

A dynamical approach to a self-similar universe

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Abstract. We write a non-relativistic Lagrangian for a hierarchical universe. The equations of motion are solved numerically and the evolution of the fractal dimension is obtained for different initial conditions. We show that our model is homogeneous at the time of the last scattering, but evolves into a self-similar universe with a remarkably constant fractal dimension. We also show that the Hubble law is implied by this model and make an estimate for the age of the universe.

Key words: cosmology: large-scale structure of Universe – cosmology: theory

1. Introduction

This is a decisive time for cosmology since our theories for the large scale structure of the universe are being seriously challenged by the ever-growing amount of data.

The CfA1 redshift survey (de Lapparent et al. 1986, 1988) was the first to reveal structures such as filaments and voids on scales where a random distribution of matter was expected. The most remarkable feature of these structures is the so-called “great wall” which is a coherent sheet of galaxies extended over an area of at least 60×170 Mpc (Geller & Huchra 1989). Later on, deep pencil beam surveys (Broadhurst et al. 1990), the redshift surveys based on IRAS catalogue (Efstathiou et al. (1990a, 1990b), Saunders et al. 1991, Fisher et al. 1996), the deep wide angle survey SSRS (da Costa 1988, 1994) and some others have shown inhomogeneities at scales where the galaxy-galaxy and cluster-cluster correlations were believed to be negligible. Of particular significance for the future are the two extensive redshift surveys, SLOAN and 2dF, about to commence, which aim to trace the 3-dimensional distribution of over one million galaxies across the northern and southern skies.

The first quantitative study of the cosmic inhomogeneity lead to the well-known 1.8 power law behaviour of the galaxy-galaxy correlation function (Groth & Peebles 1977, Peebles 1980, Davis & Peebles (1983a, 1983b)). It is worth mentioning that concerning such intermediate scales there are some papers describing the measure of fractal two-dimensional structures. In particular they show also that double power laws are observed, and galaxies may be distributed along sheets on large scales and

filaments on smaller ones, see Calzetti et al. (1992), Murante et al. (1998). Although this law has been consistently identified in different catalogues, the break away from it at larger scales and a crossover to homogeneity has not yet been established (Davis 1996, Pietronero 1987, Coleman & Pietronero 1992, Pietronero et al. 1996). Whether there is a crossover to homogeneity or not, the power law nature of the two-point and higher-order correlation functions is itself suggestive of some kind of scaling behaviour at least in some range. The simplest structure that obeys such a scaling law is a single fractal.

A theoretical model describing a non-analytic inhomogeneous scale-invariant universe is non-existent. The most-extensively-studied inhomogeneous cosmological model is Tolman spacetime (Tolman 1934, Bondi 1947). Tolman’s dust solution has been used to model a hierarchical cosmology compatible with the observational analysis of the redshift surveys (Bonnor 1974, Ribeiro 1992a, 1992b, 1993). Recently, the Einstein equation for a scale-invariant spherically symmetric inhomogeneous, but isotropic, universe, which allows a non-vanishing pressure has been solved (Abdalla & Mohayaee 1997). However, the results obtained in this way are perturbative, assume a preferred center for the universe and violate the linearity of the Hubble law. A self-similar universe avoiding such difficulties can only be constructed for a non-analytic distribution of matter. It is rather a difficult task to construct a fractal metric and to solve Einstein equation for a self-similar universe. However, many cosmological phenomena can be accurately described by the Newtonian gravity, especially in the present matter-dominated era.

In this work, we construct a self-similar universe whose dynamics is governed by the Newtonian gravity. We divide the universe into k spherical clusters each of which contains k sub-clusters which in their turn contain k sub-subclusters. This clustering cascades down all the way to the level of the galaxies which are at the lowest rung on the clustering ladder. The mass and radius of each cluster can be used to define the fractal dimension of our model.

We write the kinetic and potential energies of each cluster in terms of its center of mass energy and the internal energies of its subclusters. The thermal energy is obtained by requiring the total entropy of the canonical ensemble of the clusters and their subclusters to remain constant. The final Lagrangian is

formulated in terms of two dynamical parameters: radius of the largest cluster and its ratio to the radius of its subclusters. The radius of the largest and smallest clusters, the ratio of the mass to the critical mass contained in a sphere of radius 20 Mpc, the number of subclusters in each cluster and the ratio that characterises the relative significance of thermal and gravitational energies are left as free parameters. By fixing these to different observational values, we are able to solve the equations of motion numerically using a Pascal program. From the solutions, we can trace the evolution of the fractal dimension and verify the linearity of the velocity-distance relationship over time scales comparable to the age of the universe.

The results are remarkable. We observe that for different initial conditions a nearly homogeneous universe with a fractal dimension close to 3 evolves into a universe with a fractal dimension of the order of 2 at the present time. This fractal dimension fluctuates slightly about the value of 2 over the future times but remains on the average constant. We also show that for insignificant thermal energies, the Hubble law is closely obeyed by our model.

We also make an estimate for the age of our self-similar universe. This is one of the most challenging problems of Friedmann cosmology since the observed age of the old stars in the globular clusters is far bigger than the value estimated for the age of the universe in the nearly flat standard model. The age of the universe obtained in our model is related to the radius at which the crossover to homogeneity occurs. The farther the crossover radius the older is the universe.

This article is organized as follows. In Sect. 2, we formulate our clustering model. In Sect. 3, we obtain the kinetic, the potential and the thermal energies, write the Lagrangian and the equations of motion. In Sect. 4, we solve these equations numerically and discuss the validity and limitations of our Newtonian approximation. In Sect. 5, we show different plots of the fractal dimension for different choices of the initial condition. Hubble law is discussed in Sect. 6. In Sect. 7, we study the evolution of the scale factor and obtain a value for the age of the universe in our model. Sect. 8 is devoted to the conclusion.

2. A single-fractal model

We consider a hierarchical model of the universe, such that a spherical cluster of radius R_n and mass M_n is composed of k uniformly distributed subclusters of uniform radius R_{n-1} and mass M_{n-1} . That is

$$M_n = kM_{n-1} \quad \text{and} \quad R_n = \alpha R_{n-1} \quad , \quad (1)$$

where α stands for the ratio of the radius of the n th cluster to the radius of the $(n-1)$ th cluster. The parameters k and α do not depend on n .

The smallest constituent of the hierarchy is a galaxy of mass M_1 ¹. We suppose that the universe contains N hierarchies, where N is a free parameter in this model (Fig. 1).

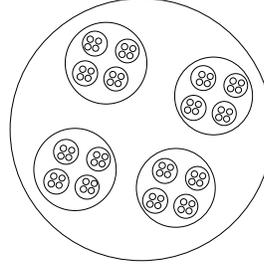


Fig. 1. A simple example is shown of a hierarchical model in which $k = 4$, $N = 4$ and $\alpha \approx 3$.

In terms of the mass M and radius R of the largest cluster, the expressions (1) can be rewritten as

$$R_n = \alpha^{n-N} R \quad , \quad M_n = k^{n-N} M \quad . \quad (2)$$

The fractal dimension D is given by (see e.g. Coleman & Pietronero 1992)

$$D = \frac{\ln k}{\ln \alpha} \quad . \quad (3)$$

We assume that the clusters neither dissociate nor collide during the evolution which leaves R , α and consequently the fractal dimension as the only dynamical quantities of this model.

3. The Lagrangian

In this section, we write down the Lagrangian of our model by computing the gravitational and thermal energies of the clusters.

The kinetic energy of a sphere of radius L expanding uniformly is

$$T = \frac{1}{2} \int_0^L \rho \dot{r}^2 dv = \frac{3}{10} M \dot{L}^2 \quad , \quad (4)$$

where we have used²

$$\dot{r} = \frac{\dot{L}}{L} r \quad . \quad (5)$$

For a composite system, such as ours, the total kinetic energy is the sum of the center of mass kinetic energy and the kinetic energies of the constituents of the system in the center of mass frame. Therefore, the kinetic energy of the $(n+1)$ th cluster in our model can be written as

$$T_{n+1} = \frac{3M_{n+1}\dot{R}_{n+1}^2}{10} + kT_n \quad , \quad (6)$$

Subsequently the total kinetic energy T is

$$T = \frac{3}{10} \sum_{n=2}^N k^{N-n} M_n \dot{R}_n^2 \approx \frac{3}{10} M R^2 \left(\frac{\dot{R}^2}{R^2} - \frac{2\dot{\alpha}\dot{R}}{\alpha^3 R} + \frac{\dot{\alpha}^2}{\alpha^4} \right) \quad (7)$$

where the approximation is made for α^2 much bigger than unity.³

² Note that the Hubble law is not assumed by this expression since \dot{L}/L is allowed to be different for different clusters.

³ This approximation is within the range given by the observational data. As an example, the radius of the local supercluster is about 10 times the radius of the local group. We shall use it here as a working hypothesis.

¹ Since galaxies do not expand, it is reasonable to assume that they do not contain subclusters.

The potential energy can be derived in a similar manner. The gravitational potential energy of a uniform sphere is $-3GM^2/5R$ which becomes $-3GM^2(1-1/k)/5R$ for a discrete mass distribution. In our case the potential energy of the $(n+1)$ th cluster is

$$U_{n+1} = kU_n - \frac{3GM_n^2 k(k-1)}{5R_{n+1}}. \quad (8)$$

We also note that

$$\begin{cases} D > 1 & \text{for } (\frac{\alpha}{k})^{N-1} \ll 1 \\ D < 1 & \text{for } (\frac{\alpha}{k})^{N-1} \gg 1 \end{cases} \quad (9)$$

The first range is compatible with the values of the fractal dimension given by the observations⁴ and can be used to approximate the potential energy to,

$$U \approx -\frac{3GM^2}{5\mu R(1-\alpha/k)}, \quad (10)$$

where $\mu = (1-k^{-1})^{-1}$.

As well as expanding, the clusters are also fluctuating randomly. This introduces a generalized thermal term into the Lagrangian. The canonical partition function for a system of \mathcal{N} particles interacting through gravitational potential is given by

$$\mathcal{Z} = \frac{1}{\mathcal{N}!} \int dq^{3\mathcal{N}} dp^{3\mathcal{N}} \exp(-\beta\mathcal{H}(p, q)), \quad (11)$$

where the Hamiltonian \mathcal{H} is

$$\mathcal{H} = \sum_{i=1}^{\mathcal{N}} \left(\frac{P_i^2}{2m} \right) + U\{q_i\} \quad (12)$$

In order to compute the integral over the potential term we observe that the volume available to one subcluster inside the $(n+1)$ th cluster is $\frac{4}{3}\pi(R_{n+1}^3 - (k-1)R_n^3)$. For k subclusters, there is an exponent of k for the complete available volume in the configuration space, thus

$$\mathcal{Z} \propto \beta^{-\frac{3\mathcal{N}}{2}} \prod_{n=2}^{\mathcal{N}} (R_n^3 - (k-1)R_{n-1}^3)^{(k^{N-n+1})} e^{-\beta U}. \quad (13)$$

To obtain the above expression we have factored out $e^{-\beta U}$ as a constant, where we have assumed that its dependence on q has already been accounted for by taking a homogeneous distribution of subclusters. Using the *standard equations of thermodynamics for the conserved entropy S :

$$S = \ln \mathcal{Z} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} \quad (14)$$

the thermal energy can be written as

$$\mathcal{E} \approx \frac{3Ma}{10} R^{-2\mu} (\alpha^3 - k + 1)^{-2\mu/3} \alpha^{3p} \quad (15)$$

for $k^{N-1} \gg 1$, implied by $\alpha^2 \gg 1$, $p = \frac{2}{3}(N\mu + \mu^2)$ and the value of the constant a depends on the initial value of the large-scale entropy.

Putting expressions (7), (10) and (15) together, the Lagrangian, is given by

$$\mathcal{L} = \frac{3M}{10} \left(\dot{R}^2 - \frac{2\dot{\alpha}\dot{R}R}{\alpha^3} + \frac{\dot{\alpha}^2 R^2}{\alpha^4} + \frac{g}{(k-\alpha)R} - aR^{-2\mu} (\alpha^3 - k + 1)^{-2\mu/3} \alpha^{3p} \right), \quad (16)$$

where $g = 2GMk/\mu$. The equation of motion for α becomes

$$\begin{aligned} \ddot{\alpha} &= 2\frac{\dot{\alpha}^2}{\alpha} + \alpha \left(\frac{\dot{R}}{R} \right)^2 - 2\frac{\dot{\alpha}\dot{R}}{R} + \frac{\alpha g(\alpha^3 - k)}{2(k-\alpha)^2 R^3} \\ &+ \left(aR^{-2\mu-2} \alpha^{3p+3} (\alpha^3 - k + 1)^{-\frac{2\mu}{3}-1} \right) \times \\ &\times \left(\mu\alpha^3 - \frac{3}{2}p(\alpha^3 - k + 1) \right) \end{aligned} \quad (17)$$

while for R we obtain

$$\begin{aligned} \ddot{R} &= \frac{\alpha^{-2}\dot{R}^2}{R} - 2\alpha^{-3}\dot{\alpha}\dot{R} + \frac{g(2\alpha - k)}{2(k-\alpha)^2 R^2} \\ &+ \left(aR^{-2\mu-1} (\alpha^3 - k + 1)^{-\frac{2\mu}{3}-1} \alpha^{3p} \right) \times \\ &\times \left(\mu\alpha^3 + \left(\mu - \frac{3p}{2} \right) (\alpha^3 - k + 1) \right). \end{aligned} \quad (18)$$

4. Numerical solutions and their limitations

In this section, we solve the equations of motion numerically. The parameters in our model are $R, \dot{R}, \alpha, \dot{\alpha}, N, k, a$ and M . The first four parameters are taken as initial conditions⁵ and the latter four as constants. These are all expressed in terms of observational quantities.

The crossover radius R marks the transition from a hierarchical distribution to a random homogeneous distribution and we take it to lie in the range 30–1000 Mpc. Subsequently, the present value of \dot{R} is fixed by the observed value of the Hubble constant H_0 which is about $55 \text{ km sec}^{-1} \text{ Mpc}^{-1}$ (Tammann 1997). We take this value as a reference, although one should keep in mind that the observational bounds indicate that using the above units $50 \leq H_0 \leq 100$. Furthermore, the vanishing of $\dot{\alpha}$ at near distances is implied by the Hubble law. However, whether this law remains valid or not over large distances is not determined by this assumption and shall be shown in Sect. 6.

The parameters α, N, k and M can be fixed by using the radius of the largest and smallest clusters (R and R_2 , respectively, where the latter is taken to be about 3 Mpc which is the size of our local group⁶), Ω which is defined to be the ratio of the mass contained in a sphere of radius 20 Mpc to the critical

⁵ We take the present time to be the zero time and evolve our model back and forth in the matter-dominated era.

⁶ Our results basically do not change for larger clusters such as Virgo cluster which has a radius of about 10Mpc.

⁴ The fractal dimension lies between 1.2 (Peebles 1980) and 2.0 (Pietronero et al. 1998).

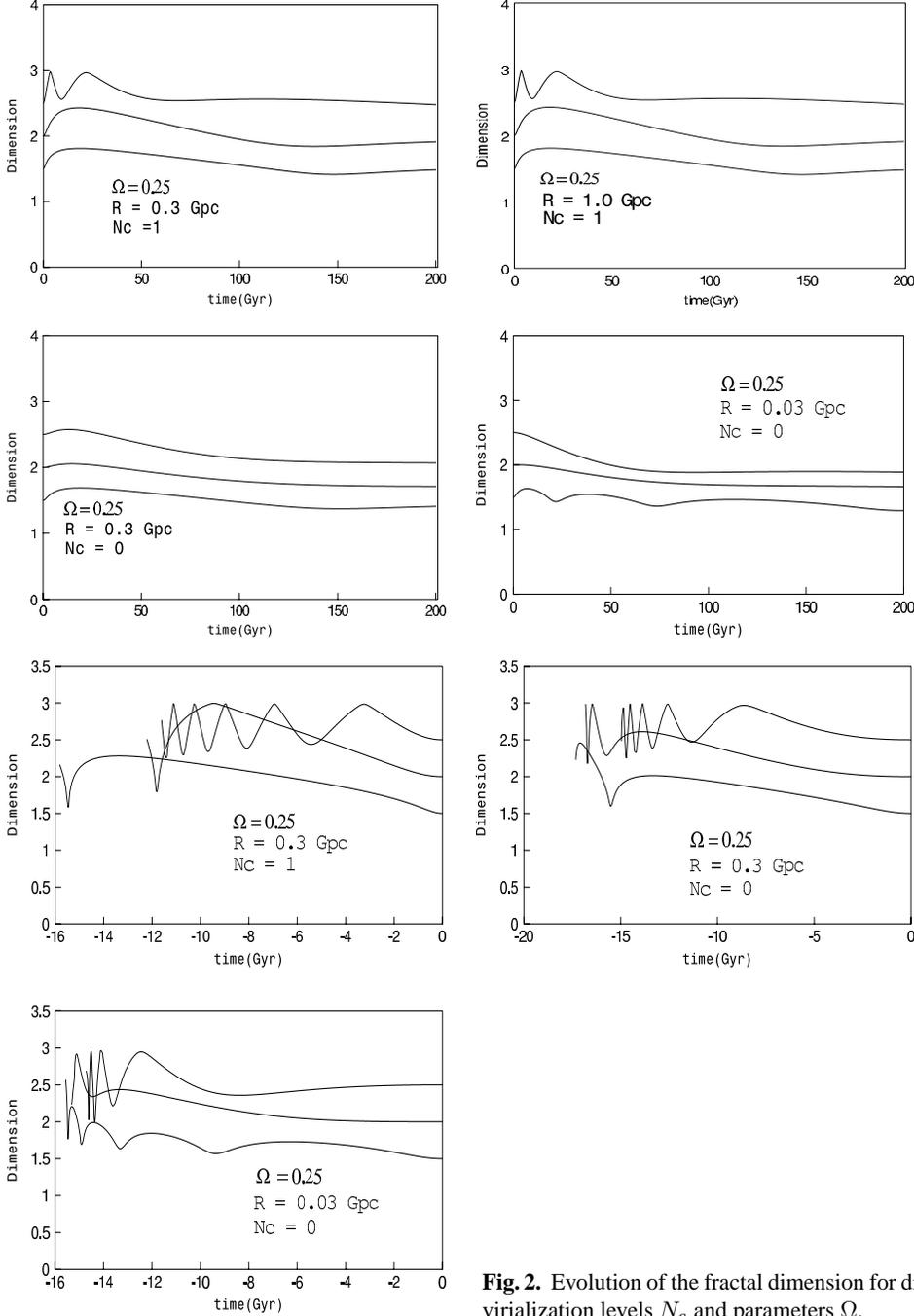


Fig. 2. Evolution of the fractal dimension for different initial dimensions, crossover radii R , virialization levels N_c and parameters Ω .

mass and is taken to lie in the range 0.25–1.0 (see e.g. Ostriker 1993), the typical mass of a galaxy which is about 10^{11} times the solar mass and the present value of the fractal dimension which is about 2 (Pietronero et al. 1998).

The parameter a can be fixed by using the hierarchical level where the virialization sets up, which is to say when $2\mathcal{E}_{N_c} = -U_{N_c}$. The virialization level N_c marks the relative weight of the gravitational potential energy and the thermal kinetic energy and replaces the parameter a in our computations. The center of mass thermal energy of each cluster is $3/2\beta = \mathcal{E}/N$. Hence, the center of mass thermal energy of subclusters contained in a

typical cluster turns out to be $\mathcal{E}_n = k^{2-N}\mathcal{E}$. Thus, the thermal energy tends to zero when the parameter N_c approaches minus infinity.

It is also worth commenting on the Newtonian and various other approximations that we have used in our model. In general the Newtonian approximation can be satisfactorily used in the matter-dominated era. However, in a self-similar model the range of such an approximation can be further constrained. That is to say as long as the time needed for light to cross the system is small compared to the characteristic time of the evolution of the dimension the Newtonian regime is applicable. Specifically,

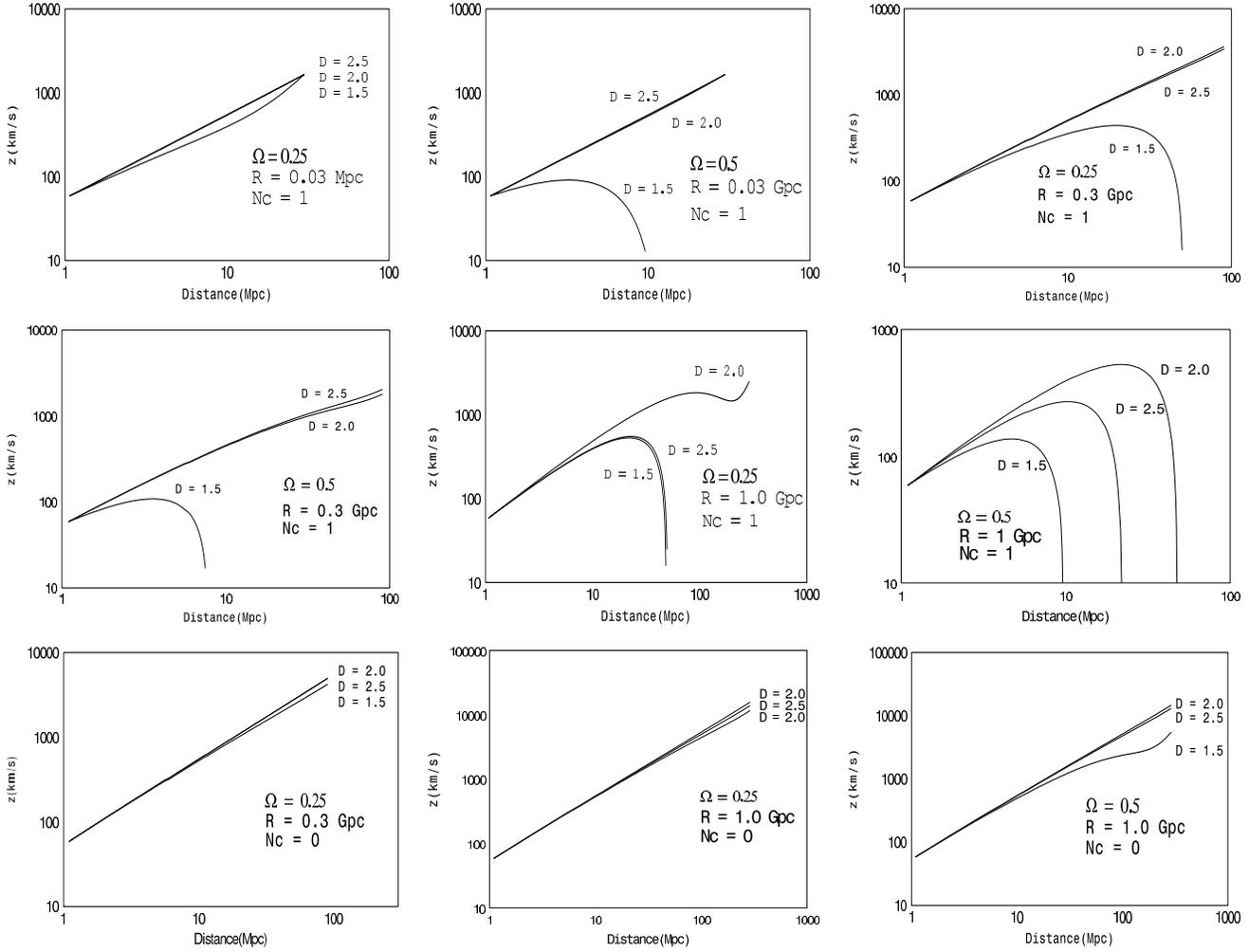


Fig. 3. The graphs of velocity versus distance

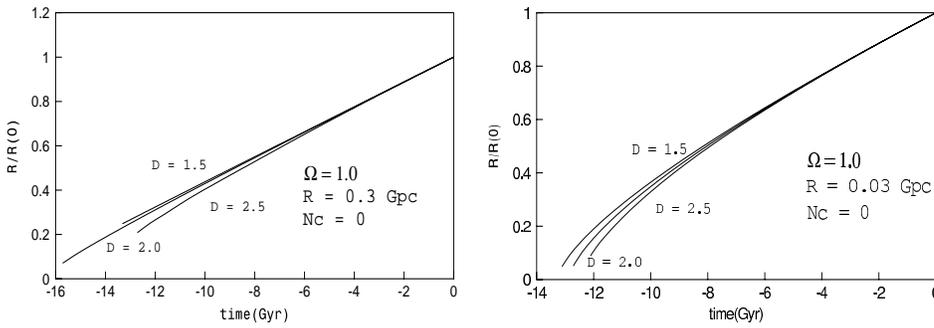


Fig. 4. The radius of the universe is plotted against time for different initial conditions.

for times later than about -15×10^9 years, or equivalently for $\frac{c\dot{\alpha}}{R\alpha} \ll 1$, the use of the non-relativistic approximation is justified. Another approximation has been the neglect of α^{-2} , which results in a maximum error of about 9% in our results for $D \approx 3$.

5. Evolution of the fractal dimension and CMBR isotropy

In this section, we study the evolution of the fractal dimension by using our numerical solutions for α , obtained in the last section. The graphs of fractal dimension versus time given at the end of

this article show the evolution of the fractal dimension, back in the matter-dominated era, until the time when the Newtonian approximation fails. All the graphs are plotted for three initial dimensions of $D = 1.5, 2.0$ and 2.5 .

From Fig. 2 we see that for shorter crossover radii the initial fluctuations die away faster and the fractal dimension settles at its almost constant value of 2 earlier.

The homogeneity at earlier times, due to the significantly large thermal energy, is remarkable since it conforms with the isotropy of the microwave background radiation. However the

full spectrum of CMB anisotropy, which has recently proved to be one of the most powerful methods to constrain the standard models of cosmology and structure formation (see e.g. Zaldarriga 1998) is beyond the capabilities of this Newtonian toy model.⁷

We also see that the transition from homogeneity to fractality is marked by oscillations. The frequency of these oscillations increases with increasing initial dimension and thermal energy and decreases with time because of the cooling caused by the adiabatic expansion of the universe. The reason why this oscillatory behaviour does not appear for the two lower initial dimensions is the breakdown of the Newtonian regime which occurs earlier for smaller initial dimensions (see Sect. 4).

A qualitative description of these oscillations can be made by studying the potential terms of the Lagrangian (16). The thermal term in the potential energy consists of a decreasing function of the parameter α , which diverges as we approach homogeneity, and a fast increasing function of α . Therefore, a minimum exists near $D = 3$ which is the origin of the observed oscillations. The gravitational potential term becomes important when D approaches unity. The negative sign of this term smooths down the sharp increase of the potential near $D = 1$ which accounts for the decrease of the frequency for small values of the initial dimension (*i.e.*, $D = 1.5$ and $D = 2.0$).

The solutions are quite stable for future times and the dimension remains at a mean value of about 2 over sufficiently long times and for a complete range of parameters. This confirms that at the present time the dimension is almost time-independent which was already assumed in solving the equations of motion. This means that we do not expect any change in the fractal structure in the future, according to this model.

6. The Hubble law

In the previous section, we have discussed the evolution of the fractal dimension and the isotropy of the microwave background radiation. This one is devoted to Hubble law.

In the preceding sections, we have assumed Hubble law to be valid at small distances. The uniform distribution of subclusters contained in each cluster implies the Hubble law for that cluster. However, this does not imply the Hubble law for a full range of distances. Using expressions (2) and (3) we obtain

$$\dot{R}_n = R_n \left(\frac{\dot{R}}{R} - \frac{\dot{D}}{D} \ln \left(\frac{R_n}{R} \right) \right) \quad (19)$$

for the velocity-distance relationship. Although the relationship is valid only for a discrete set of R_n 's, one can interpolate the velocity for the intermediate values since the observed velocity distribution is not expected to be discontinuous. In addition, because of the finiteness of the speed of light the value for \dot{D}/D

is calculated in the retarded time⁸ which is $t - R_n/c$ for a typical galaxy located approximately at the center of a cluster.⁹

Comparing graphs in Fig. 3, we see that decreasing crossover radius leads to a better fit for the Hubble law, which is compatible with the results of standard cosmology.

The presence of the second term in the last equation causes the \dot{R}_n to vanish for large $|\dot{D}/D|$. Increasing the thermal energy always results in a vanishing \dot{R}_n at a certain distance which increases with lowering Ω or increasing R (Fig. 3). The fact that the Hubble law is so well-fitted up to some hundred megaparsecs clearly rules out the earlier measurements of the dimension which was about 1.2 (Peebles 1980). In all of the graphs except for the case of $D = 2.0$ a linear behaviour is obtained which also confirms the recent observational results of $D = 2.0 \pm 0.2$ (Pietronero et al. 1998). The problem has been classically solved in the standard models by the introduction of so-called bias factor (Padmanabhan 1993) which is a phenomenological parameter to relate the galaxy distribution to the underlying mass. However in our model we do not have the assumption of biasing. One should also keep in mind that, our model being just monofractal, we are not considering refinements concerning a possible multifractal structure (see Balian & Schaeffer, 1989, and Colombi et al., 1992).

For relatively small thermal energies (*i.e.*, when $N_c = 0$) the linearity of the Hubble law is more robust although the graph for $D = 1.5$ shows deviations from the Hubble law at large distances. In general, smaller Ω and R give better fits for the Hubble law. This is because the behaviour of the universe in our model is mainly determined by the kinetic rather than the thermal term in the Lagrangian (16). Furthermore, in the absence of gravitational and thermal energies, the universe expands self-similarly with a constant dimension which, ignoring the time-retardation effect, leads to a perfect Hubble law.

7. The age of the universe

In the preceding sections, we have discussed the isotropy of the microwave background radiation and the Hubble law. The other crucial constraint, set by observations, on all viable cosmological models, is the age of the universe. The lower bound to the age of the universe, obtained from the age of the oldest stars in the globular clusters, lies far above the value estimated by the nearly flat standard model (Bolte & Hogan 1995). In this section, we make an estimate for the age of the universe in our model.

Any estimate of the age of the universe is highly dependent on the value of the parameter Ω which in our model depends on the crossover radius. A higher value of Ω leads to a younger universe and vice versa. In our model, Ω as defined in the standard model does not exist. However, we have assumed that Ω lies in

⁸ Being first order in velocity, the retardation effect is also considerable in a Newtonian model.

⁹ In fact, \dot{D}/D is only calculated for galaxies which are located approximately about the center of a cluster. For galaxies located near the edge of the clusters the retarded time is direction dependent and does not have a single value.

⁷ We can only extrapolate our model back to the time of decoupling since the Newtonian approximation cannot be used beyond approximately -15 Gyr.

the range 0.1–1 as given by the Friedmann model which is only valid for a crossover radius of about 20 Mpc. A larger crossover radius leads to a much lower value for Ω and subsequently to a much older universe.

Taking the extreme case of $\Omega = 1$, $N_c = 0$ and a crossover radius of 300 Mpc we obtain 15–17 Gyr for the age of the universe¹⁰. This result is in agreement with the maximum age estimated for the old stars which is 15.8 ± 2.1 Gyrs (Bolte & Hogan 1995). On the other hand, if we take the lower value of 30 Mpc for the crossover radius then we obtain 13 Gyrs for the age of the universe which lies below the astrophysical lower bound (see Fig. 4).

8. Conclusion

We have constructed a model for a nonrelativistic fractal universe. Our model starts off homogeneous and evolves rapidly to a self-similar universe with a remarkably constant fractal dimension. The homogeneity at the earlier times explains the isotropy of the microwave background radiation. We have also shown that the Hubble law is closely obeyed by our model for small thermal energies. We have estimated the age of the universe and have shown that it complies with the corresponding observational results. It remains an open problem to extend our model to multi-fractals and to the relativistic regime.

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¹⁰ Extrapolation of the first graph in Fig. 4 gives this value.