - \frac{\mu^2}{\rho^3 r^4} \right) \frac{d\rho}{dr} = -\mathcal{R}\frac{dT}{dr} + \mu^2 \frac{2}{\rho^2 r^5} - \frac{GM}{r^2}(1-\tilde{L}_{th}). \quad (18)$$

Eqs. (15)–(18) form a complete set.

Eq. (18) has a singular point where the left hand side of the equation vanishes. There,

$$\frac{\mathcal{R}T_{\rm cr}}{\rho_{\rm cr}} = \frac{\mu^2}{\rho_{\rm cr}^3 r_{\rm cr}^4}. \quad (19)$$

A solution passing continuously through this critical point, which is of the saddle type (Parker 1963), should nullify also the right hand side of Eq. (18) which, after using dT/dr term from Eq. (16) is represented by an algebraic relation. This point corresponds to the "isothermal sonic" point where

$$u^2 = u_{\rm s}^2 \equiv \left(\frac{\partial P}{\partial \rho} \right)_T.$$

Substituting Eqs. (3), (4), and (17) in Eq. (7), we have

$$L_{\rm th} = -4\pi r^2 \left[\lambda \frac{dT}{dr}(1-e^{-\tau}) - \frac{1}{3} acT^4 e^{-\tau} \right]. \quad (20)$$

The second singular point of the equation set (15)-(18) is situated at infinity ($r \to \infty$), where

$$T \to 0, \quad \rho \sim \frac{1}{r^2} \to 0, \text{ and } \quad u \to {\rm const} = u_\infty. \quad (21)$$

In reality, the wind may be treated as stationary only up to the limiting radius $r_{\rm lim} \simeq v_\infty t >> r_{\rm cr}$, where t is the characteristic mass-loss time of the star. So the formal solution with Eq. (21) as the outer boundary condition is very close to the real solution with conditions at $r_{\rm lim}$. This approximation is commonly used in describing a well-developed solar wind (Parker 1963).

When $\kappa \to \infty$ ($\lambda \to 0$), the two singular points of the equation system (15)-(16) become identical, coinciding with the point where the speed of the matter flow equals the adiabatic speed of sound. Making use of the non-dimensional variables

$$\tilde{T}(r) = \frac{T(r)}{T_{\rm cr}} \quad , \quad \tilde{\rho}(r) = \frac{\rho(r)}{\rho_{\rm cr}} \quad , \quad \tilde{L}_{\rm th} = \frac{L_{\rm th}}{L_{\rm Edd}}, \quad (22)$$

and $\tilde{x} = \dfrac{r_{\rm cr}}{r}$,

Eqs. (15)–(16), (18) can be transformed into a dimensionless system of equations:

$$\frac{d\rho}{dx} = \left(\frac{x^4}{\rho^3} - \frac{T}{\rho} \right)^{-1} \left\{ \frac{dT}{dx} \left(1 + A_1(1-e^{-\tau})\frac{T^3}{\rho} \right) - A_3 + \frac{1}{4}\frac{A_1 e^{-\tau}}{A_5}\frac{T^4}{x^2} + 2\frac{x^3}{\rho^2} \right\} \quad , \quad (23)$$

$$\frac{dT}{dx} = -\biggl(\frac{5}{2}T - A_3 x + \frac{1}{2}\frac{x^4}{\rho^2} + (1-e^{-\tau})A_1\frac{T^4}{\rho} \tag{24}$$

$$+ \frac{e^{-\tau}}{4A_2A_5}\frac{T^4}{x^2} + 2L^{\infty}A_3A_5\frac{x^2}{\rho} - \frac{A_4}{A_2}\biggr)\frac{A_2\rho}{T^3(1-e^{-\tau})} \quad \text{, and}$$

$$\frac{d\tau}{dx} = \frac{\rho}{A_5 x^2} \quad , \tag{25}$$

where $L^{\infty} \equiv \tilde{L}^{\infty}_{\rm th}$. To simplify further development, we henceforth omit the tilde in all variables.

The dimensionless coefficients A_i are given by

$$A_1 = \frac{4aT_{cr}^3}{3\rho_{cr}\mathcal{R}} \quad , \quad A_2 = \frac{3\kappa\mu}{4ac}\frac{\rho_{cr}\mathcal{R}}{r_{cr}T_{cr}^3}, \tag{26}$$

$$A_3 = \frac{GM}{r_{cr}\mathcal{R}T_{cr}} \quad , \quad A_4 = \frac{3\kappa L}{16ac\pi}\frac{\rho_{cr}}{r_{cr}T_{cr}^4}. \tag{27}$$

The parameters A_i have physical interpretations (see also Bisnovatyi-Kogan 1967). $A_1/4$ is the ratio of the isotropic radiation pressure to the gas pressure at the critical point. $(2A_3)^{1/2}$ is the ratio of the parabolic velocity at the critical point to the isothermal sound speed. A_1A_4/A_3 is the ratio of the full energy flux from the star to the critical Eddington luminosity at $r = r_{\rm cr}$. A_2/A_4 is equal to the ratio of the internal energy flux carried by the outflowing gas to the total energy flux at the critical point. An additional fifth parameter,

$$A_5 = \frac{1}{r_{cr}\kappa\rho_{cr}} \quad , \tag{28}$$

is of the order of the reciprocal optical depth at the critical point. The requirement that the solution makes a smooth, continuous transition through the critical point reduces the number of independent dimensionless parameters. Setting to zero the expression in the curly brackets of Eq. (23) at the critical point and taking Eq. (24) into account, we have

$$\begin{aligned} A_4 = \bigl(&4A_3A_5(1-e^{-\tau_{cr}}) + 8A_5(-1+e^{-\tau_{cr}}) \\ &+4A_1^2A_2A_5\bigl(1-2e^{-\tau_{cr}} + (e^{-\tau_{cr}})^2\bigr) \\ &+A_2\bigl(12A_5 + A_3(-4A_5 + 8A_5^2L^{\infty})\bigr) \\ &+A_1A_2\bigl(16A_5(1-e^{-\tau_{cr}}) \\ &+A_3(4A_5(-1+e^{-\tau_{cr}}) + A_5^2(8L^{\infty} \\ &-8e^{-\tau_{cr}}L^{\infty}))\bigr)+e^{-\tau_{cr}}\bigr)/4\bigl(A_5 + A_1A_5(1-e^{-\tau_{cr}})\bigr). \end{aligned} \tag{29}$$

The dimensional quantities T, ρ, and r can be expressed as functions of dimensionless quantities x, $\tilde{\rho}$, $\tilde{T}$, the parameters A_i, and dimensional combinations of physical constants (Bisnovatyi-Kogan 1967):

$$r = \left(\frac{4a\kappa}{3c}\right)^{2/5}\frac{(GM)^{7/5}}{\mathcal{R}^{8/5}}\frac{1}{(A_1^2A_2A_3)^{2/5}A_3 x},$$

$$\rho = \left(\frac{3\mathcal{R}}{4a}\right)^{1/5}\left(\frac{c\mathcal{R}^{1/2}}{\kappa GM}\right)^{6/5}(A_1^2A_2A_3)^{6/5}\frac{\tilde{\rho}}{A_1}.$$

$$T = \left(\frac{3c\mathcal{R}^{3/2}}{4a\kappa GM}\right)^{2/5}(A_1^2A_2A_3)^{2/5}\tilde{T}. \tag{30}$$

The coefficient A_5 is a function of A_1, A_2, A_3, and of a non-dimensional combination of physical constants, and thus it is not an independent parameter:

$$A_5 = \left(\frac{3}{4}\right)^{1/5}\frac{A_3^{1/5}\mathcal{R}^{4/5}}{A_1^{3/5}A_2^{4/5}\kappa^{1/5}a^{1/5}c^{4/5}(GM)^{1/5}}. \tag{31}$$

In addition to coefficients A_i, we have independent non-dimensional parameters L^{∞} and $\tau_{\rm cr}$, the optical depth at the critical point. So, before satisfying the boundary conditions at infinity, we must specify five "independent" non-dimensional parameters.

4. Numerical solution

In order to satisfy the boundary conditions far from the star, we need to integrate Eqs. (23)-(25) from the critical point outward to infinity. We cross the critical point by means of expansion formulae. Expanding the solution in the vicinity of the critical point $x = T = \rho = 1$ in powers of $y = (1-x)$, we have

$$T = 1 + ay, \tag{32}$$

$$\rho = 1 + by, \tag{33}$$

$$e^{-\tau} \simeq e^{-\tau_{cr}}(1 + \frac{y}{A_5}),$$

where a and b are

$$b = \frac{4A_5\bigl(A_4 - A_2\bigl[3 - A_1(e^{-\tau_{\rm cr}}-1) - A_3(1-2A_5L^{\infty})\bigr]\bigr) - e^{-\tau_{\rm cr}}}{4A_5(e^{-\tau_{\rm cr}}-1)}, \tag{34}$$

$$a = -\left(c_1 + (c_1^2 - 4c_2c_0)^{1/2}\right)/(2c_2). \tag{35}$$

Because of their cumbersome form, the coefficients c_i in Eq. (34) are given in Appendix.

We begin a numerical integration from the critical point, using the expansion formulas Eqs. (32) and (33) to make the first step. Integrating outward to infinity, we satisfy the boundary conditions, Eq. (21). To satisfy the condition of zero T at infinity, we find a unique relationship $A_3(A_1, A_2, L^{\infty}_{\rm th}, \tau_{cr})$.

For the given values of A_1, A_2, and $L^{\infty}_{\rm th}$, a unique value for $\tau_{\rm cr}$ is found by satisfying the condition $v^{\infty} =$ constant (or, $\rho \propto 1/r^2$). That is, when $r \to \infty$, only for a unique value of $\tau_{\rm cr}$ is the proper behavior of u (and ρ) at infinity obtained.

Results of numerical calculations are shown in Figs. 1–4. Curves in these figures follow from dimensionless parameters having the values $A_1 = 50$, $A_2 = 10^{-4}$, $A_3 = 43.88$, $\tau_{\rm cr} = 125$, and $L^{\infty}_{\rm th} = 0.6$. Physical variables at the critical point have the values $T_{\rm cr} = 1.4 \times 10^4\,K$, $r_{\rm cr} = 2.6 \times 10^{13}$cm, and $\rho_{\rm cr} = 6.6 \times 10^{-12}$ g/cm^3. We have chosen $M = 2 \times 10^{34}$ g. The velocity of the flow at the critical point is $v_{\rm cr} \approx 11\,{\rm km\,s^{-1}}$, and the mass-loss rate is $\dot{M} \approx 9 \times 10^{-3} M_{\odot}/$ yr. The spherical shell produced by such a high mass-loss rate may eventually

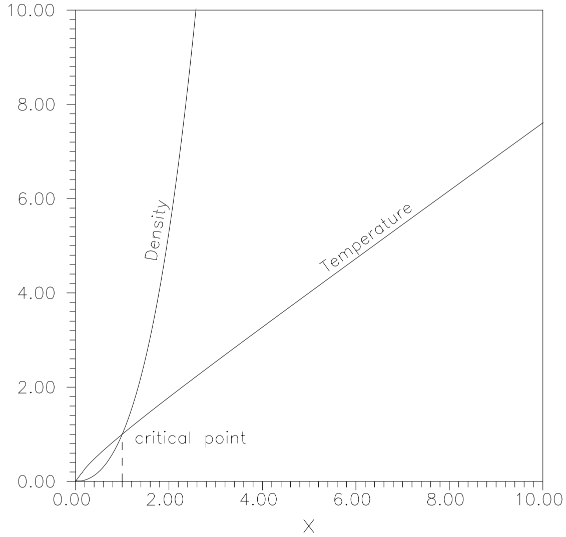

Fig. 1. Results of the integration of Eqs. (23)–(25) from the critical point $x = 1$ outward to the infinity $x = 0$. Points $x = 1$ and $x = 0$ are the singular points of our system. A solution shown for $A_1 = 50$, $A_2 = 10^{-4}$, $A_3 = 43.88$, $\tau_{cr} = 125$, $L_{th}^{\infty} = 0.6$, passes through the critical point ($T_{cr} = 1.4 \cdot 10^4\, K$, $r_{cr} = 2.6 \cdot 10^{13}$cm, $\rho_{cr} = 6.6 \cdot 10^{-12} g/\mathrm{cm}^3$) and satisfies boundary conditions (21).

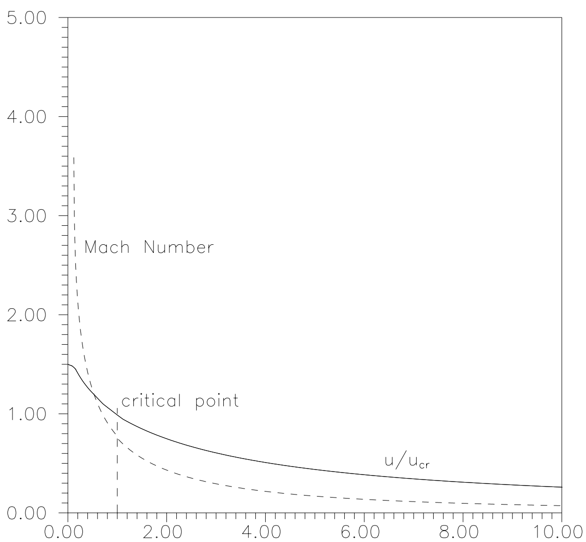

Fig. 2. Dimensionless velocity (solid line) and Mach number $= u \cdot \left(\frac{\partial P}{\partial \rho}\right)_s^{-\frac{1}{2}}$ (dashed line). Inwardly decreasing Mach number ensures matching the static core even in the case of $\kappa = const$. When $r \to \infty$ velocity tends to constant $v_\infty \simeq 16\, km\, s^{-1}$ and $\rho \sim 1/r^2$.
Solution is shown for the same values of nondimentional parameters as on Fig. 1.

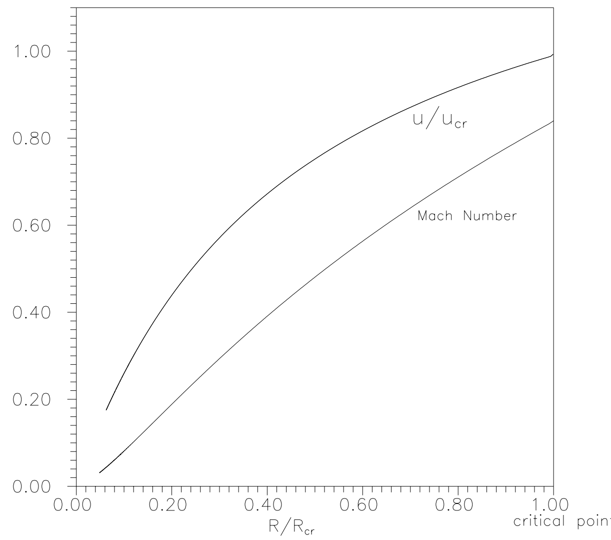

Fig. 3. Dimensionless velocity (solid line) and Mach number (dashed line). Towards the static core, the velocity tends to zero. See last paragraphs at Sect. 4.

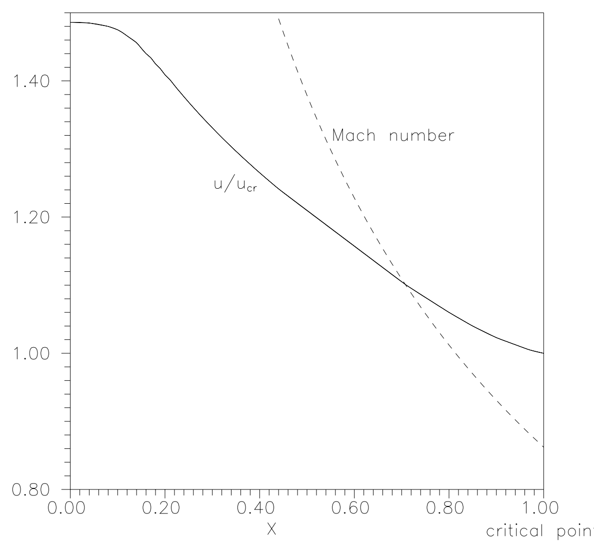

Fig. 4. Dimenisonless velocity (solid line) and Mach number (dashed line). Towards the infinity the velocity tends to the v_∞.

appear as a ring nebula, after the high speed wind from the WR star product of the shell-producing star interacts with the shell (Lozinskaya 1986).

The fact that, in the region below $r = r_{\rm cr}$, the Mach number decreases rapidly with decreasing r makes it possible to match the solution to a static interior. With a more realistic opacity, the opacity peak is situated near the critical point and the opacity and the velocity decrease more rapidly with decreasing r than when κ=constant (Bisnovatyi-Kogan & Nadyozhin 1972). Deep inside the star, all hydrodynamical solutions converge to a static one. In the static atmosphere, at L = constant and M = constant, we have $\rho \propto T^3$ and $T \propto 1/r$ (Bisnovatyi-Kogan 1973). Since $u \propto 1/(\rho r^2)$, the velocity in the subsonic region tends to zero roughly linearly with r. In reality, deeper in the star $\dot{M}$ decreases toward zero. This means that the velocity goes to zero faster than linearly with respect to r. The Mach number $M_{\rm a}$, is defined by $M_{\rm a} = u/\sqrt{\gamma P/\rho}$, where P is taken from Eqs. (9) and (11), with the anisotropic term omitted from Eq. (9):

$$P = \frac{aT^4}{3}\left(1 - e^{-\tau}\right) + \rho\mathcal{R}\mathcal{T}. \tag{36}$$

The value of γ varies from $\frac{4}{3}$ for a pure radiation field to $\frac{5}{3}$ for a fully ionized gas, and is given in general by (Chandrasekhar 1939)

$$\gamma = \beta_g + \frac{2}{3}\frac{(4-3\beta_g)^2}{8-7\beta_g},$$

where $\beta_g = P_g/P$, with P from (36) and τ is constant for adiabatic variations.

The effective temperature at the photosphere is obtained from $L_{th}^{\infty}/4\pi r^2 = \sigma T^4$. For the given set of parameters, we have $x_{\rm ph} = 0.03$, $\tau_{\rm ph} = 4$, $\tilde{T}_{\rm ph} = 0.06$, and $\tilde{\tau}_{\rm ph} = 3.75$. Transforming to physical variables, $r_{\rm ph} = 8.6 \times 10^{14}$ cm and $T_{\rm ph} = 840$ K. Thus, the model describes a very luminous infrared star with an extended outflowing atmosphere.

For realistic functions $\kappa(\rho, T)$ and $P(\rho, T)$, the corresponding model properties will, of course, be different. For temperatures below several 10^3 K, the opacity drops (e.g., Iglesias & Rogers 1996; Seaton 1996), the photosphere moves further into the star, and the effective temperature is greater.

At $r = \infty$, the solution may be represented by an expansion in powers of $1/r$, and $L_{\rm th}^{\infty}$ can be determined directly from Eq. (20), as will be shown explicitly in a subsequent paper, leaving free only two nondimensional parameters A_1 and A_2. After obtaining the solution in non-dimensional variables and parameters, we obtain from Eq. (30) corresponding dependances for dimensional values, prescribing mass M and prescribing the chemical composition in $\mathcal{R}$. Stationary solutions for the outflowing envelope may be obtained for arbitrary values of $r_{\rm cr}$ and total energy flux L. Matching the solution of the outflowing envelope to the static interior determines uniquely the values of $r_{\rm cr}$ and L, leading to a unique solution for the mass-losing star with a self-consistent mass loss rate.

Dimensionless variables in the static interior and in the outflowing envelope are not the same, so matching outflowing and static solutions should be performed on the dimensional variables ρ and T and at a value of τ large enough that $u \sim 0$ and $\dot{M}(r) \sim 0$. With this prescription, all solutions for the envelope converge to one static solution, regardless of $\dot{M}$ in the envelope. In our treatment, we did not use the expansion at infinity, and thus could not specify uniquely $L_{\rm th}^{\infty}$.

5. Discussion

The main goal of this paper has been to construct a general method to describe a spherically symmetric, stationary, outflowing envelope that is accelerated by radiation pressure at arbitrary optical depth, and which can be used for self-consistent evolutionary calculations of massive stars with mass loss. In the explicit numerical example presented for illustration, the physics has been simplified by adopting constant molecular weight and constant opacity. For real stars, of course, variable molecular weight and opacity must be taken into account, and this introduces serious technical complications. For example, the bimodal behaviour of radiation at $\tau < 1$ makes it necessary to use different κ's for the two components of the radiation. But, as shown for the optically thick case by Bisnovatyi-Kogan & Nadyozhin (1969, 1972), these complications can be surmounted without too many difficulties.

We have derived thermodynamic relations for the matter flux and have separated the radiation field into isotropic and non-isotropic components, obtaining a satisfactory description of the problem for arbitrary τ. For $\tau \to \infty$, the equation of state of the equilibrium radiation field is given by $E_r\rho = 3P_r$, and when $\tau \to 0$, the equation of state of the field is $E_r\rho \simeq P_r$, which coincides with the solution of the radiative transfer equation in the Eddington approximation. These relations have been used in the equations of radiation hydrodynamics with a constant mass flux $\dot{M}$ and with a constant total energy flux L. To satisfy boundary conditions, the solution proceeds through a critical point where the speed of the mass flow equals the local isothermal sound speed. An analytical expansion near the sonic point is used to begin the numerical solution. Two nondimensional parameters are specified to satisfy conditions at infinity. Further integrations are needed to specify one additional parameter.

In a number of earlier studies (e.g., Zytkow 1972, Kato 1985, Kato & Iben 1992), the outer boundary conditions have been oversimplified, giving the impression that regions with $\tau < 1$ have no influence on the mass outflow, even though Paczynski (1969) had demonstrated that, for static cases with extended atmospheres, regions with small τ and diluted radiation should be thoroughly taken into consideration. Zytkow imposed boundary conditions on an artificially introduced surface that is determined by the relation $L = 8\pi\sigma T_{\rm surf}^4 R_{\rm surf}^2$ taken from static solutions. However, to describe correctly the mass outflow from stars, one should satisfy boundary conditions at infinity, as shown by Parker (1963) in the case of the solar wind. To illustrate differences in the two approaches, we highlight some aspects of the Zytkow approach. For $\tau < 2/3$, Zytkow adopts the equations of motion and heat transfer:

$$\frac{d\rho}{dx} = \frac{2\rho v^2}{r} - \frac{GM\rho}{r^2} - \frac{\partial P_g}{\partial T}\frac{dT}{dr} + \frac{\kappa\rho}{4\pi c}\frac{L}{r^2}, \qquad (37)$$

where

$$\frac{dT}{dr} = -\frac{L}{4\pi r^2\lambda} - \frac{1}{2}f(\tau)T_{\rm surf}R_{\rm surf}^{1/2}r^{-3/2}, \qquad (38)$$

where $T_{\rm surf}$ is is related to the photospheric temperature by $T_{\rm surf} = T_{\rm ph}/2^{1/4}$, L is constant for $\tau < 2/3$, and $f(\tau) = 1-(3/2)\tau$. The second term in Eq. (38) is an interpolation that connects regions of small τ (with Eddington factor = 1) with regions at $\tau >> 1$ (with an Eddington factor = 1/3). Instead of the heat transfer equation with a rough interpolation between regions of large and small τ (38), we use Eq. (7), which follows exactly from the radiative transfer equation. In our approach, we do not have to specify "surface" temperature and radius, but solve Eq. (7) together with the hydrodynamical equations in the whole region of the flow up to very large radii. This is possible because we make an interpolation in the thermodynamic relations for the radiation which had not been considered at all by Paczynski (1969) in optically thin layers. Our approach permits us to describe the radiation field in the layers with $\tau < 1$, and to follow smoothly its transformation from the equilibrium, isotropic field inside the star, to the free outwardly directed flux of quanta in the supersonic flow far from the star.

6. Conclusion

1. An approximate system of equations for radiation hydrodynamics with radiative heat transfer is derived; the system gives a continuous transition between optically thick and optically thin regions of a spherically symmetric mass outflow.
2. A solution of these equations is obtained numerically; the solution is continuous through the singular sonic point, and satisfies the conditions $\rho = T = 0$ at infinity.
3. The photosphere of a star with stationary mass loss is obtained uniquely from this solution.
4. The derived equations can be used to describe outflowing envelopes of evolved massive stars, permitting calculations of stellar evolution with self-consistent determinations of the mass-loss rate as it changes during evolution. Matching between envelope solutions and interior solutions should be made in a deeply subsonic region, far below the photosphere.

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Appendix

Calculations of the coefficients c_i appearing in Eq. (35) of the text were made using Mathematica 2.2. The results are:

$$
\begin{aligned}
c_0 = \; & \big(A_1 A_2^2 A_3^2 A_5^4 96(-1+e^{-\tau_{cr}})(L^\infty)^2 \\
& +A_5^3(-40A_2^2A_3L^\infty + A_1^2A_2^2A_3(-160+320e^{-\tau_{cr}} \\
& -160(e^{-\tau_{cr}})^2)L^\infty + A_2(A_3 176(1-e^{-\tau_{cr}})L^\infty \\
& +A_3^2(-48+48e^{-\tau_{cr}})L^\infty) \\
& +A_1(A_2A_3(80+96A_4+(-160-96A_4)e^{-\tau_{cr}} \\
& +80(e^{-\tau_{cr}})^2)L^\infty + A_2^2(A_3^2 96(1-e^{-\tau_{cr}})L^\infty \\
& +A_3 392(-1+e^{-\tau_{cr}})L^\infty))) - 2(e^{-\tau_{cr}})^2 \\
& +A_5(2(e^{-\tau_{cr}})^2 + e^{-\tau_{cr}}(-2+8A_4) \\
& +A_2e^{-\tau_{cr}}(-29+8A_3) + 2A_3(1e^{-\tau_{cr}} - (e^{-\tau_{cr}})^2) \\
& +2A_1^2A_2e^{-\tau_{cr}}(1-2(e^{-\tau_{cr}}) + (e^{-\tau_{cr}})^2) \\
& +A_1(-2A_4e^{-\tau_{cr}} + 2A_4(e^{-\tau_{cr}})^2 \\
& +A_2(-7e^{-\tau_{cr}} + 7(e^{-\tau_{cr}})^2 + A_3(-2e^{-\tau_{cr}} \\
& +2(e^{-\tau_{cr}})^2)))) + A_5^2(-96 + A_2^2(-60+20A_3) \\
& -72A_4 + (192+72A_4)e^{-\tau_{cr}} - 96(e^{-\tau_{cr}})^2 \\
& +A_3(24+24A_4(-48-24A_4)e^{-\tau_{cr}} \\
& +24(e^{-\tau_{cr}})^2) + A_1^3A_2^2(-56+168e^{-\tau_{cr}} \\
& -168(e^{-\tau_{cr}})^2 + 56(e^{-\tau_{cr}})^3) \\
& +A_1^2(A_2(24+80A_4+(-72-160A_4)e^{-\tau_{cr}} \\
& +(72+80A_4)(e^{-\tau_{cr}})^2 - 24(e^{-\tau_{cr}})^3) \\
& +A_2^2(-292+584e^{-\tau_{cr}} - 292(e^{-\tau_{cr}})^2 \\
& +A_3(80-160e^{-\tau_{cr}} + 80(e^{-\tau_{cr}})^2))) \\
& +A_2(232+20A_4+A_3^2(24-24e^{-\tau_{cr}}) - 232e^{-\tau_{cr}} \\
& +A_3(-152+152e^{-\tau_{cr}} - 16e^{-\tau_{cr}}L^\infty)) \\
& +A_1(-24A_4 - 24A_4^2 + (48A_4+24A_4^2)e^{-\tau_{cr}} \\
& -24A_4(e^{-\tau_{cr}})^2 + A_2^2(-392+A_3(196-196e^{-\tau_{cr}}) \\
& +392e^{-\tau_{cr}} + A_3^2(-24+24e^{-\tau_{cr}})) \\
& +A_2(160+196A_4+(-320-196A_4)e^{-\tau_{cr}} \\
& +160(e^{-\tau_{cr}})^2 + A_3(-56-48A_4+(112 \\
& +48A_4)e^{-\tau_{cr}} - 56(e^{-\tau_{cr}})^2 + L^\infty(4e^{-\tau_{cr}} \\
& -4(e^{-\tau_{cr}})^2))))))\big)/\big(2(-1+e^{-\tau_{cr}})\big),
\end{aligned}
$$

$$
\begin{aligned}
c_1 = \; & A_5^2(8+A_2(56-20A_3) - 20A_4 + A_3(12-12e^{-\tau_{cr}}) \\
& -8e^{-\tau_{cr}} + A_1^2A_2(8-16e^{-\tau_{cr}} + 8(e^{-\tau_{cr}})^2) \\
& +A_1(-12A_4+12A_4e^{-\tau_{cr}} + A_2(48-48e^{-\tau_{cr}} \\
& +A_3(-12+12e^{-\tau_{cr}})))) + A_5^3(32A_2A_3L^\infty \\
& +A_1A_2A_3(16-16e^{-\tau_{cr}})L^\infty) + 5A_5e^{-\tau_{cr}},
\end{aligned}
$$

$$
c_2 = A_5^2\left(8 - 8e^{-\tau_{cr}}\right).
$$

References

Bisnovatyi–Kogan G.S., 1967, Prikl. Mat. Mech. 31, 762
Bisnovatyi–Kogan G.S., 1973, Ap&SS 22, 293
Bisnovatyi–Kogan G.S., Nadyozhin D.K., 1972, Ap&SS 15, 353
Bisnovatyi–Kogan G.S., Nadyozhin D.K., 1969, Nauch.Inform. 11, 27
Bisnovatyi–Kogan G.S., Zeldovich Ya.B., 1968, AZh 45, 241
Castor J., Abbott D., Klein R., 1975, ApJ 195, 157
Chandrasekhar S., Stellar structure. Chicago, 1939
Cox A., Tabor J., 1976, ApJS 31, 271
Iglesias C.A., Rogers F.J., 1996, ApJ 464, 943
Kato M., Iben I., 1992, ApJ 394, 305
Kato M., 1985, PASJ 37, 19
Lozinskaya T.A., 1986, Supernovae stars and stellar wind, interaction with the galactic gas. Moscow, Nauka
Maeder A., 1981, A&A 102, 401
Owocki S., Puls J., 1996, ApJ 462, 894
Paczynski B, 1969., Acta Astron. 19, 1
Parker E.N., 1963, Interplanetary dynamical processes. Interscience Publishers, New York - London
Pauldrach A.W.A., Kudritzki R.P., Puls J., Butler K., Hunsinger J., 1994, A&A 283, 525
Seaton M.J., 1996, MNRAS 279, 95
Schaerer D., de Koter A., Schmutz W., Maeder A., 1996, A&A 310, 837
Smith, L., 1968, In: Wolf-Rayet stars. Proc. Symp. Boulder
Sobolev, V.V., 1967, Course of theoretical astrophysics. Moscow, Nauka
Stothers R., Chin C., 1979, ApJ 233, 267
Zytkow A., 1972, Acta Astron. 22, 103
Zytkow A., 1973, Acta Astron. 23, 121