Astron. Astrophys. 344, 973–980 (1999)

ASTRONOMY AND ASTROPHYSICS

Linear adiabatic dynamics of a polytropic convection zone with an isothermal atmosphere

I. General features and real modes

F. Schmitz and S. Steffens

Astronomisches Institut der Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany

Received 18 September 1998 / Accepted 11 January 1999

Abstract. To investigate and understand basic properties of non-radial solar p-modes with high wave numbers l, it is sufficient to consider only the outer layers of the sun. As an atmosphere, the upper part of the convection zone may be approximated by a plane layer with constant gravity. A simple standard model is a polytropic convection zone with an overlying isothermal atmosphere. In this case, the adiabatic wave equation of each layer can be solved analytically. However, the dispersion relation $F(\omega, k) = 0$ of the acoustic and gravity modes of the whole layer is complicated and cannot be solved in closed form. In this paper, we present a model with a smooth transition between the poytropic convection zone and the isothermal atmosphere. For this model, using the column mass instead of the geometrical height, the adiabatic wave equation can be reduced to Whittakers differential equation. The geometrical height is a simple elementary function of the column mass. The dispersion relation $F(\omega, k) = 0$ is a fourth order algebraic equation in ω^2. In the important case of an isentropically stratified polytropic convection zone, it reduces to a cubic equation in ω^2. In any case, the dispersion curves $\omega(k)$ can be given in closed form. As in the case of a purely polytropic convection zone, the z-dependence of the waves and the modes is represented by Whittaker functions. We analyze the behavior of the dispersion curves of modes with an adiabatic exponent $\gamma = 5/3$ for layers with polytropic indices $n = 3$ and $n = 3/2$. Further, we investigate the appearance of resonances in the region of the continuous spectrum of acoustic waves. We find that these resonances are present only at frequencies slightly above the acoustic cutoff frequency of the isothermal atmosphere. The case of purely vertical wave propagation is considered separately. In the present paper, we deal only with real frequencies.

Key words: hydrodynamics – Sun: atmosphere – Sun: oscillations

Send offprint requests to: F. Schmitz (Schmitz@astro.uni-wuerzburg.de)

1. Introduction

To study basic properties of solar p-modes with $l \gg 1$, it is sufficient to consider only the upper convection zone and the atmospheric layers of the sun. For $l \gg 1$, the approach of these regions of the sun by a plane layer with constant gravity is common. A simple model of the convection zone is a polytropic layer with positive polytropic index n. In this case, the pressure and the temperature vanish at some height. The solution of the adiabatic wave equation of this layer has to satisfy a zero pressure boundary condition. By the request of vanishing pressure perturbations in the interior, a discrete spectrum of modes is obtained. This problem was investigated already by Lamb (1932). The isolated polytropic convection zone corresponds to simple polytropic stellar models with zero pressure boundary condition and moderate central densities, as far as Cowling's classification is valid.

To study the effect of an atmosphere on the modes of a convection zone, a two layer model, consisting of a polytropic layer and an overlying isothermal layer is a useful tool. There are two possibilities: To join the atmosphere by a temperature jump to the polytropic convection zone or to join the atmosphere continuously. In the literature, both options are considered.

Pekeris (1948) used the two layer model to study the propagation of waves in the atmosphere of the earth. Murray (1993) and Goldreich et al. (1994) have considered the excitation of solar p-modes by use of this two-layer model with isentropic stratification of the polytropic layer. Balmforth & Gough (1990) and Rast & Gough (1995) have investigated the behavior of forced and damped oscillations in the one-dimensional vertical case. Recently, the model has also been used by Price (1996) to calculate interference effects proposed by Kumar et al. (1990) and Kumar & Lu (1991) to explain the observed high frequency modes.

Intuitively, the joining of the layers by a temperature jump is attractive as it seems to approach the steep temperature increase of the photosphere. However, a temperature jump produces a discontinuous density inversion which introduces a Rayleigh-Taylor-instability.

The two layer model enables a simple solution of the wave equation as the analytic solutions of the wave equations of the convective layer and the isothermal atmosphere are fitted together by the condition of continuity of the vertical velocity and the Lagrangian pressure perturbation. However, the dispersion relation $F(\omega, k) = 0$ obtained by the fitting conditions is complicated and involves Whittaker functions. ("The mode behavior in the presence of an overlying atmosphere is less evident, the dispersion relation in this case being somewhat less transparent", Price (1996).)

For this reason, we present a model consisting of a convection zone and an atmosphere which are fitted by a smooth temperature transition. The convection zone becomes polytropic as $z \to -\infty$, the atmosphere becomes isothermal as $z \to +\infty$. With this model, some specific features which stem from the temperature gradient of the photosphere cannot be described. Many other effects are not essentially influenced by the photospheric temperature structure. For such effects our model is a useful tool. For example, the question of the excitation of p-modes, the origin of the ridges above the acoustic cut-off frequency first observed by Libbrecht (1988), and the existence and the meaning of modes with complex frequencies in the adiabatic case are problems basically not affected by atmospheric details.

In Sect. 2 we discuss the structure of the static layer. In Sect. 3 we present the adiabatic wave equation of the Lagrangian pressure perturbation. Sect. 4 deals with the reduction of this wave equation to Whittaker's differential equation. The general solution of this equation is considered in Sect. 5. The dispersion relation of the modes of the layer, a fourth order algebraic equation for ω^2 is derived in Sect. 6. We compare this modes with the modes of the simple polytropic layer. For this reason, in Sect. 7, we give the dispersion relation of the polytropic layer. As a real convection zone is mainly isentropic, we study the case of an isentropic stratification of the lower layer in Sect. 8. In Sect. 9 we present two examples: Dispersion curves for the non-isentropic case $\gamma = 5/3$ and $1 + 1/n = 4/3$ and for the isentropic case $\gamma = 1 + 1/n = 5/3$. Sect. 10 deals with resonances which occur for frequencies above the acoustic cut-off frequency of the isothermal atmosphere. The solutions of the special case of vertically propagating waves are given in Sect. 11.

2. The equilibrium layer

Let z be the vertical, outwards directed geometrical coordinate, g the constant gravity, n the polytropic index, m the column mass, defined by $dm = -\rho\, dz$, p the pressure, ρ the density, a the isothermal sound speed. We use the equation of state of the classical ideal gas. Then, we have $p = a^2\rho$. The equilibrium condition is $p = m\, g$. We put

$$a^2(m) = a_0^2 + \epsilon\, m^\lambda \, . \tag{1}$$

For $m \to 0$ the layer becomes isothermal with $a = a_0$, for large m the layer becomes polytropic with index n. We have $\lambda = 1/(1+n)$. In the limit $a_0 = 0$ we obtain the polytropic layer. The density is

$$\rho(m) = \frac{m\, g}{a_0^2 + \epsilon\, m^\lambda} \, . \tag{2}$$

The geometrical height $z(m)$ is obtained by integrating $dz = -\, d\, m/\rho$. We get

$$z = -\, \frac{1}{g}\, [\, a_0^2\, \ln\, m + \frac{\epsilon}{\lambda}\, m^\lambda\,] \, . \tag{3}$$

We have $m \to 0$ as $z \to +\infty$ and $m \to \infty$ as $z \to -\infty$, and

$$a^2(z) = -\, \frac{g}{1+n}\, z \quad \text{for} \quad z \to -\infty \, . \tag{4}$$

Let c be the adiabatic sound speed, γ the adiabatic exponent, c_0 the constant adiabatic sound speed at $z \to \infty$. Then, $c^2 = \gamma\, a^2$, and $c_0^2 = \gamma\, a_0^2$. For the convection zone of the sun we have $\epsilon \approx 10^{11}$ in cgs-units.

3. The adiabatic wave equation

Let $\Delta p(z,t)$ be the Lagrangian pressure perturbation. The frequency is denoted by ω, the horizontal wave number by k. We study adiabatic waves with time dependence $\exp(i\omega t)$. The wave equation of the Lagrangian pressure perturbation Δp is

$$\omega^2\, c^2\, \Big[\, \frac{d^2\,\Delta p}{dz^2} - \frac{1}{\rho}\, \frac{d\,\rho}{d\,z}\, \frac{d\,\Delta p}{dz}\, \Big] -$$

$$\Big[\, k^2\, g\, [\, g + \frac{c^2}{\rho}\, \frac{d\,\rho}{d\,z}\,] + \omega^2\, (\, c^2\, k^2 - \omega^2\,)\, \Big]\, \Delta p = 0 \, . \tag{5}$$

(Schmitz and Fleck 1994, Eq. 5. The reason for preferring the Lagrangian pressure perturbation is explained there.) The vertical Lagrangian displacement Δz is related to the pressure perturbation Δp by

$$\Delta z = \big[\, \omega^2 - \frac{k^2\, g^2}{\omega^2}\, \big]^{-1}\, \frac{1}{\rho}\, \Big(\frac{d\,\Delta p}{dz} + g\, \frac{k^2}{\omega^2}\, \Delta p \Big). \tag{6}$$

Instead of the geometrical height z we use the mass m as the independent variable. We have:

$$\Big[\, \frac{d^2\,\Delta p}{dz^2} - \frac{1}{\rho}\, \frac{d\,\rho}{d\,z}\, \frac{d\,\Delta p}{dz}\, \Big] = \rho^2\, \frac{d^2\,\Delta p}{dm^2} \, . \tag{7}$$

For the classical ideal gas with $p = a^2\,\rho$ and $c^2 = \gamma\, a^2$, the wave equation reads

$$m^2\, \frac{d^2\,\Delta p}{dm^2} + \frac{\Delta p}{\gamma\,\omega^2\, g^2} \, .$$

$$\Big[[\omega^4 - k^2 g^2 (1-\gamma)]\, a^2 - \omega^2 \gamma a^4 k^2 - k^2 g^2 \gamma m \frac{d\, a^2}{d\, m} \Big] = 0 \, . \tag{8}$$

For $a^2 = a_0^2 + \epsilon\, m^\lambda$ we obtain the equation

$$m^2\, \frac{d^2\,\Delta p}{dm^2} + [\, A\, m^{2\,\lambda} + B\, m^\lambda + C\,]\; \Delta p = 0 \, , \tag{9}$$

with

$$A = -\, \frac{k^2\,\epsilon^2}{g^2} \, , \tag{10}$$

$$B = \frac{\epsilon}{\gamma\,\omega^2\, g^2}\, [\, \omega^4 - k^2\, g^2\, (1-\gamma+\gamma\lambda) - 2\,\omega^2 k^2\, \gamma a_0^2\,] \, , \tag{11}$$

$$C = \frac{a_0^2}{\gamma\,\omega^2\,g^2}\,[\,\omega^4 - k^2\,g^2\,(1-\gamma) - \omega^2\,\gamma\,k^2\,a_0^2\,]\,. \quad (12)$$

This equation can be reduced to Whittaker's equation. The parameter A stems from the polytropic part of the layer, the parameter C from the isothermal atmosphere. Both the polytropic and the isothermal part of the layer contribute to the parameter B.

4. The reduction of the adiabatic wave equation

Introducing a new independent variable ξ by

$$\xi = m^\lambda\,, \quad (13)$$

we obtain the equation

$$\lambda^2\xi^2\,\frac{d^2\,\Delta p}{d\xi^2} + \lambda(\lambda-1)\xi\frac{d\,\Delta p}{d\xi} + [A\xi^2 + B\xi + C]\Delta p = 0. \quad (14)$$

Now we put

$$\Delta p = \xi^\nu\;\eta \quad (15)$$

to obtain:

$$\xi^2\,\frac{d^2\,\eta}{d\xi^2} + [\,2\,\nu + \frac{\lambda-1}{\lambda}\,]\,\xi\,\frac{d\,\eta}{d\xi} + \Big[\,\frac{\lambda-1}{\lambda}\,\nu + \nu\,(\nu-1) + \frac{A}{\lambda^2}\,\xi^2 + \frac{B}{\lambda^2}\,\xi + \frac{C}{\lambda^2}\,\Big]\,\eta = 0\,. \quad (16)$$

We put

$$\nu = \frac{1-\lambda}{2\,\lambda} \quad (17)$$

to eliminate the first derivative of η. Inserting the quantity A as given by Eq. (10), we obtain the equation

$$\frac{d^2\,\eta}{d\xi^2} + \Big[-\frac{k^2\,\epsilon^2}{g^2\,\lambda^2} + \frac{B}{\lambda^2}\,\frac{1}{\xi} + [\frac{C}{\lambda^2} + \frac{\lambda^2-1}{4\,\lambda^2}\,]\,\frac{1}{\xi^2}\Big]\,\eta = 0\,. \quad (18)$$

Now let us asssume that $k \neq 0$. The case $k = 0$ is considered in Sect. 11. We introduce the dimensionless variable x by

$$\xi = \frac{\lambda\,g}{2\,k\,\epsilon}\,x\,, \quad (19)$$

to obtain the equation

$$\frac{d^2\,\eta}{dx^2} + \Big[-\frac{1}{4} + \frac{gB}{2\lambda\epsilon\,k}\,\frac{1}{x} + [\frac{C}{\lambda^2} + \frac{\lambda^2-1}{4\,\lambda^2}\,]\frac{1}{x^2}\,\Big]\,\eta = 0\,. \quad (20)$$

This is Whittaker's equation

$$\frac{d^2\,\eta}{dx^2} + \Big[-\frac{1}{4} + \frac{\kappa}{x} + (\,\frac{1}{4} - \mu^2\,)\,\frac{1}{x^2}\,\Big]\,\eta = 0\,. \quad (21)$$

where

$$\kappa = \frac{1}{2\lambda\gamma g\omega^2 k}\,[\,\omega^4 - k^2 g^2\,(1-\gamma+\gamma\lambda) - 2\omega^2 k^2 c_0^2\,]\,, \quad (22)$$

$$\mu = \frac{\pm 1}{2\lambda}\,\sqrt{1 - \frac{4c_0^2}{\gamma^2\omega^2 g^2}\,[\omega^4 - k^2 g^2\,(1-\gamma) - \omega^2 k^2 c_0^2]}\,. \quad (23)$$

For real ω^2, the coefficient κ is real, μ is real or imaginary. For $\mu = 0$ we obtain the equation

$$\omega^4 + k^2 g^2\,(\gamma-1) - \omega^2\,k^2 c_0^2 - \omega^2\,\big(\frac{\gamma g}{2c_0}\big)^2 = 0 \quad (24)$$

the solutions $\omega(k)$ of which define the boundary lines separating the regions of acoustic waves, gravity waves, and evanescent waves of the $k-\omega$ - diagram of the isothermal atmosphere.

5. The solution of the wave equation

Two independent solutions of Whittaker's equation are

$$\eta_1 = e^{-x/2}\;x^{1/2+\mu}\;M(\frac{1}{2}+\mu-\kappa, 1+2\mu, x)\,, \quad (25)$$

$$\eta_2 = e^{-x/2}\;x^{1/2+\mu}\;U(\frac{1}{2}+\mu-\kappa, 1+2\,\mu, x)\,, \quad (26)$$

where M and U are the confluent hypergeometric functions. The corresponding Lagrangian pressure perturbations are

$$\Delta p_1 = c_1\;m^{1/2}\,m^{\lambda\mu}\;e^{-x/2}\;M(\frac{1}{2}+\mu-\kappa, 1+2\,\mu, x) \quad (27)$$

and

$$\Delta p_2 = c_2\;m^{1/2}\,m^{\lambda\mu}\;e^{-x/2}\;U(\frac{1}{2}+\mu-\kappa, 1+2\,\mu, x) \quad (28)$$

where

$$x = \frac{2\,k\,\epsilon}{\lambda\,g}\;m^\lambda\,. \quad (29)$$

The general solution of the wave equation is a superposition of both solutions. The factor $m^{1/2}\,m^{\lambda\mu}$ is due to the isothermal atmosphere. In the limit $m \to 0$ where $z \to \infty$ we obtain

$$m^{1/2}\,m^{\lambda\mu} = \exp\big(-\frac{z}{2H} + \frac{\lambda\mu\,z}{H}\,\big) \quad \text{with} \quad H = \frac{a_0^2}{g}\,. \quad (30)$$

In the following, instead of ω^2, we use the quantity y defined by

$$y = \frac{\omega^2}{g\,k}\,, \quad (31)$$

and instead of the wave number k we use the relative wave number k/k_0 with

$$k_0 = \frac{g}{4\,a_0^2\,\gamma^2}\,. \quad (32)$$

Written in terms of y and k/k_0, the coefficients κ and μ read:

$$\kappa = \frac{1}{2\,\lambda\,\gamma\,y}\,[\,y^2 - (1-\gamma+\gamma\,\lambda) - \frac{1}{2\,\gamma}\,\frac{k}{k_0}\,y\,]\,, \quad (33)$$

$$\mu = \frac{\pm 1}{2\lambda}\,\sqrt{1 - \frac{1}{\gamma^3}\,\frac{k}{k_0}\,\frac{1}{y}\,[\,y^2 + (\gamma-1) - \frac{1}{4\,\gamma}\,\frac{k}{k_0}\,y\,]}\,. \quad (34)$$

6. The dispersion relation of the modes

As shall be shown in Appendix A, for real μ we have to select the solution Δp_1 with $\mu > 0$. Therefore,

$$\Delta p \propto \; x^{\nu+1/2+\mu}\, e^{-\,x/2}\, M(\frac{1}{2}+\mu-\kappa, 1+2\,\mu, x)\,. \tag{35}$$

As $M = 1$ for $x = 0$, this solution approaches the evanescently decaying wave of the unbounded isothermal atmosphere for $z \rightarrow +\infty$ when $\mu > 0$.

For $z \rightarrow -\infty$ or $x \rightarrow +\infty$ we require vanishing or at least finite pressure perturbations. This assumption is common, and corresponds to the condition of vanishing non-radial pressure perturbations in the center of a star.

The asymptotic expansion of M is (Abramowitz & Stegun, 1965)

$$M(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)}\, e^{x}\, x^{a-b} \quad \text{for} \quad x \rightarrow \infty\ . \tag{36}$$

We obtain

$$\Delta p \propto x^{\nu-\kappa}\, e^{x/2}\, \frac{\Gamma(1+2\,\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)} \quad \text{for} \quad x \rightarrow \infty\ . \tag{37}$$

Therefore, the criterion for convergence at $x \rightarrow \infty$ is:

$$\frac{1}{2}+\mu-\kappa = -\,j\ , \qquad j = 0,1,2,3.... \tag{38}$$

In this case, the function M reduces to a polynomial of degree j, a generalized Laguerre-polynomial. The pressure perturbation decays exponentially for $x \rightarrow \infty$. Otherwise, the pressure perturbation diverges. Taking the square of

$$\mu = \kappa - j - \frac{1}{2}\,, \tag{39}$$

and inserting κ and μ, we finally obtain the dispersion relation:

$$y^4 - 2\,\gamma\,\lambda\,(2j{+}1)\; y^3 +$$

$$\Big[\lambda^2\gamma^2(2j{+}1)^2 + \lambda\frac{k}{k_0}(2j{+}1) -\; 2(1{-}\gamma{+}\gamma\lambda) - \gamma^2\Big]\, y^2 +$$

$$\Big[\lambda\,\frac{k}{k_0} + 2\lambda\gamma\,(1{-}\gamma{+}\gamma\lambda)(2j{+}1)\Big]\, y + \,(1{-}\gamma{+}\gamma\lambda)^2 = 0. \tag{40}$$

For $j = 0$, a solution is $y = -1$. Therefore, the dispersion relation of the p_0-mode and the g_0-mode can immediately be reduced to an equation of third order:

$$y^3 - (1+2\gamma\lambda)\, y^2 + [\,\lambda\,\frac{k}{k_0} + \lambda^2\gamma^2 - (1-\gamma)^2\,]\, y +$$

$$(1-\gamma+\gamma\lambda)^2 = 0\ . \tag{41}$$

7. The polytropic layer

For $a_0 = 0$ we obtain the known dispersion relation of the polytropic layer (Lamb 1932). By $\Delta p = e^{k\,z}\; w(z)$ and $\zeta = -2\,k\,z$ with $z \leq 0$, the wave equation (5) reduces to the confluent hypergeometric equation

$$\zeta\, \frac{d^2\, w}{d\zeta^2} \;+ (b-\zeta)\, \frac{d}{d\zeta}\, w - a\, w = 0\ , \tag{42}$$

with $b = -\,n$ and

$$a = \frac{kg}{2\omega^2}\,[1 - \frac{(\gamma-1)}{\gamma}(1+n)] - \frac{\omega^2(1+n)}{2\,k\,\gamma\, g} - \frac{n}{2}\,.$$

The boundary condition $\Delta p = 0$ for $z = 0$ is fulfilled by the solution

$$w(\zeta) = \zeta^{1+n}\; M(1+a+n, 2+n, \zeta)\ . \tag{43}$$

We obtain:

$$\Delta p \propto p_0(z)\; e^{k\,z}\; M(1+a+n, 2+n, -2\,k\,z) \tag{44}$$

for $-\infty < z < 0$. This solution is also obtained from the solution (27) in the limit $a_0 = 0$. There we have $\mu = 1/(2\lambda)$, and thus $1 + 2\,\mu = 1+n$. The boundary condition $\Delta p = 0$ as $\zeta \rightarrow \infty$ or $z \rightarrow -\infty$ is fulfilled by putting $1 + a + n = -\,j$ with $j = 0, 1, ...$ This condition yields the equation

$$y^2 - y\;[\,\gamma\,\lambda\,(2j+1) + \gamma\,] - (1-\gamma+\gamma\,\lambda) = 0\ . \tag{45}$$

This quadratic equation yields the frequencies of pressure modes and gravity modes:

$$\frac{\omega^2}{k\,g} = \frac{\gamma}{1{+}n}(1{+}\frac{n}{2}{+}j)$$

$$\pm\, \sqrt{(\frac{\gamma}{1{+}n})^2\,(1{+}\frac{n}{2}{+}j)^2 + \frac{\gamma}{1{+}n}\,[1 - \frac{(\gamma{-}1)}{\gamma}\,(1{+}n)]}. \tag{46}$$

for $j = 0, 1, 2, 3, ...$. For $\gamma = 1 + 1/n$ there are no gravity modes, and the dispersion relation of the acoustic modes is:

$$\frac{\omega^2}{k\,g} = (2\gamma-1) + 2\,(\gamma-1)\,j\ , \qquad j = 0, 1, 2, ...\ . \tag{47}$$

We present these known results, as the dispersion relations of the polytropic layer are displayed in the figures of Sect. 9. to show the influence of the atmosphere.

8. Isentropic stratification of the lower layer

A real convection zone is nearly in adiabatic equilibrium. In the case of isentropic stratification of a polytropic layer, we have $\gamma = 1 + 1/n$. In principle, our layer becomes isentropic only in the limit $z \rightarrow -\infty$ when we put $\gamma = 1 + 1/n$. Then we have

$$\lambda = \frac{\gamma-1}{\gamma}\ . \tag{48}$$

As $1{-}\gamma{+}\gamma\,\lambda = 0$, the dispersion relation reduces to an equation of third degree,

$$y^3 - 2\,\gamma\,\lambda\,(2j+1)\, y^2 +$$

$$[\,\lambda^2\,\gamma^2\,(2j{+}1)^2 + \lambda\,\frac{k}{k_0}\,(2j\,{+}1) - \gamma^2\,]\, y + \lambda\,\frac{k}{k_0} = 0\,. \tag{49}$$

For a given wavenumber k there is only one real root y which fulfils condition (38) if $0 < k < k_{max}(j)$. The case $j = 0$ is simple and exemplatory. Here we obtain:

$$y^3 - 2(\gamma-1)\,y^2 + \left[(\gamma-1)^2 + \lambda\frac{k}{k_0} - \gamma^2\right] y + \lambda\frac{k}{k_0} = 0\,, \quad (50)$$

where we have used $\lambda = (\gamma - 1)/\gamma$.

Separating the solution $y = -1$, this equation reduces to the quadratic equation

$$y^2 - y\,(2\gamma - 1) + \lambda\,\frac{k}{k_0} = 0\,, \quad (51)$$

which can also be obtained from Eq. (41) for $1-\gamma+\gamma\,\lambda = 0$. The solution fulfilling the convergence condition (38) with $\mu > 0$ is

$$\omega^2 = \frac{g\,k}{2}\,[2\gamma - 1 + \sqrt{(2\gamma-1)^2 - 16(\gamma-1)\gamma\frac{k\,a_0^2}{g}}\;] \quad (52)$$

for $0 \le k \le k_{max}$. Here, k_{max} is the value where the function $\omega(k)$ reaches its maximum. Details are given in Appendix A.

9. Examples of dispersion curves

In the following we consider two cases: Convective equilibrium of the layer in the limit $z \to -\infty$ with $n = 3/2$ and $\gamma = 5/3$, and a general case with $n = 3$, i.e. $1 + 1/n = 4/3$ and $\gamma = 5/3$. As the dispersion relation is linear with respect to k/k_0, a simple procedure to display a mode is to consider k/k_0 as a function of y. We however have calculated the roots of both the cubic equation and the quartic equation, to obtain immediately $\omega(k)$. This procedure is more appropriate with regard to further investigations, for example the question of the existence of complex frequencies. However, complex frequencies are not subjects of the present paper. As the procedure of calculating the roots of a cubic and a quartic equation is elementary, we do not present the corresponding formulas. By taking the square of the condition (35), additional branches $\omega(k)$ are obtained which fulfil the relation $\frac{1}{2} - \mu - \kappa = -j$, instead of the relation $\frac{1}{2} + \mu - \kappa = -j$. These solutions which represent downward directed waves, must be excluded. Only those branches $\omega(k)$ which fulfil the relation $\frac{1}{2} + \mu - \kappa = -j$ with $\mu > 0$ represent real modes. We take $c_0 = 7\ \mathrm{km\,s^{-1}}$ and $g = 274\ \mathrm{m/s^2}$. The dispersion curves $\omega(k)$ are independent of the parameter ϵ. Fig. 1 displays the isentropic case $n = 3/2$. Figs. 2 and 3 show acoustic modes and gravity modes for the non-isentropic case $n = 3$. Both the p-modes and the g-modes tangent the corresponding boundary curves.

10. Resonances for real frequencies above the acoustic cut-off frequency

A real frequency ω lies above the acoustic cut-off frequency of the $k-\omega$ - diagram of the isothermal layer, when the exponent μ is imaginary. In this case, the second solution U of Whittaker's equation is appropriate. We have

$$\Delta p = e^{-x/2}\,x^{\nu+1/2}\,x^{i\,\alpha}\,U(\frac{1}{2} + i\,\alpha - \kappa, 1 + 2\,i\,\alpha, x)\,, \quad (53)$$

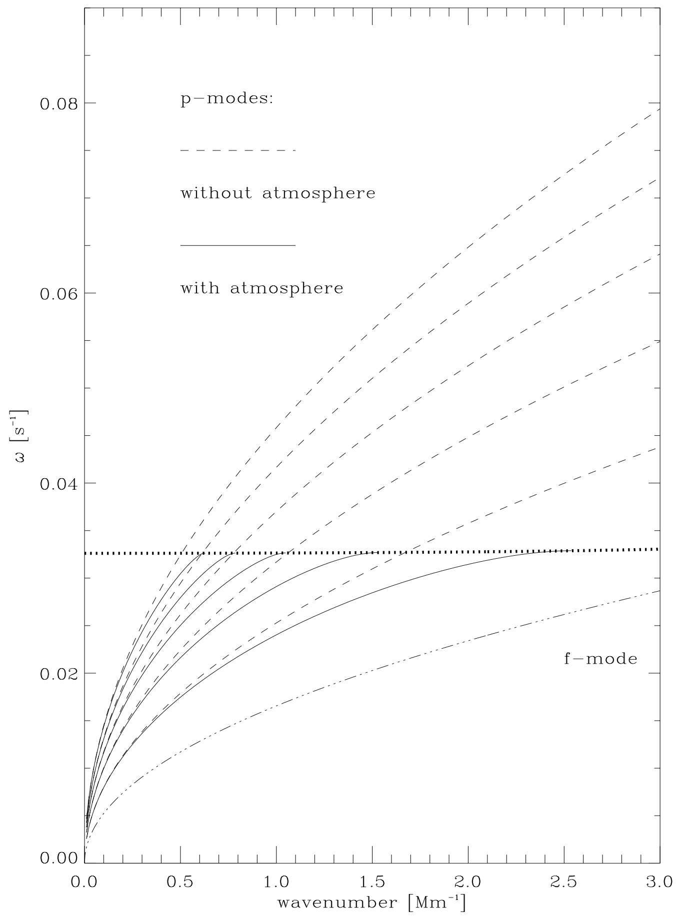

Fig. 1. p-mode dispersion curves for $n = 3/2$.

with

$$\alpha = \frac{\pm 1}{2\lambda}\sqrt{\frac{1}{4\,\gamma^3}\,\frac{k}{k_0}\,\frac{1}{y}\,[\,y^2 + (\gamma-1) - \frac{1}{4\,\gamma}\,\frac{k}{k_0}\,y\,] - 1}\,. \quad (54)$$

We have (Abramowitz & Stegun 1965):

$$U(a,b,x) = x^{-a} + \mathcal{O}(|x|^{-a-1}) \qquad x \to \infty\,. \quad (55)$$

Thus, for $x \to \infty$, with $a = 1/2 - \kappa + i\,\alpha$, we obtain:

$$\Delta p = e^{-x/2}\,x^{\nu+\kappa} \quad (56)$$

For real ω, the coefficient κ is real. The pressure perturbation Δp decays non-oscillatory for $x \to \infty$ or $m \to \infty$.

To study the asymptotic behavior of Δp for $z \to \infty$ or $x \to 0$, we need the following representation of U:

$$U(a,b,x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}\,M(a,b,x) + \frac{\Gamma(b-1)}{\Gamma(a)}\,x^{1-b}\,M(1+a-b, 2-b, x) \quad (57)$$

(Abramowitz & Stegun 1965). For $x \to 0$ we obtain

$$U(a,b,x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \frac{\Gamma(b-1)}{\Gamma(a)}\,x^{1-b} \quad (58)$$

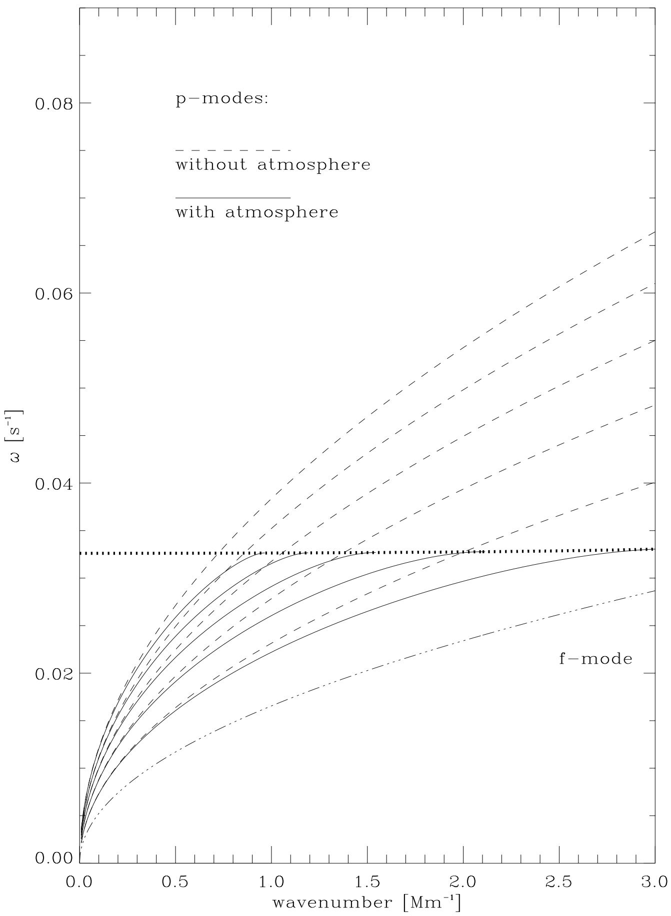

Fig. 2. p-mode dispersion curves for $n = 3$.

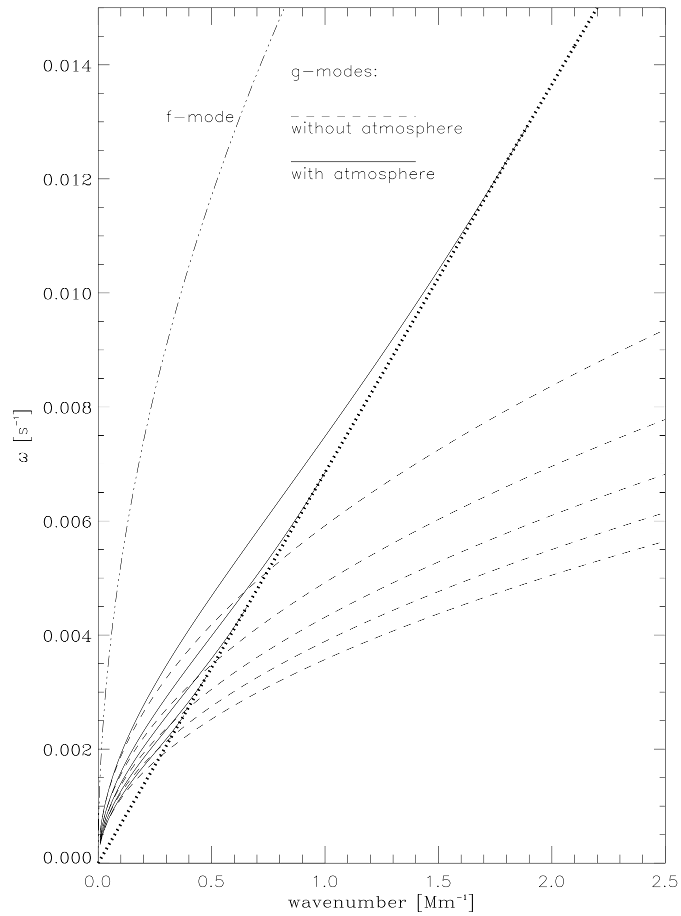

Fig. 3. g-mode dispersion curves for $n = 3$.

or:

$$U(a,b,x) = \frac{\Gamma(-2i\alpha)}{\Gamma(\frac{1}{2}-\kappa-i\,\alpha)} + \frac{\Gamma(+2i\alpha)}{\Gamma(\frac{1}{2}-\kappa+i\,\alpha)}\; x^{-2\,i\,\alpha}\,. \quad (59)$$

Thus, for $x \to 0$ we have:

$$\Delta p = x^{1/2\lambda}\left[\frac{\Gamma(-2i\alpha)}{\Gamma(\frac{1}{2}-\kappa-i\alpha)}\,x^{i\,\alpha} + \frac{\Gamma(+2i\alpha)}{\Gamma(\frac{1}{2}-\kappa+i\alpha)}\,x^{-\,i\,\alpha}\right]. \quad (60)$$

The amplitude of the pressure perturbation is

$$\Delta p_1 = 2\,x^{1/2\lambda}\left|\frac{\Gamma(2\,i\,\alpha)}{\Gamma(\frac{1}{2}-\kappa+i\,\alpha)}\right|\,. \quad (61)$$

By use the identity

$$|\Gamma(i\,y)|^2 = \frac{\pi}{y\,\sinh(\pi\,y)}\,, \quad (62)$$

we replace $|\Gamma(2\,i\,\alpha)|$. We finally obtain

$$\Delta p_1 = m^{1/2}\left[\frac{2k\epsilon}{\lambda g}\right]^{\frac{1}{2\lambda}}\sqrt{\frac{2\pi}{\alpha\sinh(2\pi\alpha)}}\;\frac{1}{\left|\Gamma(\frac{1}{2}-\kappa+i\alpha)\right|} \quad (63)$$

for $m \to 0$.

From Eq. (56) we obtain the asymptotic form of the pressure perturbation for $m \to \infty$:

$$\Delta p_2 = m^{\frac{1}{2}}\left[\frac{2k\epsilon}{\lambda g}\right]^{\frac{1}{2\lambda}+(\kappa-\frac{1}{2})} m^{\lambda(\kappa-\frac{1}{2})}\;\exp\left[-\frac{k\epsilon}{\lambda g}\,m\lambda\right]. \quad (64)$$

Therefore the ratio q of the amplitudes Δp_1 at $m_1 \to 0$ and Δp_2 at $m_2 \to \infty$ is given by:

$$q = \sqrt{\frac{2\pi}{\alpha\sinh(2\pi\alpha)}}\;\frac{1}{\left|\Gamma(\frac{1}{2}-\kappa+i\alpha)\right|}\left[\frac{\lambda g}{2k\epsilon}\right]^{\kappa-\frac{1}{2}}\,. \left[\frac{m_1}{m_2}\right]^{\frac{1}{2}} m_2^{\lambda(\frac{1}{2}-\kappa)}\;\exp\left[\frac{k\epsilon}{\lambda g}\,m_2^{\lambda}\right]. \quad (65)$$

Only the second factor of this expression can be oscillatory. The function $\left|\Gamma(\frac{1}{2}-\kappa+i\alpha)\right|$ oscillates for $\kappa > 1/2$ and $|\alpha| \ll 1$. We study only this factor. Fig. 4 displays $1\,/\,\left|\Gamma(\frac{1}{2}-\kappa+i\alpha)\right|$. Studies of the global behavior of the function U indicate that the asymptotic behavior of U for large m is a measure of the amplitudes of U in the whole polytropic layer. Therefore the quantity q points to the behavior of the ratio of the amplitudes in the atmosphere and in the convection zone. In the region of gravity waves there are only small amplitude modulations for $k < 1\ \mathrm{Mm}^{-1}$.

11. Vertical propagation of waves, k = 0

In the case of vertical propagation where $k = 0$, Eq. (18) reduces to

$$\xi^2\,\frac{d^2\,\eta}{d\xi^2} + \left[\frac{B}{\lambda^2}\,\xi + \left[\frac{4\,C+\lambda^2-1}{4\,\lambda^2}\right]\right]\eta = 0 \quad (66)$$

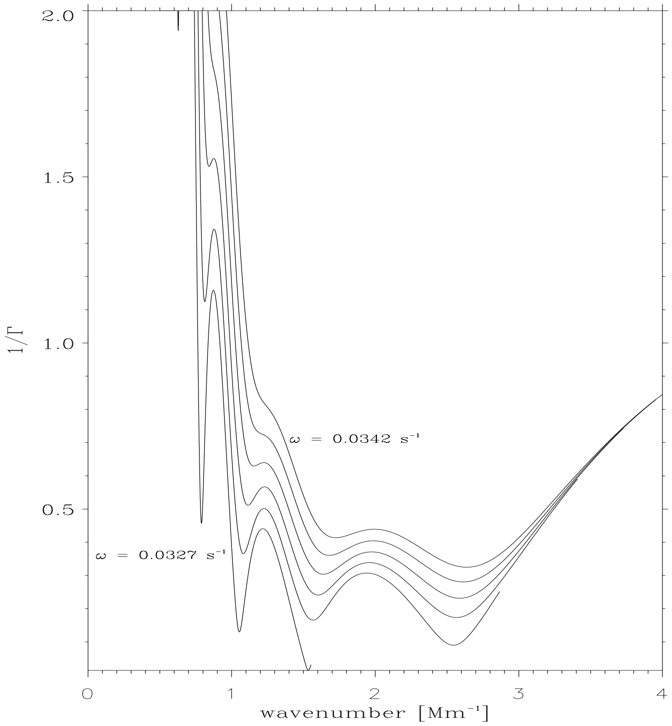

Fig. 4. Modulations in the acoustic region for $n = 3/2$. The cut off-frequency is $\omega = 0.0326\ s^{-1}$.

with

$$B = \frac{\epsilon\,\omega^2}{\gamma\,g^2} \quad \text{and} \quad C = \frac{a_0^2\,\omega^2}{\gamma\,g^2}\ .$$

The solution of this equation is (Abramowitz & Stegun 1965)

$$\eta = \sqrt{\xi}\ Z_\nu\Big(\frac{2\,\omega}{\lambda\,g}\sqrt{\frac{\epsilon}{\gamma}}\sqrt{\xi}\Big)\ , \tag{67}$$

with

$$\nu = \frac{1}{\lambda}\sqrt{1 - \frac{\omega^2}{\omega_0^2}} \quad \text{and} \quad \omega_0 = \frac{\gamma\,g}{2\,c_0}\ ,$$

where Z_ν is a Bessel function. For $\omega < \omega_0$, the solution fulfilling the condition $\Delta p = 0$ at $\xi = 0$ finally yields

$$\Delta p = m^{1/2}\ J_\nu\Big(\frac{2\,\omega}{\lambda\,g}\sqrt{\frac{\epsilon}{\gamma}}\,m^{\lambda/2}\Big)\ . \tag{68}$$

For $m \to 0$ or $z \to \infty$, the behavior of this standig wave approaches the evanescently decaying wave of the unbounded isothermal atmosphere. This is obtained from the limiting form $J_\nu(x) = (x/2)^\nu\,/\Gamma(1+\nu)$.

For $\omega > \omega_0$ where ν is imaginary, we have to use the functions J_ν and $J_{-\nu}$ or J_ν and Y_ν to construct real travelling waves.

12. Conclusions

We have presented a simple analytic model of a convection zone with an overlying isothermal atmosphere. As regards the representation of the dispersion relation $F(\omega, k) = 0$, this model has not the shortcommings of the familiar two layer model. For the new model, the three-dimensional adiabatic wave equation can be solved analytically by reduction to Whittaker's differential equation. In the general case with $\gamma \neq 1 + 1/n$, the dispersion relation is a polynomial of forth degree in ω^2. The dispersion relation of an isentropic convection zone with an isothermal atmosphere is a cubic equation for ω^2. So, the function $\omega(k)$ can be given in closed form. We have presented dispersion curves of acoustic and gravity modes with $\gamma \neq 1 + 1/n$ and of acoustic modes with $\gamma = 1 + 1/n$. As the stratification of the layer becomes asymptotically isentropic for $\gamma = 1 + 1/n$, no gravity waves exist in this case. We have studied the existence of resonances in the region of the continuous spectrum of acoustic waves. We find strong amplitude modulations only at frequencies slightly above the acoustic cut-off frequency. The model is appropriate for further investigations; for example, to study the problem of the generation of the observed ridges above the acoustic cut-off frequency. As in the case of the simple polytropic layer, waves are given in terms of Whittaker functions. The mathematical procedure is very similar to the procedure used in the case of the simple polytropic layer. We plan to investigate the excitation of modes and the existences of modes with complex frequencies.

Acknowledgements. We thank the referee for useful comments. S. Steffens gratefully acknowledges the financial support by the Deutsche Forschungsgemeinschaft under grant De 226/11-2.

Appendix A: selection of the physically acceptable solutions

In Sect. 5 we have presented two independent solutions of the wave equation. To select the physically acceptable pressure perturbations, we study the behavior of these solutions for $m \to 0$ or $z \to \infty$. The first solution Δp_1 behaves as

$$\Delta p_1 = c_1\ m^{1/2}\,m^{\lambda\mu} \quad \text{for} \quad m \to 0\ . \tag{A1}$$

To select the decaying evanescent wave we have to put $\mu > 0$. Now we study the behavior of the second solution Δp_2 for $m \to 0$. For real parameters a and b, from Eq. (58) we obtain:

$$U(a,b,x) = \frac{\Gamma(b-1)}{\Gamma(a)}\ x^{1-b} \quad \text{for} \quad b > 1\ , \quad x \to 0 \tag{A2}$$

and

$$U(a,b,x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \quad \text{for} \quad b < 1\ , \quad x \to 0\ . \tag{A3}$$

We have $b = 1 + 2\,\mu > 1$ for $\mu > 0$ and $b = 1 + 2\,\mu < 1$ for $\mu < 0$. With $x \propto m^\lambda$, the Lagrangian pressure perturbation

$$\Delta p_2 = c_2\ m^{1/2}\,m^{\lambda\mu}\ e^{-x/2}\ U(\frac{1}{2} + \mu - \kappa, 1 + 2\,\mu, x) \tag{A4}$$

behaves as

$$\Delta p_2 \sim c_2\ m^{1/2}\,m^{-\lambda\mu} \quad \text{for} \quad \mu > 0\ , \quad m \to 0 \tag{A5}$$

and

$$\Delta p_2 \sim c_2\ m^{1/2}\,m^{+\lambda\mu} \quad \text{for} \quad \mu < 0\ , \quad m \to 0\ . \tag{A6}$$

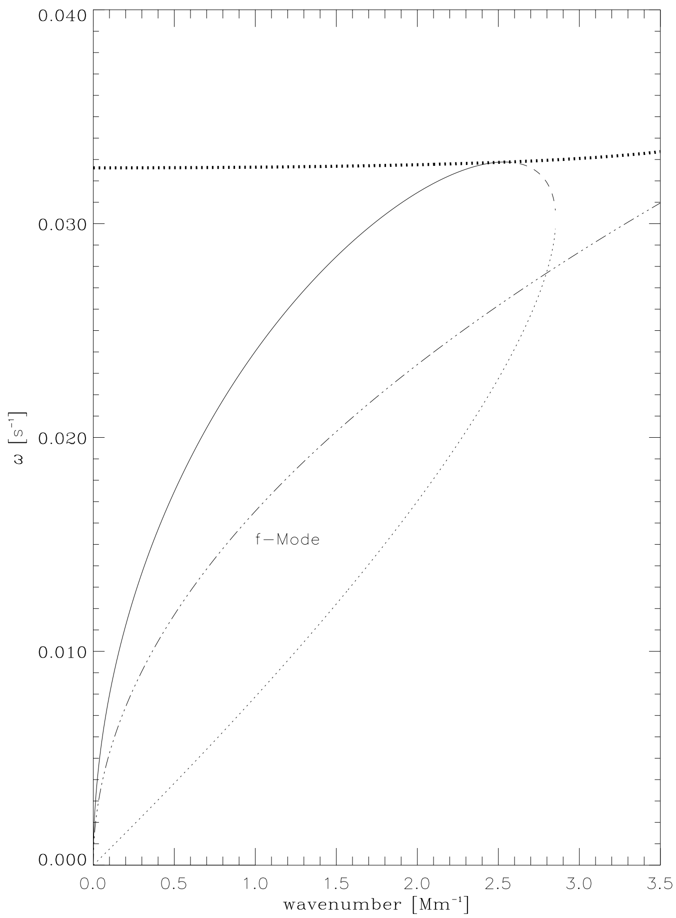

Fig. A1. The three branches of the $j = 0$ dispersion relation for $n = 3/2$.

Thus, for real μ the solution Δp_2 always represents a downgoing or reflected evanescent wave which must be excluded.

The dispersion relation (40) of the general case is a quartic equation, the dispersion relation (49) of the isentropic case a cubic equation. Excluding solutions $y(k)$ which do not fulfil Eq. (39) for $\mu > 0$ we obtain the results presented in Sect. 9.

As an example we take the dispersion relation (51) which leads to two solutions

$$\omega^2 = \frac{g\,k}{2}\,[2\gamma - 1 \pm \sqrt{(2\gamma-1)^2 - 16(\gamma-1)\gamma\frac{k\,a_0^2}{g}}\;] \,. \quad \text{(A7)}$$

Fig. A1. displays these solutions. The dotted curve represents the solution with the $-$ sign, the solid and the dashed curves the solution with the $+$ sign. The dashed curve and the dotted curve, however, are solutions with $\mu < 0$. Also the third solution $\omega^2 = -g\,k$ of the original cubic equation (50) fulfils the condition (39) only for $\mu < 0$. (The occurrence of this solution stems from the condition of the existence of the divergence-free mode, $\omega^4 = k^2\,g^2$, which has two roots: $\omega^2 = \pm g\,k$. The fact that the divergence-free solution is not governed by the wave equation of the Lagrangian pressure perturbation plays no role, as there is a second solution for $\omega^4 = k^2\,g^2$ with a non-vanishing Lagrangian pressure perturbation. Further, the dispersion relation is essentially invariant with respect to the dependent variable. So, the relation $\omega^2 = g\,k$ fulfils the general dispersion relation (40) when we put $j = -1$.) Therefore, only the solution represented by the solid line is physically acceptable, and it represents the p_0-mode.

References

Abramowitz M., Stegun I.A., 1965, Handbook of Mathematical functions. Dover Publ. Inc., New York
Balmforth N.J., Gough D.O., 1990, ApJ 362, 256
Goldreich P., Murray N., Kumar P., 1994, ApJ 424, 466
Kumar P., Duvall T.L. Jr., Harvey J.W., et al., 1990, In: Osaki Y., Shibahashi H. (eds.) Lectures Notes in Physics, Progress of Seismology of the Sun and Stars. Springer, Heidelberg, p. 87
Kumar P., Lu. E., 1991, ApJ 375, L35
Lamb H., 1932, Hydrodynamics. University Press, Cambridge
Libbrecht K., 1988, ApJ 334, 510
Murray N., 1993, In: Brown T.M. (ed.) Seismic Investigations of the Sun and Stars. ASP Conference Series Vol. 42, p. 3
Pekeris C.L., 1948, Phys. Rev. 73, 145
Price G., 1996, ApJ 458, 850
Rast M.P., Gough D.O., 1995, In: Ulrich R.K., Rhodes Jr. E.J., Däppen W. (eds.) Helio-and Astero-Seismology. ASP Conference Series Vol. 76, p. 322
Schmitz F., Fleck B., 1994, A&AS 106, 129