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Precise modeling of the precession-nutation of Mars

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Abstract. This paper aims to present the theoretical study of the precession and of the nutation of the planet Mars considered as a rigid body, in a rigorous way, by using canonical equations related to Hamiltonian theory, and by taking into account all the coefficients of nutation up to 0.1 mas. The equations are solved by taking into account the leading influence of the Sun, but also those due to Jupiter, to the Earth, and to the Martian satellites Phobos and Deimos. Opplolzer terms which make the separation from the axis of angular momentum to the figure axis as well as to the axis of rotation, are also determined, as well as semi-diurnal terms coming from the triaxial asymetry of the planet. Calculations and important remarks related to the accuracy of the determination of the variation of the obliquity at a long periodic time scale complete the results above.

Key words: celestial mechanics, stellar dynamics – planets and satellites: individual: Mars

1. Introduction

The accurate theoretical study of the rotation of the planets other than the Earth is particularly recent. One of the reason is that the main observational parameter related to this study is the period of rotation of each planet. Even in that point of view, concerning Mercury and Venus this period of rotation has been definitely known only in the 60's. The first observations which clearly showed the slow retrograde rotation of Venus have been done by Carpenter (1964), whereas the 2/3 resonance of Mercury is known only from 1965, starting from Doppler- spread measurements by Petengill & Dyce (1965), after the planet's spin was often believed, for a long time, to be exactly equal to its orbital period, as it is the case for the Moon. Because of the variety of details on the surface of Mars, its period of rotation was determined with good accuracy from a long time ago with the help of the observational data acquired from big telescopes. The complementary information concerning the rotation of Mars other than the spin rate, that is to say the motion of precession and nutation of its figure axis in space, is much more difficult to determine observationally.

The precession constant of Mars was calculated theoretically by Struve at the end of the 19th. century, but its observational determination at the present time cannot be done very accurately, as it was shown by Pitjeva (1996), who found a value of $750'' \pm 36''/cy$. Nevertheless the very recent results from the Mars Pathfinder mission lead to a much more accurate determination of the precession, that is to say -757.6 $\pm 3.5''/cy$ (Folkner et al., 1997) starting from spacecraft data.

Concerning the theoretical approach of the precession-nutation motion of Mars, an important progress was done by Borderies (1980) by considering only the leading torque due to the Sun. Hilton (1991) has shown in fact that the effects on the nutation related to the action of the two small but very close artian satellites Phobos and Deimos are not negligible in comparison with the main nutation term, with a roughly 1% relative order of amplitude. Notice that the evaluations of the nutations coming from the three main and recent theoretical studies (Borderies, 1980; Reasenberg & King, 1979; Hilton, 1991) give some difference at a relative 10^{-3}.

Various reasons lead us to calculate here with the best accuracy the coefficients of the precession and of the nutation of Mars: one is that the launch of several spacecraft missions on and around Mars are already achieved or in the way of a launch in the near future. No doubt that the accumulation of data coming from the tracking of the probes will require an improved analytical determination of the rotational parameters of the planet, thus improving the knowledge about its physical and dynamical characteristics.

For instance, the dynamical ellipticity H_d^{Ma} is well-known in the case of the Earth, for it is determined from an accurate observational value of the precession constant by the way of a straightforward relationship (Kinoshita, 1972; Souchay & Kinoshita, 1997). In fact H_d^{Ma}, in the case of Mars, was determined until now only from an hypothetic modeling of the internal structure of the planet. Consequently, the variations of this fundamental parameter according to the theoretical modelings (without taking into account Pathfinder's results), were about $\pm 5\%$. The determination of H_d^{Ma} which will be done in the following starting from the new value of the precession constant (Folkner et al., 1997), will be much more accurate.

Then we can expect that future probe explorations of the red planet will bring enough information to get better constraints concerning the value of parameters like $\frac{C}{MR^2}$ and H_d^{Ma}, following a more accurate value of the precession constant and an improved modeling of the planet's interior.

In a reciprocical way, it seems important in our analytical calculations carried out here to choose a level of truncation smaller than that already adopted in the precedent series of the nutation of Mars, in order to improve the modelisation of the motion of the figure axis in the space and to avoid truncation problems when using these series after taking into account more and more accurate data analysis.

For this aim, we have decided to apply a theoretical way of calculation of the rotation of the Earth, considered as a rigid body, to the planet Mars. The basic analytical principles related to this are taken from the work of Kinoshita (1977), improved by Kinoshita & Souchay (1990), and constructed starting on Hamiltonian canonical equations. In the case of the Earth, the most accurate observations (essentially based on the VLBI technique) show that the difference between the values of the main coefficients of the nutation in the case of a rigid body and a non-rigid one, are of the order of a relative 10^{-5}. There is no real reason to believe that for a telluric planet as Mars, the relative difference will be noticeably bigger, that is to say by one order or more, for the Martian characteristics (size, rotation, internal structure) are relatively close to that of the Earth.

Then it seems that the accuracy concerning the determination of the rotation of the planet, which should be obtained from missions around Mars in the very near future, will hardly enable to detect the influence of non rigidity.

One of the new topics included in this paper is the computation of the Oppolzer terms, which give the angular spatial offset between the axis of figure and the axis of angular momentum. Moreover we evaluate in the following the influence of the planets in addition to that of the satellites Phobos and Deimos. Notice that this influence can be ranged into two categories: the direct one, that is to say the influence related to the direct torque exerted by the planets on the equatorial martian bulge; and the indirect one, caused by the perturbation of the planets on the orbital motion of Mars around the Sun, and consequently on the gravitational potential exerted by the Sun on the planet.

The ephemerides used in order to compute the potential exerted by the external bodies are VSOP87 (Bretagnon & Francou, 1988) for the Sun and the other planets, and those calculated by Chapront-Touzé (1990) concerning Phobos and Deimos.

2. The materialization of the motion of rotation

The canonical variables that we choose here in order to solve the equations of motion of Mars by the way of the Hamiltonian equations are the Andoyer variables equivalent to those used in the case of the study of the motion of rotation of the Earth, as done by Kinoshita (1977). Refer to Fig. 1. The basic plane (P_M^t) is the mean orbit of Mars for the date t, which is sightly moving with respect to an inertial plane, which is the mean orbit of Mars (P_M^0) at the epoch J2000.0. Nevertheless, the basic point used in order to measure the motion of the precession and nutation in longitude of Mars is not here an equinox, which is classically chosen in the case of the Earth, but a point called the "depature point" D_t along (P_M^t). The choice of this point is justified by the fact that as considering the condition of non-rotation which characterizes it, and which is described in detail by Guinot (1979) and Capitaine et al. (1986), D_t is the natural point to measure any motion along the moving plane (P_M^t). This is particularly the case of the motion of the true martian equinox whose determination is one of the aims of our study together with the variation of the obliquity ε_A^M, which is defined as the angle between the plane of the true martian equator with respect to (P_M).

Let P be the node of the equator of figure with respect to the plane perpendicular to the angular momentum vector, and Γ the descending node of the plane perpendicular to the angular momentum vector with respect to (P_M^t) (in fact one of the two equinoxes of Mars when considering the equator perpendicular to the angular momentum, not the true equator).

Still refer to Fig. 1. The angle variables l, g, and h and the action variables L, G and H, are defined as in the following (Kinoshita, 1977):

.l is the angle between P and the principal axes of Mars corresponding to the minimum moment of inertia.

.g is the angle between Γ and P, along the plane perpendicular to the angular momentum.

.h is the angle along the mean orbit of Mars (P_M^t), between the departure point D_t and the node Γ.

.L is the component of the angular momentum on the axis along the axis of figure.

.G is the total angular momentum of Mars.

.H is the component of the angular momentum along the axis perpendicular to (P_M^t).

Notice that only h has a different meaning from that in Kinoshita (1977), for it is defined from D_t, not from an equinox. Nevertheless we choose the departure point D_0 at J2000.0 in coincidence with the equinox Γ_0 at this epoch. Moreover, by calling J the angle between the true equatorial plane of Mars (perpendicular to the fugure axis) and the plane perpendicular to the angular momentum, and $I \approx -\varepsilon_A^M$ the angle between this last plane and (P_M^t), we have the following relationships:

$$H = G \times \cos I \quad (1)$$

$$L = G \times \cos J \quad (2)$$

As the variables h and I enable to give the location of the plane perpendicular to the angular momentum vector, with respect to (P_M^t), in a similar way the variables quoted as h_f and I_f will enable to give the location of the equator of figure with respect to (P_M^t). Notice that h_f and I_f are two of the classical Euler angles used to represent the rotation of the Earth (Woolard, 1953) or any other planet, the third one being generally quoted as ϕ.

The link between the Andoyer variables l, g, and h, and the Euler angles h_f, I_f, is given by the following classical relationships, derived from the spherical triangle, neglecting the second order of J (Kinoshita, 1977):

$$h_f \approx h + \frac{J}{\sin I} \sin g \quad (3.1)$$

$$I_f \approx I + J \cos g \quad (3.2)$$

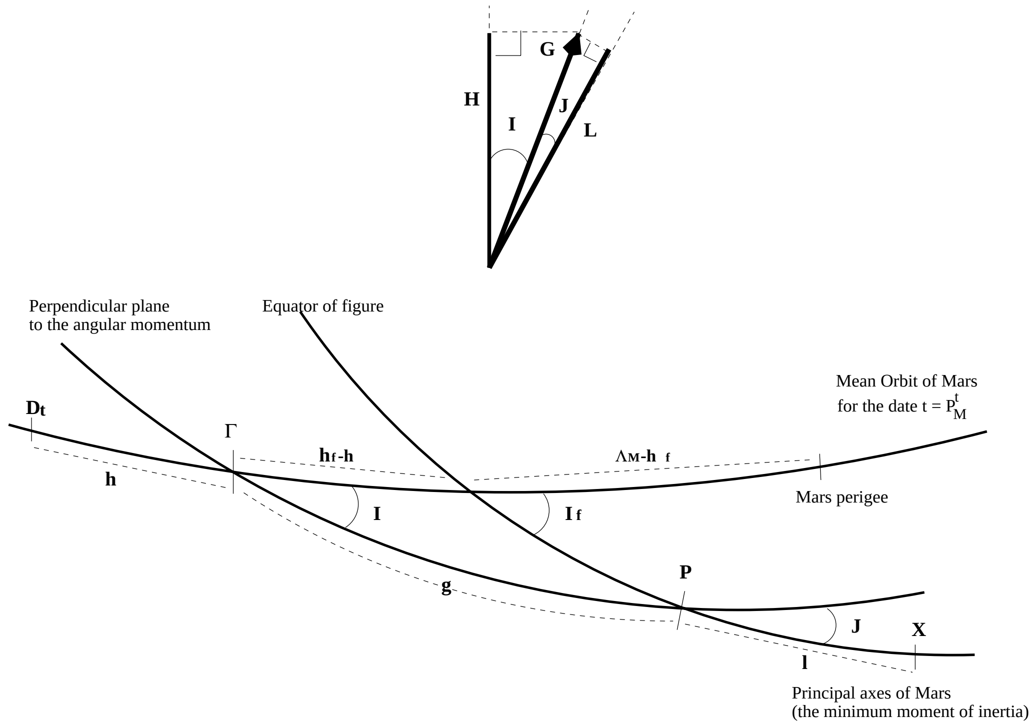

Fig. 1. Andoyer variables

$$\phi \;=\; l + g - J \cot I \sin g \tag{3.3}$$

Notice that the angles I and I_f characterize the obliquity, whereas h and h_f are the combination of the general precession in longitude and of the nutation in longitude, so that we can write:

$$h_f \;=\; -p_A^M - \Delta\psi \tag{4.1}$$
$$I_f \;=\; -\varepsilon_A^M - \Delta\varepsilon \tag{4.2}$$

p_A^M and ε_A^M being the notations corresponding respectively to p_A and ε_A for the Earth, as can be found in Lieske et al. (1977).

3. The Hamiltonian of the system and the equation of the motion of rotation

The way chosen for the parametrization of the problem being equivalent to that chosen by Kinoshita (1977), and by more recent studies (Kinoshita & Souchay 1990; Souchay & Kinoshita 1996; Souchay et al., 1998) of the rotation of the Earth, naturally the Hamiltonian related to the rotational motion of Mars can be written in a similar manner, by:

$$K = F + \sum_i U_i + E \tag{5}$$

where F is the Hamiltonian for the free motion, $\sum_i U_i$ represents the potential due to the forced motion, that is to say to the gravitational action exerted by the external bodies as the Sun, the satellites Phobos and Deimos, and the other planets, and E can be considered as a complementary term which is due to the fact that our reference plane, that is to say the plane of the mean orbital motion of Mars, is slightly moving with respect to an inertial reference system. F has the following form:

$$F = \frac{1}{2}\left[\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right] \times \left(G^2 - L^2\right) + \frac{1}{2C} \times L^2 \tag{6}$$

where A, B and C are the principal moments of inertia of Mars. For each of the external bodies represented by an index i, the disturbing potential is given at the first order by:

$$U_i \;=\; \frac{\kappa^2 M_i}{r_i^3} \times \left[\left(\frac{2C - A - B}{2}\right) P_2^0\left(\sin\delta_i\right) + \left(\frac{A-B}{4}\right) P_2^2\left(\sin\delta_i\right)\cos 2\alpha_i\right] \tag{7}$$

where M_i is the mass of the perturbing body and r_i is the distance between its barycenter and the barycenter of Mars. α_i and δ_i are respectively the marsocentric longitude and latitude of the perturbing bodies, with respect to the meridian given as an origin and which is crossed by the axis of minimum moment of inertia A, and to the mean martian equator of the date. $P_2^0(\sin\delta_i)$ and $P_2^2(\sin\delta_i)$ are the modified Legendre polynomials.

At last, the complementary component E is expressed as:

$$E \;=\; G \sin I \times \left[\sin\pi \cos(h - D_t\bar{M})\dot{\Pi} - \sin(h - D_t\bar{M}) \times \dot{\pi}\right] \tag{8}$$

Let us refer to Fig. 2. π is the angle between the two orbital planes (P_M^t) and (P_M^0), and Π is defined as the angle between the martian equinox of the epoch Γ_0 and the ascending node of (P_M^t) with respect to (P_M^0). π and Π are the martian corresponding variables to the classical variables π_A and Π_A adopted conventionally for the Earth (Lieske et al., 1977). Notice that for the sake of simplicity, we have decided in the precedent section

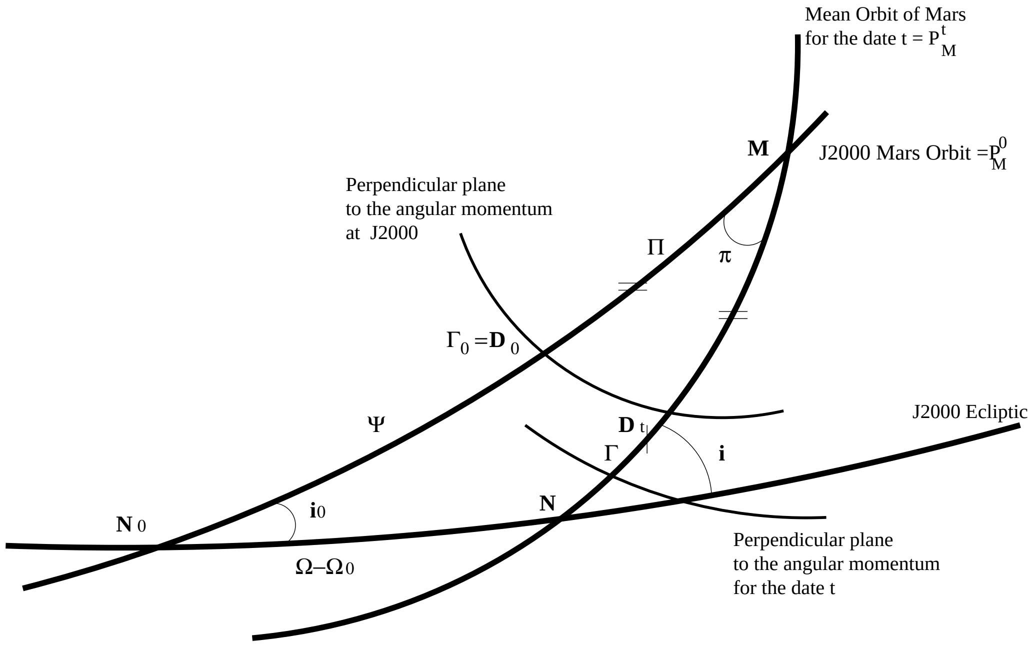

Fig. 2. Moving reference plane

that at the epoch J2000.0, the *a priori* arbitrary point D_0 corresponds to the martian equinox Γ_0 which is the ascending node of the martian orbit w.r.t. the mean equator at the epoch J2000.0. The numerical expressions for π and Π will be calculated and given in a next section.

The free motion of a planet as studied from the principle of the Hamiltonian has been abundantly studied by Kinoshita (1972), in the case of the Earth. It is rather complex and involves elliptic functions at the second order, but it has no important effect on the angles h and I. Moreover it is not studied here, for we are only concerned with the forced motion of rotation of Mars.

After the determination of U_i and E, the angles h and I are given by the means of canonical equations (Kinoshita, 1977):

$$\dot{h} \; = \; \frac{\partial K}{\partial H} = -\frac{1}{G\sin I}\frac{\partial K}{\partial I} \approx -\frac{1}{G\sin I}\left(\frac{\partial E}{\partial I}+\frac{\partial U_i}{\partial I}\right) \quad (9.1)$$

$$I \; = \; \frac{\partial K}{\partial h} = \frac{1}{G}\left(\frac{\partial K}{\partial h}\frac{1}{\sin I}-\frac{\partial K}{\partial g}\frac{1}{\tan I}\right) \quad (9.2)$$

As it is the case in the method developed by Kinoshita (1977), as well as in its improvements by Kinoshita & Souchay (1990) concerning the study of the rotation of the Earth, we separate the periodic part of the Hamiltonian from the secular one, in order to derive directly the quantities related to the nutation. This can be done by using an averaging algorithm close to Lie transformations (Hori, 1966). The coefficients of the nutation are then given in a straightforward manner, by the following formula:

$$\Delta\psi \; = \; -\Delta h = -\frac{\partial W_1}{\partial H}-\frac{\partial W_2}{\partial H}-\frac{1}{2}\left\{\frac{\partial W_1}{\partial H},W_1\right\} \quad (10.1)$$

$$\Delta\varepsilon \; = \; -\Delta I = -\left[\frac{1}{G\sin I}\right]\left(\frac{\partial W_1}{\partial h}+\frac{\partial W_2}{\partial h} -\frac{1}{2}\left\{\frac{\partial W_1}{\partial h},W_1\right\}\right)+\frac{1}{2}(\Delta I)^2\cot I \quad (10.2)$$

Notice that some of the terms in (10.1) and (10.2) are completely negligible w.r.t. our level of truncation of our coefficients, as the Poisson brackets and the last term at the r.h.s. of (10.2). In the case of the Earth (Souchay et al., 1998) they are relatively much larger, for they involve important effects between the Moon and the Earth, as croosed nutation effects and coupling effects between the orbital motion of the Moon and the Earth flattening. W_1 and W_2 are respectively the components of the determining function at the first and second order. The determining function at the first order $W_{i,1}$ is obtained by integrating the potential, at the first order $U_{i,1}$, exerted by each perturbing body, whose expression is given by (7):

$$W_{i,1} = \int U_{i,1}\times dt \quad (11)$$

As it was the case for the Earth (Kinoshita, 1977) the development of the disturbing function $U_{i,1}$ as given by (7) can be done in function of the coodinates of the perturbing body r_i, λ_i and β_i with respect to the moving orbital plane of Mars, and to the departure point D_t on this plane instead of the equinox. For that we use the same transformations of $P_2^0(\sin\delta)$ and $P_2^2(\sin\delta)$ as Kinoshita (1977), based on the Jacobi polynomials (Kinoshita et al., 1974), that is to say:

$$\begin{aligned}
&P_2^0(\sin\delta)\\
&=\frac{1}{2}(3\cos^2 J-1)\left[\frac{1}{2}(3\cos^2 I-1)P_2^0(\sin\beta)\right.\\
&\quad -\frac{1}{2}\sin(2I)P_2^1(\sin\beta)\sin(\lambda-h)\\
&\quad \left.-\frac{1}{4}\sin^2(I)P_2^2(\sin\beta)\cos 2(\lambda-h)\right]\\
&\quad +\sin 2J\left[-\frac{3}{4}\sin(2I)P_2(\sin\beta)\cos g-\sum_{\varepsilon=\pm1}\frac{1}{4}(1+\varepsilon\cos I)\right.\\
&\quad \times(-1+2\varepsilon\cos I)P_2^1(\sin\beta)\sin(\lambda-h-\varepsilon g)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\varepsilon=\pm1} \frac{1}{8} \sin I (1+\varepsilon\cos I) P_2^2(\sin\beta)\cos(2\lambda - 2h - \varepsilon g)\Big] \\
&+ \sin^2 J \Big[\frac{3}{4}\sin^2(I) P_2^0(\sin\beta)\cos 2g \\
&+ \frac{1}{4}\sum_{\varepsilon=\pm1} \varepsilon \sin(I)(1+\varepsilon\cos I)P_2^1(\sin\beta)\sin(\lambda - h - 2\varepsilon g) \\
&- \frac{1}{16}\sum_{\varepsilon=\pm1}(1+\varepsilon\cos I)P_2^2(\sin\beta)\cos 2(\lambda - h - \varepsilon g)\Big] \qquad (12)
\end{aligned}
$$

In the same way, we get:

$$
\begin{aligned}
&P_2^2(\sin\delta)\cos 2\alpha \\
&= 3\sin^2 J\Big[-\frac{1}{2}(3\cos^2 I - 1)P_2^0(\sin\beta)\cos 2l \\
&+ \frac{1}{4}\sum_{\varepsilon=\pm1}\sin 2I P_2^1(\sin\beta)\sin(\lambda - h - 2\varepsilon l) \\
&+ \frac{1}{8}\sin^2 I P_2^2(\sin\beta)\cos 2(\lambda - h - \varepsilon l)\Big] \\
&+ \sum_{\rho=\pm1}\rho\sin J(1+\rho\cos J)\Big[-\frac{3}{2}\sin(2I) \\
&\times P_2^0(\sin\beta)\cos(2\rho l + g) \\
&- \sum_{\varepsilon=\pm1}\frac{1}{2}(1+\cos I)(-1+2\varepsilon\cos I)P_2^1(\sin\beta) \\
&\times\sin(\lambda - h - 2\rho\varepsilon l - \varepsilon g) \\
&- \sum_{\varepsilon=\pm1}\frac{1}{4}\varepsilon\sin I(1+\varepsilon\cos I)P_2^2(\sin\beta) \\
&\times\cos(2\lambda - 2h - 2\rho\varepsilon l - \varepsilon g)\Big] \\
&+ \sum_{\rho=\pm1}\frac{1}{4}(1+\rho\cos J)^2\Big[-3\sin^2(I)P_2(\sin\beta) \\
&\times\cos(2l + 2\rho g) - \sum_{\varepsilon=\pm1}\varepsilon\sin I(1+\varepsilon\cos I) \\
&\times P_2^1(\sin\beta)\sin(\lambda - h - 2\rho\varepsilon l - 2\varepsilon g) \\
&+ \sum_{\varepsilon=\pm1}\frac{1}{4}(1+\varepsilon\cos I)^2 P_2^2\cos 2(\lambda - h - \rho\varepsilon l - \varepsilon g)\Big] \qquad (13)
\end{aligned}
$$

Notice that the second component in the parenthesis in the r.h.s. (7) involves the potential at the second order $U_{i,2}$; it is not considered in the present section, and gives birth to the terms of nutation coming from the triaxial asymmetry of the planet, which will be treated in the Sect. 4.6.

The three Legendre polynomials $P_2^0(\sin\beta)$, $P_2^1(\sin\beta)$ and $P_2^0(\sin\beta)$ present in Eq. (12) can be written as follows:

$$P_2^0(\sin\beta) \;=\; \frac{1}{2}\times(-1+3\sin^2\beta) \qquad (14.1)$$

$$P_2^1(\sin\beta) \;=\; 3\sin\beta\cos\beta \qquad (14.2)$$

$$P_2^2(\sin\beta) \;=\; 3\sin\beta\cos\beta \qquad (14.3)$$

Moreover the expression $\frac{\kappa^2 M_i}{r_i^3}$ in (7) can be advantageously replaced by $\frac{\kappa^2 M_i}{a_i^3}\times\frac{a_i^3}{r_i^3}$ where a_i is the semi-major axis of the Mars motion given by the following relationship, related to the keplerian motion:

$$n^2 a_i^3 = \kappa^2(M_S + M_M) \qquad (15)$$

where κ^2 is the constant of the gravitation.

Then the first part of the expression $\frac{\kappa^2 M_i}{a_i^3}$ is a constant term. In the opposite, the second part $\frac{a_i^3}{r_i^3}$ can be developed as a function of Mars excentricity and of its mean longitude, when the perturbing body is the Sun. In the other cases (Phobos, Deimos and planets), this development contains also the excentricity and mean longitude of the perturbing body.

Contrary to the Earth for which the leading influence on the nutation is coming from its satellite, the Moon for which the amplitude is roughly 20 times that of the Sun, the nutation of the planet Mars is largely dominated by the gravitational action of the Sun, the influence of Phobos and Deimos being of the order of 1/100 in comparison, and the influence of the planets being still significantly smaller.

4. Results of the various contributions to the martian nutation

4.1. *The main terms of nutation of Mars due to the Sun*

The coefficients of the nutation of Mars due to the Sun are computed in the following manner: the solar potential at the first order U_S^1 as expressed by the first part of the formula at the r.h.s. of (7) is calculated by the way of the transformations given by (12). The analytical expressions for the coordinates λ_S, β_S, and r_S are taken from the ephemerides VSOP87 (Bretagnon & Francou, 1988). More precisely the rectangular coordinates of Mars w.r.t. the Sun extracted from these ephemerides have been converted into the geocentric spherical solar ecliptic coordinates above. It must be noticed that β_S which represents the latitude of the Sun with respect to the mean ecliptic of the date, is in fact non equal to 0, for we must take into consideration the small oscillations of the orbital plane of the Sun (that is to say the plane of the true ecliptic). Even if the terms of nutation coming from this contribution have a very small amplitude they should be taken into consideration, as was demonstrated by Souchay & Kinoshita (1997) in the case of the Earth. Moreover the nutation coefficients coming from the second component of the potential at the r.h.s. of (7) are not considered here, for they can be neglected for two reasons: at first $A-B$ is very small in comparison with $2C-A-B$, and at second (this is the main explanation) they have a period close to half a martian day, which means that the corresponding component in W_S, that is to say after integration in (11), becomes very small itself because of the high frequency at the denominator. Nevertheless, we will give a rough estimation of the largest semi-diurnal term related to this component, in Sect. 4.6.

Table 1.1. The solar influence on the nutation of Mars, longitude $\Delta\psi$

sin (")	cos (")	Period (year)	M	Λ_M	Borderies Values sin (")	Hilton values sin (")
−1.09689	0.00006	0.940	2	2	−1.0431	−1.0962
0.63460		1.881	1		0.6031	0.6357
−0.23971	0.00001	0.627	3	2	−0.2278	−0.2401
0.10463	−0.00001	1.881	1	2	0.0994	0.1047
−0.04076		0.470	4	2	−0.0387	−0.0409
0.04432		0.940	2		0.0421	0.0445
−0.00630		0.376	5	2	−0.0061	−0.0063
0.00405		0.627	3		0.0038	0.0041
−0.00093		0.313	6	2		−0.0009
0.00041		0.470	4			
−0.00013		0.269	7	2		

Table 1.2. The solar influence on the nutation of Mars, obliquity $\Delta\varepsilon$

sin (")	cos (")	Period (year)	M	Λ_M	Borderies Values cos (")	Hilton values cos (")
0.00003	0.51589	0.940	2	2	0.4908	0.5158
0.00001	0.11274	0.627	3	2	0.1072	0.1130
	−0.04921	1.881	1	2	−0.0468	−0.0493
	0.01917	0.470	4	2	0.0182	0.0193
	0.00296	0.376	5	2	0.0029	0.0030
	0.00044	0.313	6	2		

Table 1.3. The solar influence on the nutation of Mars, mixed terms, longitude t x $\Delta\psi$

sin (")	cos (")	Period (year)	α (t^α) 1000 years	λ_{Ea}	M	λ_{Ju}	λ_{Sa}	Λ
	7590.39671		1					
0.00627	−0.04917	1.881	1		1			
−0.00223	0.01860	0.627	1		3			2
0.00959	0.00550	0.940	1		2			1
0.00101	0.00809	1.881	1		1			2
−0.00077	0.00632	0.470	1		4			2
0.00086	−0.00687	0.940	1		2			
−0.00377	0.00105	1.881	1		1			1
0.00287	0.00179	−1783.395	1	4	−8	3		
0.00127	0.00297	1.881	1		1			1
0.00212	0.00119	0.627	1		3			1
0.00163	−0.00160	−883.270	1			2	−5	
−0.00018	0.00146	0.376	1		5			2

In Tables 1.1 and Table 1.2, we show respectively in longitude and in obliquity the coefficients deduced from the Eqs. (10.1) and (10.2) and characterizing the nutation of Mars due to the Sun, together with the corresponding argument and period. Borderies (1980) and Hilton (1991) calculated the same coefficients by different approachs. Borderies used an Hamiltonian method with orbit elements given by Struve and Hilton an Eulerian method by using the ephemerides VSOP82 (Bretagnon, 1982). Their values are given in the two last columns of Table 1.1 and 1.2. Our values are very close to Hilton's ones. The difference with Borderies results are more important. But Borderies took $H_d^{Ma} = 5.103 \times 10^{-3}$ while we took $H_d^{Ma} = 5.363 \times 10^{-3}$. The coefficients are proportional to the value of H_d^{Ma}. The difference between our H_d^{Ma} and Borderies one explains a large part of the differences between the amplitudes of the nutation coefficients.

Moreover, in Tables 1.3 and 1.4, we show the mixed secular terms in the form $t \times \cos$ or $t \times \sin$ which result from our computations. They result themselves from the presence of mixed secular terms in the expression of the coordinates λ_S, β_S and

Table 1.4. The solar influence on the nutation of Mars, mixed terms, obliquity t x $\Delta\varepsilon$

sin (")	cos (")	Period (year)	α (t^α) 1000 years	M	λ_{Ju}	λ_{Sa}	Λ_M
0.00875	0.00105	0.627	1	3			2
−0.00332	0.00579	0.940	1	2			1
0.00381	−0.00048	1.881	1	1			2
0.00297	0.00036	0.470	1	4			2
	0.00242		1				
−0.00063	−0.00227	1.881	1	1			1
0.00180	−0.00077	1.881	1	1			−1
−0.00072	0.00128	0.627	1	3			1
0.00019	0.00104	−877.785	1		2	−5	−1

r_S. We are taking into account all the coefficients with absolute amplitude up to 0.1 milliarcsecond (mas), in combined absolute amplitude (sine and cosine). Notice that, as it can be expected by similarity to the case of the Earth, the leading component of nutation, with amplitude $1.09512''$ and $0.51532''$ respectively in longitude and in obliquity, has a period corresponding to half the period of revolution of the planet around the Sun, that is to say 343.49 d, the relative argument being $2M+2\Lambda_M$, where M is the mean anomaly of Mars and Λ_M corresponds to the angle between the point D_t and the Mars perigee, along the moving orbit of Mars (see Fig. 1). Notice that Λ_M must not be confused with the longitude of the perigee, for it is determined along the mean martian orbit and the martian equinox, and not the ecliptic and the equinox of the Earth.

The second term in decreasing order of amplitude has no component in obliquity, but its amplitude in longitude is $-0.63358''$ and its period is 687.0 d, for the corresponding argument is the mean anomaly of Mars. At total 11 coefficients are present for $\Delta\psi$ and 6 coefficients for $\Delta\varepsilon$, above our level of truncation, that is to say 0.1 mas. The value of H_d^{Ma} which serves to the determination of the scaling factor:

$$K' = 3\frac{\kappa^2 M_S}{a_S^3\omega} \times H_d^{Ma}$$

from which the values of the coefficients of nutation are depending directly, is dicussed in Sect. 6.

4.2. *The indirect planetary effect on the nutation of Mars*

As it is the case for the nutation of the Earth (Kinoshita & Souchay, 1990; Souchay & Kinoshita, 1996) the solar potential which gives birth to the nutation of Mars is influenced by the planets by the intermediary of their perturbation on the orbital motion of Mars, and consequently on the relative motion of the Sun with respect to Mars. Therefore, some terms of nutation involving the mean longitude of planets other than Mars appear when the truncation threshold in the series of nutation is small enough. These terms are sometimes called the "indirect planetary effect", although this terminology does not seem really adequate, as was dicussed by Souchay et al. (1998).

Table 2.1. The indirect planetary effects on the nutation of Mars, longitude $\Delta\psi$

sin (")	cos (")	Period (year)	λ_{Ve}	λ_{Ea}	M	λ_{Ju}	λ_{Sa}	Λ_M
−0.00039	0.00019	2.235			1	−1		
0.00028	−0.00035	2.754			1	−2		
0.00030	0.00007	−0.940		4	−10	3		2
0.00014	−0.00027	0.941		4	−6	3		2
0.00015	−0.00017	1.118			2	−2		
−0.00004	−0.00024	−877.785				2	−5	1
0.00028	0.00003	−15.781		1	−2			
−0.00019	−0.00003	11.862				1		
0.00014	−0.00007	0.662			3	−1		2
−0.00008	0.00009	0.511			4	−2		2
−0.00008	0.00010	0.701			3	−2		2
0.00011	0.00004	2.135		1	−1			
0.00009	0.00008	2.470		2	−3			
−0.00002	0.00011	32.835	1		−3			2
0.00005	−0.00009	0.627		4	−5	3		2

Table 2.2. The indirect planetary effects on the nutation of Mars, obliquity $\Delta\varepsilon$

sin (")	cos (")	Period (year)	λ_{Ea}	M	λ_{Ju}	Λ_M
−0.00003	0.00014	−0.940	4	−10	3	2
−0.00013	−0.00007	0.941	4	−6	3	2

Their argument is a linear combination of the mean longitude of Mars itself and of the other planets. These terms appear simultaneously with the main terms of nutation due to the Sun, for the planetary perturbations of the relative ecliptic coordinates α_S, β_S and λ_S of the Sun in (7) and (12) are included as Fourier series involving the mean longitudes of the other planets, together with terms characterizing a keplerian motion.

Notice that here the most influent planets are Jupiter and the Earth, whereas for the nutation of the Earth, these are Venus and Jupiter. The leading terms of the indirect planetary effects are gathered in Tables 2.1 and 2.2 respectively in longitude and in obliquity.

4.3. *The nutation of Mars due to Phobos and Deimos*

Phobos and Deimos, the two satellites of Mars, are bodies of very small dimensions, with mean radius respectively about 11.1 km and 6.2 km. But they are very closed to the planet, for their mean distance corresponds respectively to 2.76 and 6.91 Mars radii. In order to compute the nutation due to these two bodies, we have used the series ESAPHO and ESADE (Chapront-Touzé,1990). The means orbital elements of the two satellites are the following ones, according to this last paper:

$$a_{Ph} = 9373.713\,km$$
$$e_{Ph} = 0.015146$$

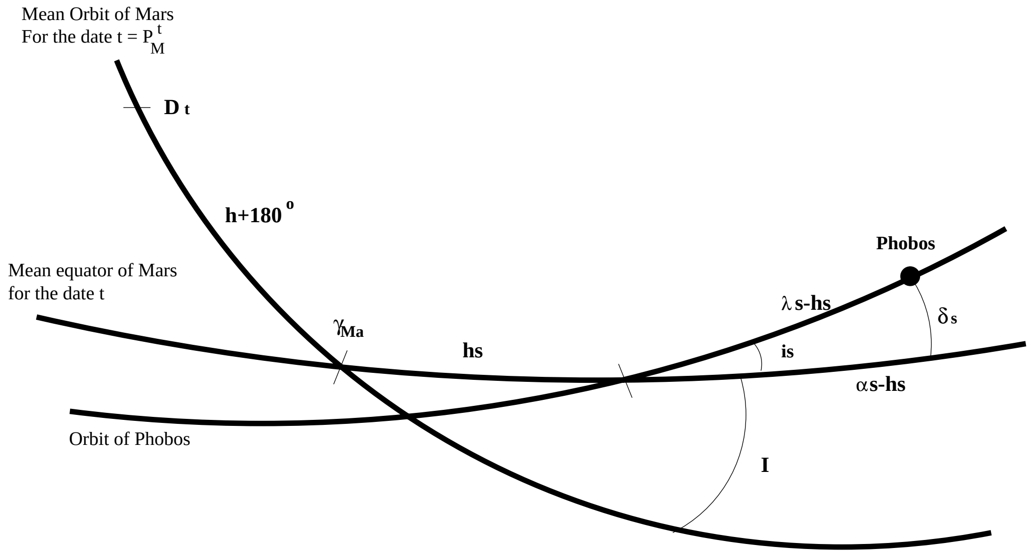

Fig. 3. Equatorial Phobos elements

$$
\begin{aligned}
i_{Ph} &= 1.067639° \\
\dot{h}_{Ph} &= 0.436025°/d \\
\dot{\lambda}_{Ph} &= 1128.84476°/d
\end{aligned}
$$

and:

$$
\begin{aligned}
a_{De} &= 23457.06\,km \\
e_{De} &= 0.00096 \\
i_{De} &= 1.78896° \\
\dot{h}_{De} &= 0.018001°/d \\
\dot{\lambda}_{De} &= 285.161875°/d
\end{aligned}
$$

Because of the smallness of the influence of the two satellites on the nutation, the second part of the potential at the r.h.s. of (7) is completely negligible and not considered here. Thus for the two satellites the perturbing potential can be expressed as:

$$U_s \approx \frac{k^2 M_S}{r_s^3} \times \left[\left(\frac{2C - A - B}{2} \right) P_2\left(\sin\delta_s\right) \right] \tag{16}$$

the indice s being associated with the given satellite. Then, we must notice some particularity in the way of computation of the nutation here, which looks like an extension with respect to the usual one as set up by Kinoshita (1977) and used precedently to calculate the coefficients of the nutation due to the Sun. The reason is that the coordinates (X,Y,Z) of the satellites in the ephemerides ESAPHO and ESADE (Chapront-Touzé, 1990) used here, are determined with respect to the equatorial plane of the planet and to the ascending node of the mean equator with respect to the mean orbital plane P_M^t. We call this reference system (R_M). Therefore we want to avoid to compute the coordinates λ_s and β_s of the satellites with respect to the mean orbital plane of the planet, and to compute the Legendre polynomial $P_2(\sin\delta_s)$ by the intermediary of the transformations in Eq. (12). In other words, we use the same canonical equations as those given by (9.1) and (9.2), but, by keeping, for further developments the original angle δ_s, which is the declination of the satellite with respect to the mean equator. Notice that the expression of the Legendre polynomial $P_2(\sin\delta_s)$ is:

$$P_2(\sin\delta_s) = \frac{3}{2}\sin^2\delta_s - \frac{1}{2} \tag{17}$$

As our procedure here is a new one, we develop it exhaustively in the following: let us consider (Fig. 3) the angles h_s, λ_s and i_s which enable to give the position of the satellite (Phobos or Deimos) with respect to the mean equator of the planet of the date and to the node between the mean equator of the date and the mean orbit of the planet of the date γ_{Ma}. h_s is the longitude of the ascending node of the orbit of the perturbing satellite with respect to the mean orbit of Mars and counted from γ_{Ma}. λ_s is the mean longitude of the satellite and i_s is the inclination of the orbit of the satellite, with respect to the equator of Mars.

Then, the coordinates (X, Y, Z) of the perturbing satellite, with respect to the equatorial frame (R_M), are related to its coodinates (x,y,z) with respect to the frame materialized by the mean orbital plane and γ_{Ma}, by the way of the following transformation:

$$[X, Y, Z] = M_x(I) M_z(h + \Pi)[x, y, z] \tag{18}$$

where M_x and M_z are respectively the rotations around the x-axis and the z-axis.

Then after (17) we find:

$$\frac{\partial P_2}{\partial h} = \frac{\partial P_2(\sin\delta_s)}{\partial h} = 3\sin\delta_s \frac{\partial \sin\delta_s}{\partial h} \tag{19.1}$$

$$\frac{\partial P_2}{\partial I} = \frac{\partial P_2(\sin\delta_s)}{\partial I} = 3\sin\delta_s \frac{\partial \sin\delta_s}{\partial I} \tag{19.2}$$

Then, with the help of (18), we can write:

$$\frac{\partial \sin\delta_s}{\partial h} = \sin I \cos\delta_s \cos\alpha_s \tag{20.1}$$

$$\frac{\partial \sin\delta_s}{\partial I} = -\cos\delta_s \sin\alpha_s \tag{20.2}$$

Notice that α_s is the equivalent of the right ascension of the satellite with respect to the planet's equatorial reference frame and equinox. Then, the combination of Eqs. (19.1), (19.2), (20.1) and (20.2) enables to write:

$$\frac{\partial P_2}{\partial h} = 3\sin I \sin\delta_s \cos\delta_s \cos\alpha_s \quad (21.1)$$
$$\frac{\partial P_2}{\partial I} = -3\sin\delta_s \cos\delta_s \sin\alpha_s \quad (21.2)$$

In a second step, we write the equations above as a function of X, Y and Z, with the help of trigonometric transformations:

$$X = \cos\delta_s \cos(\alpha_s - h_s) = \cos(\lambda_s - h_s) \quad (22.1)$$
$$Y = \sin(\alpha_s - h_s)\cos\delta_s = \sin(\lambda_s - h_s)\cos i_s \quad (22.2)$$
$$Z = \sin\delta_s = \sin(\lambda_s - h_s)\sin i_s \quad (22.3)$$

Thus, by combining the Eqs. (21.1–2) and (22.1–3), we find the following relationships:

$$\frac{\partial P_2}{\partial h} = 3Z(X\cos h_s - Y\sin h_s)\sin I \quad (23.1)$$
$$\frac{\partial P_2}{\partial I} = -3Z(Y\cos h_s + X\sin h_s) \quad (23.2)$$

or, by substitution of X, Y and Z:

$$\frac{1}{\sin I_e}\frac{\partial P_2}{\partial h} = -\frac{3}{4}\sin 2i_s \sin h_s + \frac{3}{4}\sin i_s(1+\cos i_s)\sin(2\lambda_s - h_s) + \frac{3}{4}\sin i_s(1-\cos i_s)\sin(2\lambda_s - 3h_s) \quad (24.1)$$
$$\frac{\partial P_2}{\partial I} = -\frac{3}{4}\sin 2i_s \cos h_s + \frac{3}{4}\sin i_s(1+\cos i_s)\cos(2\lambda_s - h_s) - \frac{3}{4}\sin i_s(1-\cos i_s)\cos(2\lambda_s - 3h_s) \quad (24.2)$$

In the following we will assimilate the orbit of the two satellites to a circle, for their excentricity is very small (respectively 0.015146 and 0.00096 for Phobos and Deimos). Thus, by choosing a_s as the mean distance of the satellite, the derivatives of the perturbing potential U_s w.r.t. the variables h and I can be written in the following form:

$$\frac{\partial U_s}{\partial h} = \left(\frac{\kappa^2 M_S}{a_s^3}\right)\left(\frac{2C-A-B}{2}\right)\left(\frac{\partial P_2}{\partial h}\right) \quad (25.1)$$
$$\frac{\partial U_s}{\partial I} = \left(\frac{\kappa^2 M_S}{a_s^3}\right)\left(\frac{2C-A-B}{2}\right)\left(\frac{\partial P_2}{\partial I}\right) \quad (25.2)$$

Then, the integration of the canonical equations at the first order gives:

$$\Delta h = -\frac{1}{G\sin I}\int \frac{\partial U_s}{\partial I} \quad (26.1)$$
$$\Delta I = \frac{1}{G\sin I}\int \frac{\partial U_s}{\partial h} \quad (26.2)$$

Let us call K_s the coefficient:

$$K_s = \left(\frac{3\kappa^2 M_S}{4a_s^3\omega}\right) \times \left(\frac{2C-A-B}{2C}\right)$$

Then the nutations $\Delta\psi = -\Delta h$ and $\Delta\varepsilon = -\Delta I$ are given by:

$$\Delta\psi = -\frac{K_s}{\sin I}\left[\left[\frac{\sin 2i_s}{\dot{h}_s}\right] \times \sin h_s - \left[\frac{\sin i_s\,(1+\cos i_s)}{2\dot{\lambda}_s - \dot{h}_s}\right]\sin(2\lambda_s - h_s) + \left[\frac{\sin i_s(1-\cos i_s)}{2\dot{\lambda}_s - 3\dot{h}_s}\right]\sin(2\lambda_s - 3h_s)\right] \quad (27.1)$$
$$\Delta\varepsilon = -K_s\left[\left[\frac{\sin 2i_s}{\dot{h}_s}\right] \times \cos h_s - \left[\frac{\sin i_s(1+\cos i_s)}{2\dot{\lambda}_s - \dot{h}_s}\right]\cos(2\lambda_s - h_s) - \left[\frac{\sin i_s(1-\cos i_s)}{2\dot{\lambda}_s - 3\dot{h}_s}\right]\cos(2\lambda_s - 3h_s)\right] \quad (27.2)$$

The leading components of the nutation due to Phobos and Deimos are by far those which involve the longitude of the node of the satellites, in Eqs. (27.1) and (27.2). Thus at the 0.1 milliarcsecond truncation level, we find only two terms both in longitude and obliquity:

$$\Delta\psi = -0''.01209 \times \sin h_{Phobos} - 0''.00439 \times \sin h_{Deimos}$$
$$\Delta\varepsilon = 0''.00514 \times \cos h_{Phobos} + 0''.00187 \times \cos h_{Deimos}$$

Notice that the respective periods are 2.260 y for h_{Phobos} and 54.754 y for h_{Deimos}.

4.4. The nutation of Mars due to the direct action of the planets

The attraction of the other planets on Mars causes nutations which can be calculated in the same manner as the nutation coming from the attraction of the Sun, by replacing the marsocentric coordinates of the Sun r_S, λ_S and β_S by the marsocentric coordinates r_p, λ_p and β_p, inside the Eqs. (7), (12), (14.1–3). These coordinates are obtained by substraction of the heliocentric coordinates of Mars from the heliocentric coordinates of the given planet, as they are expressed analytically by the ephemerides VSOP87 (Bretagnon & Francou, 1988). The coefficients of nutation are still here obtained after integration of the potential to provide the determining function W_1^p by the way of Eqs. (10.1) and (10.2). The Earth and Jupiter are the sole planets to give significant influence at the 0.1 mas level. The precession due to these two planets is listed together with the main nutation terms in Tables 3.1 (longitude) and 3.2 (obliquity).

4.5. The Oppolzer terms and the nutation and precession of the figure axis

All the coefficients of nutation which have been calculated in the precedent chapters concerned the axis of angular momentum, following the Eqs. (9.1), (9.2), (10.1) and (10.2). Nevertheless, it must be noticed that generally users are much more interested by the nutation of the axis of figure, rather than by the nutation of the angular momentum, for the former enables to give directly the orientation of the planet w.r.t. a given reference frame. The difference between the orientation of the axis of figure and that

Table 3.1. The direct influence of the planets on the nutation of Mars, longitude $\Delta\psi$

coefficient (")	Period (year)	arguments	influence
−0.22154	–	t (1000 y.)	Jupiter
−0.00019	5.930	$\sin(2\lambda_{Ju})$	Jupiter
−0.00003	5.930	$\cos(2\lambda_{Ju})$	Jupiter
−0.08153	–	t (1000 y.)	Earth
0.00014	−15.781	$\sin(\lambda_{Ea} - 2\lambda_{Ma})$	Earth
−0.00008	−15.781	$\cos(\lambda_{Ea} - 2\lambda_{Ma})$	Earth

Table 3.2. The direct influence of the planets on the nutation and precession of Mars, obliquity $\Delta\varepsilon$

coefficient (")	Period (year)	argument	influence
−0.00622	–	t (1000 y.)	Jupiter
0.00009	5.930	$\sin(2\lambda_{Ju})$	Jupiter
−0.00002	5.930	$\sin(2\lambda_{Ju})$	Jupiter
0.00249	–	t (1000 y.)	Earth

Table 4.1. The Oppolzer terms making the difference between the axis of angular momentum and the axis of figure, longitude, $\Delta\psi$

sin (")	Period (year)	M	Λ_M
−0.00361	0.940	2	2
−0.00119	0.627	3	2
−0.00027	0.470	4	2
0.00017	1.881	1	2

of the axis of angular momentum is given by the following expressions, obtained with the help of Eqs. (3.1) and (3.2):

$$\Delta\psi_f - \Delta\psi = -(\Delta h_f - \Delta h) = -\Delta\left(\frac{J}{\sin I} \times \sin g\right) \quad (28.1)$$

$$\Delta\varepsilon_f - \Delta\varepsilon = -(\Delta I_f - \Delta I) = -\Delta\left(J\cos g\right) \quad (28.2)$$

The development of the expressions at the r.h.s. of (27.1) and (27.2) by the intermediary of canonical equations giving both ΔJ and Δg, and the numerical treatment related to this development, are still here exactly the same as in the case of the calculation of Oppolzer terms for the Earth, as analytically carried out by Kinoshita (1977). Thus we can refer to that last paper to know in detail the whole procedure. Four coefficients are found and listed in Tables 4.1 and 4.2, up to 0.1 mas in absolute amplitude.

Thus we can refer to that last paper to know in detail the whole procedure. Four coefficients are found and listed in Tables 4.1 and 4.2, up to 0.1 mas in absolute amplitude. These coefficients have to be added to the corresponding one (with the same argument) for the axis of angular momentum, in order to get the nutation for the figure axis. Notice that the argument of the leading coefficients for the Oppolzer terms correspond to the leading ones for the axis of angular momentum or the axis

Table 4.2. The Oppolzer terms making the difference between the axis of angular momentumand the axis of figure, obliquity, $\Delta\varepsilon$

cos (")	Period (year)	M	Λ_M
−0.00139	0.940	2	2
−0.00046	0.627	3	2
0.00040	1.881	1	
−0.00010	0.470	4	2

of figure, as can be seen from Tables 1.1 and 1.2. At last, the Oppolzer terms which give the orientation of the axis of rotation with respect to the axis of angular momentum are much smaller in absolute amplitude, than those which have been calculated above for the determination of the axis of figure, that is to say by a ratio (C-A)/C. Thus they are at the order of the microarcsecond level which is negligible in the frame of our present study, and consequently they are not studied here. In other words, we can assimilate the axis of rotation to the axis of figure.

4.6. The influence of the triaxiality on the rotation of Mars

As it is the case for the Earth, Mars is not an axisymetric body, which means that the moment of inertia A and B are not identical. The consequence is the presence of semi-diurnal (which means here with period corresponding to half a martian day) components of the nutation. The way of calculation of these components starting from Hamiltonian theory is quite identical to that already used by Kinoshita (1977) and more precisely by Souchay & Kinoshita (1997). Then the final formula giving the nutation in longitude and obliquity are the same, the values of the parameters being changed. The two leading terms can be written as follows:

$$\begin{aligned}\Delta\psi_{1/2d} = {} & \frac{B-A}{2C-A-B} \times K'\Bigg[\frac{1}{8}(1+\cos I) \\ & \times \left[\frac{1}{(\dot g + \dot l - \dot M - \dot h)}\right] A_{2M+2\Lambda_M} \\ & \times \sin\left(2g + 2l - 2M - 2\Lambda_M\right) \\ & -\frac{1}{2}\cos I\left[\frac{1}{(\dot g + \dot l)}\right] A_0 \times \sin(2g+2l)\Bigg] \quad (29.1)\end{aligned}$$

$$\begin{aligned}\Delta\varepsilon_{1/2d} = {} & \frac{B-A}{2C-A-B} \times K'\Bigg[\frac{1}{8}\sin I(1+\cos I) \\ & \times \left[\frac{1}{(\dot g + \dot l) - (\dot M + \dot h)}\right] A_{2M+2\Lambda_M} \\ & \times \cos\left(2g + 2l - 2M - 2\Lambda_M\right) \\ & -\frac{1}{4}\sin 2I\left[\frac{1}{(\dot g + \dot l)}\right] A_0 \times \cos(2g+2l)\Bigg] \quad (29.2)\end{aligned}$$

where the coefficient K' has the following expression:

$$K' = 3\frac{\kappa^2 M_S}{a^3\omega} \times \left[\frac{2C-A-B}{2C}\right]$$

$A_{2M+2\Lambda_M}$ being the coefficient of the term with argument $2M + 2\Lambda_M$ in the development of the expression $(\frac{a_s}{r_S})^3 \cos^2\beta_s \cos 2\lambda_s$ and A_0 the constant term in the expression $\frac{1}{2}(\frac{a_s}{r_s})^3(1-3\sin^2\beta_s)$.

We can observe that to determine numerically the Oppolzer terms, we need to know the value of the angular rate of the variables l and g whose definitions have been given at the beginning of the paper. The value of this angular rate is determined when studying the free motion of rotation of the planet. By analogy with the free motion of the Earth studied extensively by Kinoshita (1972), we can approximate $\dot{g}$ and $\dot{l}$ by the following simple expressions:

$$\dot{g} = \frac{1}{2}\left[\frac{C}{A} + \frac{C}{B}\right] \times \omega \tag{30}$$

$$\dot{l} = \omega - \dot{g} \tag{31}$$

These approximations are available at the condition that the triaxiality expressed by the parameter e is small, where:

$$e = \frac{\left[\frac{C}{A} - \frac{C}{B}\right]}{2 - \left[\frac{C}{A} + \frac{C}{B}\right]} \tag{32}$$

A rough calculation leads to: $e = 0.065$ which gives the following numerical values of the main semi-diurnal nutations, after substitution of aguments:

$$\Delta\psi_{1/2d} = 0''.00011 \sin(2\lambda_{Ma} - 2g - 2l - 2h) - 0''.00011 \sin(2g + 2l)$$

and:

$$\Delta\varepsilon_{1/2d} = -0''.00005 \cos(2\lambda_{Ma} - 2g - 2l - 2h) + 0''.00005 \cos(2g + 2l)$$

The periods are respectively $0.514d$ and $0.513d$. λ_{Ma} is the mean longitude of Mars, and (l+g+h) has a period equal to the sidereal rotation of the planet.

These expressions hold for the axis of angular momentum. As it is the case for the Earth, the corresponding expressions for the nutation of the axis of figure for these semi-diurnal coefficients are opposite, at the level of precision of these coefficients.

5. The planetary precession and the secular variation of the obliquity

As the Eqs. (5), (8), (9.1) and (9.2) are pointing out, the additional component E in the Hamiltonian, which is related to a secular motion of the plane of reference to compute the nutation terms, that is to say the mean orbital plane of Mars, has to be taken into account. E depends on the angles π and Π, which have been determined in the following of (8).

These angles can be expressed as a function of the elements Ω and i which are respectively the longitude of the node of Mars orbit and the inclination of the orbit w.r.t. ecliptic of J2000.0. The numerical expressions for these two parameters have been calculated by Bretagnon (1982), and we take these values. Thus, we can write:

$$\sin(\Pi + \psi)\sin\pi = \sin(\Omega - \Omega_0)\sin i \tag{33.1}$$

$$\cos(\Pi + \psi)\sin\pi = -\cos i \sin i_0 + \sin i \cos i_0 \cos(\Omega - \Omega_0)\sin i \tag{33.2}$$

The variable ψ corresponds to the angle N_0D_0 in the Fig. 2. And the values found for the angles are:

$$\pi = 897.4480134T - 1.380904T^2 - 0.091923T^3 \tag{34.1}$$

$$\Pi = -446819.38155 + 12842.087336T - 4.352970T^2 + 15.967223T^3 \tag{34.2}$$

The angle from the node N between ecliptic J2000.0 and (P_m^t), and the departure point is:

$$(ND) = -520206.882048 + 27780.958512T + 5.827291T^2 + 8.236316T^3 \tag{34.3}$$

The units are arcseconds and thousand of years for the time.

The partial derivatives of E with respect to the variables h and I are obtained with the help of (8):

$$\frac{\partial E}{\partial I} = -G\cos I\left[\dot{\pi}\sin\left(h - \bar{DM}\right) - \dot{\Pi}\sin\pi\cos(h - \bar{DM})\right] \tag{35}$$

$$\frac{\partial E}{\partial h} = G\sin I\left[-\dot{\Pi}\sin\pi\sin\left(h - \bar{DM}\right) - \dot{\pi}\cos(h - \bar{DM})\right] \tag{36}$$

Notice that the variations of these angles are very small: h, π and Π have respectively the periods $1.71 \times 10^5 y$, $2.87 \times 10^6 y$ and $1.59 \times 10^5 y$. We can consider them as constant terms for a short span of time.

The rate of precession $\dot{h}_E$ coming from the motion of the mean orbit of Mars, as well as the secular variation of the obliquity $\dot{\varepsilon}$ of Mars equator with respect to the mean ecliptic of the date, are given by:

$$\dot{h}_E = \frac{-1}{G\sin I}\frac{\partial E}{\partial I} \tag{37}$$

$$= \cot I\left[\dot{\pi}\sin\left(h - \bar{DM}\right) - \dot{\Pi} \times \sin\pi\cos\left(h - \bar{DM}\right)\right]$$

$$\dot{I} = \frac{1}{G\sin I}\frac{\partial E}{\partial h} \tag{38}$$

$$= -\dot{\Pi} \times \sin\pi\sin\left(h - \bar{DM}\right) - \dot{\pi}\cos\left(h - \bar{DM}\right)$$

These last equations lead to:

$$\dot{h}_E \approx \cot I \sin(h - \bar{DM}) \times \dot{\pi} \tag{39}$$

$$\dot{I} \approx -\cos(h - \bar{DM}) \times \dot{\pi} \tag{40}$$

where the value of $\sin\pi$ has been neglected because of the smallness of the angle π. The following resulting numerical determinations are: $\dot{h}_E \approx -1.57970''/y$ and: $\dot{\varepsilon}_{Ma} = -\dot{I} = -0.50336''/y$

At the first order the martian planetary precession $\dot{\chi}_{Ma}$ which is the equivalent expression of the planetary precession $\dot{\chi}_A$ for the Earth as defined in Lieske et al. (1977), is related to $\dot{h}_E$ by the straightforward formula:

$$\dot{\chi}_{Ma} = -\frac{\dot{h}_E}{\cos\varepsilon_{Ma}} = 1.74571''/y$$

Notice that these values are respectively for the Earth: $\dot{\varepsilon} = -0.46815''/yr$ and $\dot{\chi} = 0.10553''/yr$

6. The contributions to the precession in longitude and its relationships with the dynamical ellipticity

As it has been defined for the Earth, the general precession in longitude in the case of Mars characterizes the motion, with respect to the slightly moving plane of the Mars orbital motion of the date, of the intersection of the ascending node of this plane with the mean equator of Mars. We call in the following this general precession in longitude p_{Ma}

In a very similar manner as for the Earth (Kinoshita & Souchay, 1990; Williams, 1994; Souchay & Kinoshita, 1996) p_{Ma} is the sum of several contributions, which can be enumerated as: the leading one due to the gravitational attraction of the Sun, the contribution due to the gravitational attraction of the planets (direct planetary effect) as well as to the gravitational attraction of Phobos and Deimos, and the geodesic precession, which is a small correction related to a relativistic effect (De Sitter,1938). At last we must take into account the rather big component which is coming from the fact that the general precession in longitude is not measured from an inertial plane (for instance the orbital plane for the epoch of reference J2000.0), but from the moving orbital plane of the date. This component corresponds in fact to the expression $\dot{\chi}\cos\varepsilon_{Ma}$ as calculated in the precedent chapter, where $\dot{\chi}$ characterizes the planetary precession.

From the Viking/Pathfinder experiments the Mars' rate of the precession in longitude ψ_{Ma}, with respect to the fixed mean orbital plane of Mars at the epoch J2000.0, has been determined with a good accuracy, that is to say $7.576 \pm 0.035''$/y. It must not be confused with p_{Ma}; neverthelsee, to get the sole contribution of the Sun to ψ_{Ma}, as quoted as ψ^S_{Ma}, we must make the substitutions of all the other components which have been enumarated above, excepted the leading one due to the change of reference plane.

The influence of Phobos and Deimos on the precession of Mars derived from our calculations are not present, which seems to indicate that because the plane of their orbit is very closed to the equatorial plane of the planet, this influence does not exist.

The influence of Jupiter is -0.00022$''$/y and that of the Earth -0.00008$''$/y (the sign "-" indicates that the related motion is a retrograd one). Thus all these effects are considerably small (the cumulative effect of these contributions is only about 1/10000 that of the total value of the precession). Moreover, we can neglect the direct effects of the other planets as the Earth and Jupiter.

In order to determine the value of the geodesic precession for Mars, we can refer to a basic formula as it has been established by De Sitter(1938) and calculated by Barker and O'Connell (1975) as well as by Williams (1994) for the Earth, that is to say:

$$p_{geod.} = \frac{3}{2} \times \left[\frac{na}{c}\right]^2 \times \left[\frac{n}{1-e^2}\right] \qquad (41)$$

Where n, a, and e are respectively the mean motion, the semi-major axis and the excentricity of the Mars orbit, and c is the speed of light. The numerical value thus found is: $p_{Geod} = 0''.67547/cy$.

In a similar way as it was the case for the Earth (Souchay & Kinoshita, 1996), ψ^S_{Ma} can be expressed as a function of the dynamical ellipticity of Mars $H^{Ma}_d = \frac{2C-A-B}{2A}$, and of parameters which are known with a relatively very good accuracy, as the mean motion, the mean rotation, and the orbital characteristics of the planet. By using the same formula as for the Earth (Kinoshita & Souchay, 1990; Souchay & Kinoshita, 1996) but by taking into account the sole solar contribution, we can write the relationship between $\dot{\psi}^S_{Ma}$ and H^{Ma}_d:

$$\dot{\psi}^S_{Ma} = -3 \times H^{Ma}_d \times \left[\frac{M_{Ma}}{M_{Ma}+M_S}\right] \times \left[\frac{n^2_{Ma}}{\omega_{Ma}}\right] S_0 \cos\varepsilon_{Ma} \qquad (42)$$

Where S_0 is the constant term in the expression of the potential given in (14.1). Some additioning relatively small contributions already calculated in the case of the Earth (Kinoshita & Souchay, 1990) will not be considered here because of their expected smallness as it is the case for the contribution due to the solar interaction with J_4. Unless future determinations of the ratio $\frac{J_4}{J_2}$ look astonishingly relatively large for Mars in comparison with the Earth, we can neglect this last influence. Notice that for the Earth it amounts only to $-0''.0026/cy$, that is to say a relative 5×10^{-7} ratio with respect to the total precession value.

At last some other influences present in the case of the Earth, as coupling effects between the planet and its satellite (Kinoshita & Souchay, 1990) which can be divided into two parts as crossed-nutation and tilt-effects (Williams, 1994) are not considered here, because the corresponding coupling interaction between Mars and its satellites Phobos and Deimos look completely negligible in comparison: the effects of Phobos and Deimos on the first order precession and nutations are less than 1% that of the Sun, whereas in comparison the effects of the Moon on the precession of the Earth are twice that of the Sun and more than ten times in the case of the nutation. Finally, our value for H^{Ma}_d as deduced from (42) is: $H^{Ma}_d = 0.3669 \pm 0.0017$ This value of H^{Ma}_d is a little different of that calculated by Folkner et al., that is to say: $H^{Ma}_d = 0.3662 \pm 17$, but notice that we use this last value in the computation of our series. Thus the series calculated with our value H^{Ma}_d may be obtain in a simple manner by multiplying the series of this paper by the ratio of the values of the factor H^{Ma}_dtaht is to say 1.0019.

Table 5. Values of the parameters and angles used in this paper

name of the variable	value	uncertainty	unit	origin
ω	350.89198226	± 8	$^\circ$ /d	Folkner et al. (1997)
n	1886.51820925		$''$/day	conventional value of IAU.
$\kappa^2 M_S$	$1.32712438 \times 10^{20}$		$m^3 \times sec^2$	conventional value of IAU.
M_S/M_{Ma}	3098710		–	conventional value of IAU.
J_2	0.001964		–	conventional value of IAU.
J_3	0.000036		–	conventional value of IAU.
$S_{2,2}$	0.000031		–	conventional value of IAU.
$C_{2,2}$	-0.000055		–	conventional value of IAU.
C/MR^2	0.3662	± 17	–	Folkner et al. (1997)
H_d	0.005363	± 25	–	this paper
$(B-A)/C$	0.0006896	± 32	–	
C/A	1.005741	± 27	–	this paper
C/B	1.005044	± 24	–	this paper
R_{ma}	3397.2	± 40	km	conventional value of IAU.
$\dot{g}$	352.7842	± 88	$\circ$ /d	this paper
$\dot{l}$	-1.8922	± 88	$\circ$ /d	this paper
e	0.0646	± 5	–	
K'	16557	± 77	$''$/1000 ans	
ε_{Ma}	25.189417	± 35	$\circ$	Folkner et al. (1997)
α_{2000}	317.68143	± 1	$\circ$	Folkner et al. (1997)
δ_{2000}	52.88650	± 3	$\circ$	Folkner et al. (1997)
Λ_0	-108.994438		$\circ$	
a_{Ph}	9373.713		km	Chapront-Touzé (1990)
e_{Ph}	0.015146		–	Chapront-Touzé (1990)
v_{Ph}	1128.845		$\circ$ /d	Chapront-Touzé (1990)
i_{Ph}	1.067639		$\circ$	Chapront-Touzé (1990)
$\kappa^2 M_{Ph}$	6.38825×10^{15}		$m^3 \times d^2$	Chapront-Touzé (1990)
$\dot{h}_{Ph}$	0.436025		$\circ$ /d	Chapront-Touzé (1990)
K_{Ph}	383.8	± 18	$''$/1000 ans	this paper
a_{De}	23.457		km	Chapront-Touzé (1990)
e_{De}	0.000196		–	Chapront-Touzé (1990)
v_{De}	285.1619		$\circ$ /d	Chapront-Touzé (1990)
i_{De}	1.78896		$\circ$ /d	Chapront-Touzé (1990)
$\kappa^2 M_{De}$	8.96375×10^{14}		$m^3 \times d^2$	Chapront-Touzé (1990)
$\dot{h}_{De}$	0.018001		$\circ$ /d	Chapront-Touzé (1990)
K_{De}	3.436	± 16	$''$/1000ans	this paper
M_{Ju}/M_S	1047.355		–	conventional value of IAU.
M_{Terre}/M_S	328900.5		–	conventional value of IAU.

7. Long time scale evolution of the motion of rotation of Mars

In order to study on a long time scale the variation of the parameters related to the rotation of Mars, especially that of the obliquity ε_{Ma} and of the precession rate in longitude $\dot{\psi}_{Ma}$, we first express the long period and the secular parts of the Hamiltonian related to this rotation. For this purpose, we do not consider here the Hamiltonian related to the free motion as well as its component related to the short-periodic forced motion. With a suitable change of canonical variables, the Hamiltonian F_{LP} becomes (LP for Long Period and SP for Short Period):

$$F_{LP} = \frac{L^2}{2C} + G\sin I\left[\sin i\cos\left(h-\bar{DN}\right)\dot{\Omega} - \sin(h-\bar{DN})\frac{di}{dt}\right] + \frac{k^2M'}{r^3}\times\frac{2C-A-B}{2}\times\left[P_2(\sin\delta)\right]_{sec} \quad (43)$$

with:

$$\left[P_2\left(\sin\delta\right)\right]_{sec} = -\frac{1}{4}\left(\frac{r}{a}\right)^3\left(3\cos^2 I - 1\right)\left(1-e^2\right)^{-3/2} \quad (44)$$

The term E that stay in this Hamiltonian F_{LP} seems to differ from the term E use in the previous parts of this paper. In fact, the only difference is the choice of the inertial plane. For the short period study, we choose the mean orbital plane of Mars at J2000 as inertial plane. The choice of this inertial plane of reference comes from the intention to describe the motion of precession-nutation of Mars in a way analogous to what is chosen in general for the Earth, that is to say the mean orbital plane of the Earth or ecliptic, at J2000.0.

For the long period study, we choose the ecliptic of the epoch J2000.0. This plane is easier to use because the ephemerides of Mars relate to it. In both cases, the term E is written:

$$E = -\Omega_1 \times L_1 - \Omega_2 \times L_2 - \Omega_3 \times L_3 \tag{45}$$

$\Omega_i, i = 1, 2, 3$ are the coordinates of the instantaneous vector of rotation of the reference frame linked to the mean orbit of Mars w.r.t. the reference frame linked to the inertial plane. We remark that for the long period study as well as for the short period study, the origin of the angle h is the departure point D_t. Therefore Ω_3 is equal to zero. For the long period study, the Ω_i are written with the help of the variables i et Ω describe in chapter 5. Until chapter 6, the Ω_i are written with the help of the variables π et Π. So we have the relation:

$$\begin{aligned} E_{LP} &= G \sin I \left[\sin i \cos(h - \bar{DN})\dot{\Omega} - \sin(h - \bar{DN}) \frac{di}{dt} \right] \\ &= E_{SP} = G \sin I \left[\sin \pi \cos\left(h - \bar{DM}\right) \dot{\Pi} \right. \\ &\qquad \left. - \sin\left(h - \bar{DM}\right) \dot{\pi} \right] \end{aligned} \tag{46}$$

The same thing occurs for the definition of the variation of the departure point D_t. We have the two following relations that determine the same motion:

$$\dot{\bar{ND}} = -\cos i \dot{\Omega}$$

or

$$\dot{\bar{MD}} = -\cos \pi \dot{\Pi} \tag{47}$$

With the Hamiltonian F_{LP}, the two equations of the system can be written

$$\begin{aligned} \dot{h} &= -\frac{1}{G \sin I} \frac{\partial F}{\partial I} \\ \dot{h} &= -\frac{\cos I}{\sin I} \left[\sin i \cos(h - \bar{DN})\dot{\Omega} - \sin(h - \bar{DN}) \frac{di}{dt} \right] \\ &\quad -\frac{3}{2} \frac{k^2 M'}{G a^3} \left(\frac{2C - A - B}{2} \right) \left(1 - e^2\right)^{\frac{-3}{2}} \cos I \end{aligned} \tag{48}$$

$$\begin{aligned} \dot{I} &= \frac{1}{G \sin I} \frac{\partial F}{\partial h} \\ &= -\left[\sin i \sin\left(h - \bar{DN}\right) \dot{\Omega} + \cos\left(h - \bar{DN}\right) \frac{di}{dt} \right] \end{aligned} \tag{49}$$

Practically the coordinates $\Omega_i, i = 1, 2, 3$ can be expressed as a function of the variables p and q given by the long period ephemerides of Mars (Bretagnon, 1984; Laskar, 1988). They are given by:

$$p = \sin\left(\frac{i}{2}\right) \sin \Omega \tag{50.1}$$

$$q = \sin\left(\frac{i}{2}\right) \cos \Omega \tag{50.2}$$

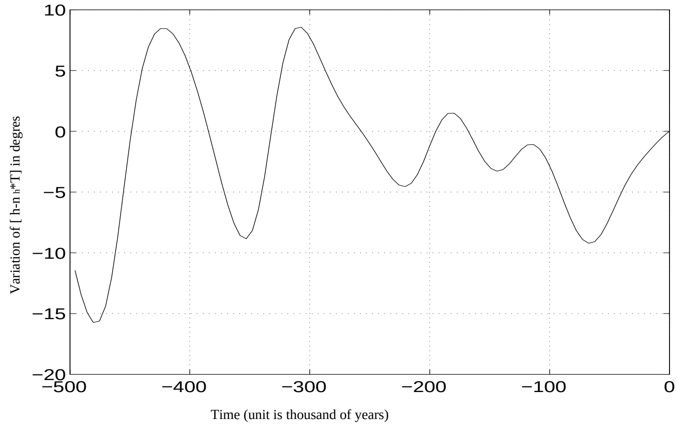

Fig. 4. Motion of variable h over 500 000 years

The components $\Omega_i, i = 1, 2, 3$ can then be written in the following form:

$$\Omega_1 = \frac{2(p\dot{p} + q\dot{q})}{\sqrt{p^2 + q^2}\sqrt{1 - p^2 - q^2}} \tag{51.1}$$

$$\Omega_2 = \frac{2(\dot{p}q - \dot{q}p)\sqrt{1 - p^2 - q^2}}{\sqrt{p^2 + q^2}} \tag{51.2}$$

$$\Omega_3 = 0 \tag{51.3}$$

With the help of Eqs. (51.1–3), we can express the Ω_i with respect to p and q. Moreover the expression $(1 - e^2)^{\frac{-3}{2}}$ in (48) can be determined from the angles $h = e \sin \tilde{\omega}$ and $k = e \cos \tilde{\omega}$ given by the long period ephemeris:

$$\left(1 - e^2\right)^{\frac{-3}{2}} = \left(1 - h^2 - k^2\right)^{\frac{-3}{2}} \tag{52}$$

Thus the differential Eqs. (48) and (49) can be expressed as a function of the time, and a numerical integration can be carried out, leading to the determination of the rotation angles h and I. We use a numerical integrator of Runge-Kunta at 8th. order with variable step. In the continuation, we use the Laskar ephemerides. On a 500 000 years period the differences with the Bretagnon ephemerides are negligible for this study. The limitation to this time span is principally due to the uncertainty on the frequencies of the orbital motion of the planets (that is to say roughly 0.1″/y). This uncertainty brings a phase shift of about 15° on the longitudes after 1/2 million years. Moreover, Laskar & Robutel (1993) have shown that Mars is located inside a chaotic zone, and that after 5 million years time interval, the determination of the orientation of the planet cannot be modelized.

As a first step we integrate the Eqs. (47), (48) and (49) with $\frac{C}{MR^2} = 0.3662$ that is to say the value determined by Folkner et al. (1997) from a precession value set to 7.576″/y. Fig. 4 shows the variation of ($h - \dot{h} \times T$), (where T is the time), from $-500\,000$ years until nowdays. Fig. 5 shows the variation of $\varepsilon_{Ma}(= -I)$ on the same period. At last, Fig. 6 shows the variation of $(\bar{D_t N} - \bar{D_0 N_0} - \bar{N_0 N}) = V(D)$ that represent the motion of the departure point on P_M^t.

Variations of the obliquity I of Mars on a very long time scale have already been done by Ward (1979). Borderies (1980)

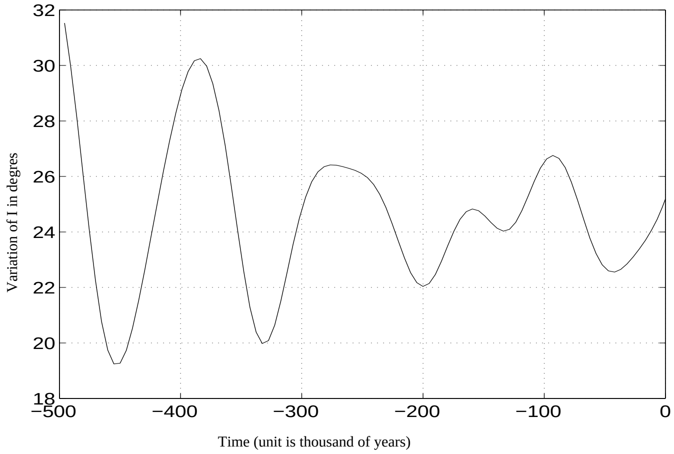

Fig. 5. Motion of variable $\epsilon_{Ma}(= -I)$ over 500 000 years

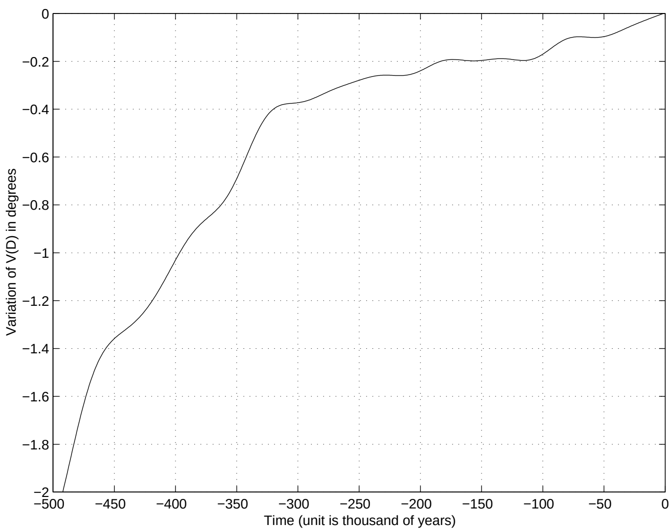

Fig. 6. Motion of departure point D_t over 500 000 years

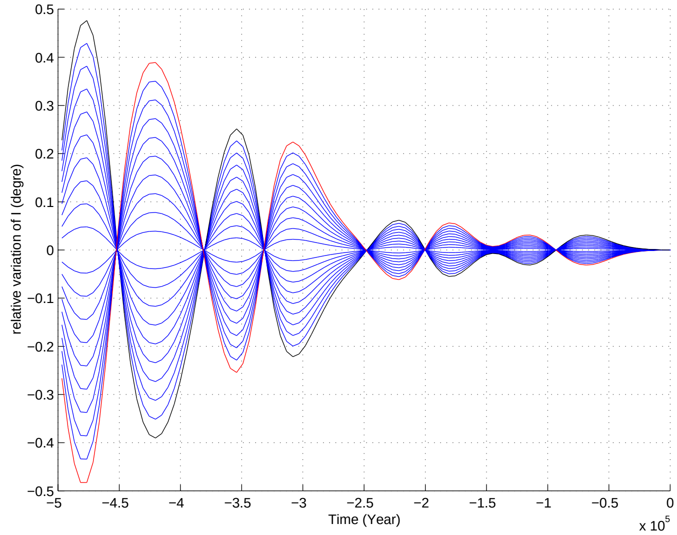

Fig. 7. Relative motion of I with different Pathfinder values of C/MR^2

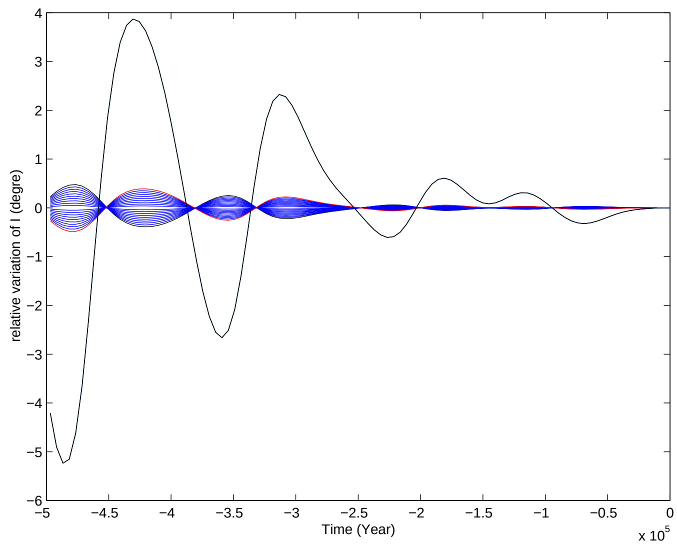

Fig. 8. Relative motion of I with Pathfinder and Viking values of C/MR^2

has also studied the variations of the obliquity and of the precession in longitude of Γ measured from the parameter h, on the same time scales, but with an origin for the determination of the angle h the point N_t which has not the same propperties as the point D_t. However, the choice of the origin does not affect directly the obliquity, which is the more significant variable for the past history of Mars weather. Although they are different ephemerides, the variations of obliquity determined by Ward and Borderies were very similar to our results. The main differences between the results are due to the choice of $(\frac{MR^2}{C})$ of which uncertainly has been improved recently by the Pathfinder mission, (see Fig. 8).

The large variations of the obliquity (Fig. 5) come from the commensurability between the frequency of the precession $\dot{h}$ and the linear combinations of the frequencies of the revolutions of the planets. Notice that the Earth is taken away from resonant zones, because of the dynamical couple exerted by the Moon, the precession caused by this last body being roughly $30''$/y. Thus the variations in obliquity for the Earth are about $1°$ whereas for Mars they reach $15°$.

Moreover, the frequency of the precession $\dot{h}$ depends on the ratio $(\frac{MR^2}{C})$ as a scaling factor. The rotation of Mars at a long time scale depends also strongly from this parameter because of resonances closed to this frequency. As we already mentioned it before, the accuracy on this parameter has been drastically improved by the measurements done by Pathfinder (Folkner et al., 1997). Nevertheless, despite the better delimitation of the range of the values of I and h in a 500 000 y time scale, Mars could still cross resonant zones. Thus we have studied this possibility; in Fig. 7 we have represented the evolution w.r.t. the time of the difference between the determinations of obliquities calculated with different values of the coefficient $(\frac{C}{MR^2})$ located inside the uncertainty range, and that calculated with the nominal value $(\frac{C}{MR^2} = 0.3662)$. Then we show that no resonance occur on a period of 500 000 y (the curves do not overlap but present the same form with gradually increasing amplitudes).

The Pathfinder data has allowed to reduce by a factor ten the uncertainty of the $(\frac{MR^2}{C})$ parameter since Viking missions. The Fig. 8 shows the drastic improvement of the accuracy in the determination of the martian obliquity from -500000 y until now. The same curves as in Fig. 7 are shown in comparison with the isolated curve which shows the uncertainty on the value of I before the mission Pathfinder. We can then remark that the accuracy on the determination of I has also been improved by a factor 10. However, the peak to peak uncertainty after 500,000 years of integration is still about one degree. Variations of amplitude of the Earth obliquity are the main reason of ice age for our planet. Then it looks very interesting to constrain at best the time evolution of such a parameter as the obliquity for Mars, as we did here.

8. Conclusion

In this paper, we carried out a complete study of the combined motion of precession and nutation of the planet Mars by choosing basic canonical equations based on the Hamiltonian for the motion of rotation of the planet analogous to that set up by Kinoshita (1977) for the Earth. We have used very recent parameters of the planet coming from the Pathfinder mission (Folkner et al., 1997), especially those concerning the moments of inertia and the precession constant ($7''.567$ /y). We have calculated the coefficients of the nutation at the 0.01 milliarcsecond level for both the axis of angular momentum, and the axis of figure, when taking into account the Oppolzer terms which make the difference between these two axes. We have considered not only the main terms of nutation which are due to the influence of the Sun, but also those which are coming from the direct action of the planets. Moreover we have made some suitable theoretical transformations to calculate the nutation due to the two satellites Phobos and Deimos, whose the amplitude is at the level of 1% w.r.t. the main nutation term due to the Sun. We have also evaluated the effect of the triaxial component of the potential of the planet on the nutation, characterized by half diurnal coefficients.

At last we have studied the evolution of Mars obliquity and precession rate at a very long time scale (500 000 y), by the way of a numerical integration. Thus we have shown that the improved accuracy of data concerning Mars by the intermediary of the Pathfinder mission leads a much better delimitation, by roughly a factor 10, of this evolution. Then we can observe that the uncertainty of the Martian obliquity for the time scale above does not exceed 1 degree peak to peak. We hope that this work might be useful to determine at best the orientation of the axis of figure in space in the perspective of future missions as NETLANDER.

References

Barker B.M., O'Connell R.F., 1975, In: Proceed of the Enrico Fermi Summer School, Varenna, Italy

Borderies N., 1980, A&A 82, 129

Bretagnon P., 1982, A&A 114, 278

Bretagnon P., 1984, In: Berger A.L., et al. (eds.) Milankovitch and climate. p. 1

Bretagnon P., Francou G., 1988, A&A 202, 309

Capitaine N., Guinot B., Souchay J., 1986, Celest. Mech 39, 283

Carpenter R.L., 1964, AJ 69, 2

Chapront-Touzé M., 1990, A&A 240, 159

De Sitter W., 1938, Bull. Astron. Netherlands 8, 213

Folkner W.M., Yoder C.F., Yuan D.N., Standish E.M., Preston R.A., 1997, Sci 278, 1749

Guinot B., 1979, In: Time and the Earth's rotation. Proceed. of the Eighty-second Symposium, San Fernando, Spain, p. 7

Hilton J.L., 1991, AJ 102, 1510

Hori G., 1966, PASJ 18, 827

Kinoshita H., 1972, PASJ 24, 423

Kinoshita H., Hori G., Nakai H., 1974, Annals Tokyo Astr. Obs. 14, 14

Kinoshita H, 1977, Celest. Mech. 33, 26

Kinoshita H., Souchay J., 1990, Celest. Mech. 48, 187

Laskar J., 1988, A&A 198, 341

Laskar J., Robutel P., 1993, Nat 361, 608

Lieske J.H., Lederle T., Fricke W., Morando B., 1977, A&A 58, 1

Pettingill G.H., Dyce R.B., 1965, Nat 206, 1241

Pitjeva E.V., 1996, In: Proceed. IAU Colloqium 165 Poznan, Poland, 251

Reasenberg R.D., King R.W., 1979 J. Geophys. Res. 84, 6231

Souchay J., Kinoshita H., 1996, A&A 312, 1017

Souchay J., Kinoshita H., 1997, A&A 318, 639

Souchay J., Loysel B., Kinoshita H., Folgueira M., 1998, A&A 134, 243

Ward W.R., 1979, J. Geophys. Res. 84, 243

Williams J.G., 1994, AJ 108, 711

Woolard E.W., 1953, Astron. Papers Amer. Eph. 15, 1