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Instability of high-frequency acoustic waves in accretion disks with turbulent viscosity

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Abstract. The dynamics of linear perturbations in a differentially rotating accretion disk with a non-homogeneous vertical structure is investigated. We find that turbulent viscosity results in instability of both pinching oscillations, and bending modes. Not only the low-frequency fundamental modes, but also the high-frequency reflective harmonics appear to be unstable. The question of the limits of applicability of the thin disk model (MTD) is also investigated. Some differences in the dispersion properties of the MTD and of the three-dimensional model appear for wave numbers $k \lesssim (1\text{–}3)/h$ (h is the half-thickness of a disk). In the long-wavelength limit, the relative difference between the eigenfrequencies of the unstable acoustic mode in the 3D-model and the MTD is smaller than 5%. In the short wavelength case ($kh > 1$) these differences are increased.

Key words: accretion, accretion disks – instabilities – Magnetohydrodynamics (MHD) – turbulence

1. Introduction

Accretion disks (AD) are important in many observable astrophysical objects (close binary systems, quasars, active galactic nuclei, young stars, protoplanetary disks). It is usually assumed that the disk is geometrically thin, and has turbulent viscosity (Shakura & Sunyaev 1973; Lightman & Eardley 1974). The physical mechanism of turbulent viscosity can be connected with various instabilities.

The basis for various viscous models of AD is the assumption that the dynamic viscosity η is caused by turbulence of the medium and that $\eta \sim \rho u_t \ell_t$ (u_t is the characteristic amplitude of the most large-scale turbulent velocity, ℓ_t is the spatial scale, ρ is the density) (Shakura & Sunyaev 1973). In this connection emphasis is given to the search of multimode instabilities, which result in a complex perturbations structure. Different spatial and temporal scales may lead to the development of turbulence in the disk.

The thin disk model (MTD) is the widely used for studies of the accretion disk dynamics. The MTD involves averaging over the z coordinate of the three-dimensional hydrodynamic equations, assuming a number of additional conditions to be fulfilled (Shakura & Sunyaev 1973; Gor'kavyi & Fridman 1994; Khoperskov & Khrapov 1995). In the context of two-dimensional models without magnetic field, four unstable modes of oscillations are present: two acoustic modes (Wallinder 1991; Wu & Yang 1994; Khoperskov & Khrapov 1995; Wu et al. 1995), a thermal mode and a viscous one (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Szuszkiewicz 1990; Wallinder 1991; Wu & Yang 1994). The instability growth rate of acoustic waves increases with smaller wavelengths λ. However, the thin disk model imposes a restriction on wavelength, $\lambda \gg h$, and therefore it is necessary to consider AD z-structure for a correct treatment when $\lambda \lesssim h$. The thin disk model applies only to pinching oscillations. Then the perturbed pressure is a symmetric function, and the displacement of gas does not change the mass centre in a disk with respect to the plane of symmetry ($z = 0$). The MTD cannot account for bending oscillations (AS-mode) because then the perturbed pressure is antisymmetric. Also, the high-frequency (reflective) harmonics with characteristic spatial scales in the z-direction $\lesssim (0.5\text{–}1)h$ cannot be investigated in the MTD.

In this paper we investigate the dynamics of acoustic perturbations taking into account the non-homogeneous z-structure of a viscous disk. Apart from the special problem of the limits of MTD applicability, the main question is the existence of short-wave instabilities and, in addition, the stability of the high-frequency harmonics. In Sect. 2 we define the AD model and we choose the viscosity law. In Sect. 3 we consider the dynamics of linear acoustic perturbations, and formulate the mathematical problem of eigenfrequency determination for various unstable modes in the disk. Lastly we discuss in Sects. 4 and 5 results of the numerical solution of the boundary problem and summarize the main conclusions.

2. Model and basic equations

We consider an axisymmetric differentially rotating gas disk in the gravitational field of a mass M. Without including self-gravity and relativistic effects, and adopting cylindrical coordinates we have:

$$\Psi(r,z) = -\frac{GM}{(r^2+z^2)^{1/2}} \simeq -\frac{GM}{r} + \frac{1}{2}\Omega_k^2 z^2 \,, \tag{1}$$

G is the gravitational constant, $\Omega_k = \sqrt{GM/r^3}$ is the Keplerian angular velocity.

We use the axisymmetric hydrodynamic equations in view of viscosity. The equations of motion and continuity have the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{v^2}{r} = \frac{1}{\rho}\left(-\frac{\partial P}{\partial r} + \frac{\partial r\sigma_{rr}}{r\partial r} - \frac{\sigma_{\varphi\varphi}}{r} + \frac{\partial \sigma_{rz}}{\partial z}\right) - \frac{\partial \Psi}{\partial r}\,, \quad (2)$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{uv}{r} = \frac{1}{\rho}\left(\frac{\partial \sigma_{\varphi z}}{\partial z} + \frac{\partial r^2\sigma_{r\varphi}}{r^2\partial r}\right)\,, \quad (3)$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{\rho}\left(-\frac{\partial P}{\partial z} + \frac{\partial r\sigma_{rz}}{r\partial r} + \frac{\partial \sigma_{zz}}{\partial z}\right) - \frac{\partial \Psi}{\partial z}\,, \quad (4)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (r\rho u)}{r\partial r} + \frac{\partial (\rho w)}{\partial z} = 0\,, \quad (5)$$

where $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}$, $\boldsymbol{V} = (u, v, w)$ is the velocity, P is the pressure, ρ is the volume density of matter in the disk, σ_{ij} are the components of the symmetric viscous stress tensor ($\sigma_{ij} = \sigma_{ji}$).

We add the thermal equation to the system of Eqs. (2)–(5) as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{Q}{T}\,, \quad (6)$$

where S is the entropy, T is the temperature, and the variable Q defines the sources of heat.

2.1. Equilibrium model and viscosity law

We shall assume that the equilibrium velocity in the disk has only r and φ components: $\boldsymbol{V}_0 = (U_0,\, V_0,\, 0)$. The components of the viscous stress tensor can be written in the following form:

$$\sigma_{ij} = -\alpha_{ij}\, P\,, \qquad \alpha_{ij} = \mathrm{const} > 0\,, \quad (7)$$

where $i, j = (r, \varphi, z)$. As if we ignore the second (volume) viscosity the trace of the viscous tensor is equal to zero ($\delta_{ij}\sigma_{ji} = 0$), $\alpha_{ii} = 0$ ($\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{zz} = 0$).

The parameters α_{rz}, $\alpha_{\varphi z}$ and $\alpha_{r\varphi}$ determine the level of turbulence in the disk. Moreover α_{rz} and $\alpha_{\varphi z}$ are caused by the shear character of flow in the z-direction, and the value $\alpha_{r\varphi}$ is connected to the differential rotation in the disk plane and coincides with the α-parameter of the standard theory of disk accretion (Shakura & Sunyaev 1973). For α_{ij} one can write:

$$\alpha_{r\varphi} = -\frac{\eta_0}{P_0}\, l_\Omega \Omega\,, \alpha_{rz} = -\frac{\eta_0}{P_0}\,\frac{\partial U_0}{\partial z}\,, \alpha_{\varphi z} = -\frac{\eta_0}{P_0}\,\frac{\partial V_0}{\partial z}\,, \quad (8)$$

where $l_\Omega = \partial(\ln\Omega)/\partial(\ln r)$. This is limited to the case of weak dependence of equilibrium velocities (U_0 and V_0) on the z-coordinate for a correct transition to the thin disk model. This is possible if the following conditions: $\alpha_{rz} \ll \alpha$ and $\alpha_{\varphi z} \ll \alpha$ are fulfilled. Therefore one can ignore the terms containing σ_{rz} and $\sigma_{\varphi z}$. The conditions $U_0 \simeq \alpha(h/r)^2 V_0$ and $h/r \ll 1$ are assumed in most models of accretion disks. With these assumptions the equilibrium balance of the forces is defined by:

$$\frac{V_0^2}{r} = \frac{1}{\rho_0}\,\frac{\partial P_0}{\partial r} + \frac{\partial \Psi}{\partial r}\,, \quad (9)$$

$$U_0\,\frac{\partial r V_0}{r\partial r} = -\frac{\alpha}{\rho_0}\,\frac{\partial r^2 P_0}{r^2\partial r}\,, \quad (10)$$

$$\frac{\partial P_0}{\partial z} = -\rho_0\,\frac{\partial \Psi}{\partial z}\,. \quad (11)$$

The equilibrium functions can be written in the self-similar form: $f_0(r, z) = f_{01}(r) f_{02}(z)$, if it is assumed $h/r \ll 1$. The disk z-structure depends on the equation of state and on the energy flux. For the sake of simplicity we limit ourselves to a polytropic model

$$\frac{\partial}{\partial z}\left\{\frac{P_0(r, z)}{[\rho_0(r, z)]^n}\right\} = 0\,. \quad (12)$$

Then the solutions for the pressure and density are:

$$P_0(r, z) = P_0(r, 0)\, F(z)^a\,,\ \rho_0(r, z) = \rho_0(r, 0)\, F(z)^b\,, \quad (13)$$

where $F(z) = 1 - z^2/h^2$, $a = n/(n-1)$, $b = 1/(n-1)$, n is the polytropic index, and h defines the disk boundary — at the points $z = \pm h$ the equilibrium pressure and density are equal to zero. We assume also that $h(r) = const$.

The relation

$$C_s^2(r, z) = \frac{\gamma P_0}{\rho_0} = C_s^2(r, 0)\, F(z)\,,\ C_s^2(r, 0) = \frac{\gamma}{a}\,\Omega_k^2 h^2\,, \quad (14)$$

defines the adiabatic sound speed (γ is the adiabatic index). For the equilibrium velocities (V_0 and U_0) we obtain from Eqs. (9), (10) taking into account Eqs. (13) and (14):

$$V_0(r, z) = r\Omega(r, z) = \sqrt{r^2\Omega_k^2 + l_P C_s^2/\gamma}\,, \quad (15)$$

$$U_0(r, z) = -\frac{\alpha(2 + l_P)}{\gamma(1 + l_V)}\,\frac{C_s^2}{V_0}\,, \quad (16)$$

where $l_P = \partial(\ln P_0)/\partial(\ln r)$, $l_\rho = \partial(\ln \rho_0)/\partial(\ln r)$, $l_V = \partial(\ln V_0)/\partial(\ln r)$.

3. Linear analysis

In the framework of the standard linear analysis the pressure, density and velocity are represented as:

$$u = U_0(r, z) + \tilde{u}(r, z, t)\,,$$

$$v = V_0(r, z) + \tilde{v}(r, z, t)\,,$$

$$w = \tilde{w}(r, z, t)\,,$$

$$P = P_0(r, z) + \tilde{P}(r, z, t)\,,$$

$$\rho = \rho_0(r, z) + \tilde{\rho}(r, z, t)\,.$$

In the linear approximation ($|\tilde{f}| \ll |f_0|$) from Eqs. (1)–(5) the linearized equations become:

$$\frac{\partial \tilde{u}}{\partial t} + U_0\frac{\partial \tilde{u}}{\partial r} + \tilde{u}\frac{\partial U_0}{\partial r} + \tilde{w}\frac{\partial U_0}{\partial z} - \frac{2V_0}{r}\,\tilde{v} =$$

$$-\frac{1}{\rho_0}\left(\frac{\partial \tilde{P}}{\partial r}-\frac{\tilde{\rho}}{\rho_0}\frac{\partial P_0}{\partial r}\right), \tag{17}$$

$$\frac{\partial \tilde{v}}{\partial t}+U_0\frac{\partial(r\tilde{v})}{r\partial r}+\tilde{u}\frac{\partial(rV_0)}{r\partial r}+\tilde{w}\frac{\partial V_0}{\partial z}=$$

$$-\frac{\alpha}{\rho_0}\left(\frac{\partial(r^2\tilde{P})}{r^2\partial r}-\frac{\tilde{\rho}}{\rho_0}\frac{\partial(r^2P_0)}{r^2\partial r}\right), \tag{18}$$

$$\frac{\partial \tilde{w}}{\partial t}+U_0\frac{\partial \tilde{w}}{\partial r}=-\frac{\partial \tilde{P}}{\rho_0\partial z}-g\frac{\tilde{\rho}}{\rho_0}, \tag{19}$$

$$\frac{\partial \tilde{\rho}}{\partial t}+\frac{\partial}{r\partial r}\left[r(U_0\tilde{\rho}+\rho_0\tilde{u})\right]+\frac{\partial(\rho_0\tilde{w})}{\partial z}=0, \tag{20}$$

where $g\equiv\partial\Psi/\partial z$.

As we study only the dynamics of acoustic oscillations, we set $Q=0$. Then the thermal Eq. (6) becomes:

$$\frac{\partial \tilde{S}}{\partial t}+U_0\frac{\partial \tilde{S}}{\partial r}+\tilde{u}\frac{\partial S_0}{\partial r}+\tilde{w}\frac{\partial S_0}{\partial z}=0, \tag{21}$$

$S_0=c_V\ln(P_0/\rho_0^\gamma)$ being the entropy of the gas at equilibrium. We eliminate $\tilde{S}$ from Eq. (21) with the help of the equation of state $\tilde{S}=\tilde{S}(\tilde{P},\tilde{\rho})$. In the linear approximation we obtain:

$$\tilde{S}=\left(\frac{\partial S}{\partial P}\right)_\rho\tilde{P}+\left(\frac{\partial S}{\partial \rho}\right)_P\tilde{\rho}=c_V\frac{\tilde{P}}{P_0}-c_P\frac{\tilde{\rho}}{\rho_0}, \tag{22}$$

where $c_V=T(\partial S/\partial T)_\rho$, $c_P=T(\partial S/\partial T)_P$ are the specific heats at constant density and pressure.

The short-wave approximation in the radial direction ($kr\gg 1$, k being the radial wave number) allows to write the solution as:

$$\tilde{f}(r,z,t)=\hat{f}(z)\,\exp\{\mathrm{i}kr-\mathrm{i}\omega t\}, \tag{23}$$

ω being the complex eigenfrequency.

Taking into account Eqs. (22) and (23), the system Eqs. (17)–(21) is reduced to two ordinary differential equations for the amplitudes of the perturbed pressure $\hat{P}(z)$ and the material displacement $\hat{\xi}(z)$ from an equilibrium position:

$$\frac{\mathrm{d}\hat{\xi}}{\mathrm{d}z}=\frac{D_1}{\hat{\omega}q^2C_s^2}\frac{\hat{P}}{\rho_0}+\left\{\frac{kD_2}{\hat{\omega}q^2}+\frac{g}{C_s^2}\right\}\hat{\xi}, \tag{24}$$

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}z}=\rho_0\left\{\hat{\omega}^2-\frac{g}{\gamma}D_3\right\}\hat{\xi}-\frac{g}{C_s^2}\left\{1+D_4\right\}\hat{P}, \tag{25}$$

where $\hat{\omega}=\omega-kU_0$, $q^2=\hat{\omega}^2-æ^2$, $æ=\Omega\sqrt{2(2+l_\Omega)}$ being the epicyclic frequency, $D_1=-\hat{\omega}^3+\hat{\omega}(æ^2+k^2C_s^2)+2\mathrm{i}\alpha\Omega k^2C_s^2$, $D_2=\mathrm{i}\omega F_V-2\Omega F_U$, $D_3=S_z-\frac{\mathrm{i}S_r}{\hat{\omega}q^2}(\hat{\omega}F_U+2\mathrm{i}\Omega F_V)$, $D_4=\frac{\mathrm{i}kS_r}{\gamma\hat{\omega}q^2}(\hat{\omega}+2\mathrm{i}\alpha\Omega)$, $F_V=\frac{l_PC_s^2}{r\gamma^2}S_z-2\Omega V_0'$, $F_U=\frac{\alpha(2+l_P)C_s^2}{r\gamma^2}S_z+\frac{æ^2}{2\Omega}U_0'$, $S_z=\frac{1}{c_V}\frac{\partial S_0}{\partial z}=(\ln P_0)'-\gamma(\ln\rho_0)'$, $S_r=\frac{1}{c_V}\frac{\partial S_0}{\partial r}=(l_P-\gamma l_\rho)/r$. The prime sign means differentiation with respect to z-coordinate ($f'\equiv\partial f/\partial z$). $\hat{\xi}$ is the complex amplitude of the material z-displacement from equilibrium state. We have $\tilde{w}=\mathrm{d}\tilde{\xi}/\mathrm{d}t=-\mathrm{i}\hat{\omega}\,\tilde{\xi}$.

Boundary conditions must be defined for solving Eqs. (24) and (25). Given the symmetry, it is natural to consider two types of oscillations: 1) the symmetric oscillations ($\hat{\xi}(z)=-\hat{\xi}(-z)$ or $\hat{P}(z)=\hat{P}(-z)$), and therefore

$$\hat{\xi}(0)=0\qquad\text{or}\qquad\frac{\mathrm{d}\hat{P}}{\mathrm{d}z}\Big|_{z=0}=0, \tag{26}$$

(such oscillations correspond to the pinch-mode or S-mode); 2) the antisymmetric oscillations ($\hat{\xi}(z)=\hat{\xi}(-z)$ or $\hat{P}(z)=-\hat{P}(-z)$), and therefore

$$\hat{P}(0)=0\qquad\text{or}\qquad\frac{\mathrm{d}\hat{\xi}}{\mathrm{d}z}\Big|_{z=0}=0, \tag{27}$$

that correspond to the bending oscillations or AS-mode. On the disk unperturbed surface the following condition should be fulfilled:

$$\hat{P}(h)+\frac{\partial P_0}{\partial z}\Big|_{z=h}\hat{\xi}(h)=0. \tag{28}$$

Solving the system of Eqs. (24), (25) with the boundary conditions Eqs. (28) and (26) (or (27)), we find the eigenvalues of complex frequency ω for a given distribution of the equilibrium parameters along the vertical coordinate in the disk. A positive imaginary part of the frequency (growth rate) means that the eigen-mode is unstable. This will define the "3D-model", the thin disk model being called the "2D-model".

3.1. *Dispersion relation*

For a homogeneous z-distribution of the equilibrium quantities, the system of Eqs. (24) and (25) is reduced to the following dispersion relation:

$$\omega^4-\omega^2\left[æ^2+C_s^2(k^2+k_z^2)\right]$$
$$-2\mathrm{i}\omega\alpha\Omega C_s^2k^2+æ^2C_s^2k_z^2=0, \tag{29}$$

where k_z is the wave number in z-direction. We stress that Eq. (29) at $k_z=0$ coincides with the dispersion relation obtained earlier for the 2D-model if we replace γ by the 2D adiabatic index $\Gamma=\Gamma_1=(3\gamma-1)/(\gamma+1)$ (Wallinder 1991; Khoperskov & Khrapov 1995). Kovalenko and Lukin (1998) have obtained a more accurate formula for the 2D adiabatic index $\Gamma=\Gamma_2=(3\gamma-1-\gamma\delta)/(\gamma+1-\delta)$ ($\delta=æ^2/\Omega_k^2$). This relation takes into account the Keplerian rotation of a thin disk. The adiabatic sound speed in the disk plane is given by the following expression:

$$C_s^2=\Gamma\,\frac{\int_0^h P(z)\,\mathrm{d}z}{\int_0^h\rho(z)\,\mathrm{d}z}. \tag{30}$$

Thus, taking into account Eq. (30), the dispersion relation Eq. (29) with $k_z=0$ describes the dynamics of perturbations within the limits of the flat model (MTD). Obviously, these oscillations correspond to the S-mode.

In the general case where $\alpha > 0$, this dispersion relation gives two unstable acoustic branches of oscillations and two damping modes (viscous and thermal). The damping of the viscous or thermal modes depends on whether or not dissipation and radiative processes are taken into account in the thermal equation. If these factors are taken into account, then the viscous and thermal low-frequency oscillatory branches can be unstable (Shakura & Sunyaev 1976), but this will not change the dispersion properties of sound waves.

4. Results

It is convenient to characterize the properties of the considered acoustic oscillations in the disk by a dimensionless frequency $W = \omega/\Omega_k$ and a dimensionless wave number kh. We define the basic model as follows: $\alpha = 0.2$, $\gamma = 5/3$, $h/r = 0.05$, $l_P = -3/2$, $l_\rho = -1/2$, $l_V = -1/2$ $(l_\Omega = -3/2)$, $n = 5/3$. If not specified otherwise, the parameters take these values.

4.1. Fundamental S-mode in the 2D- and 3D-models

Our analysis confirms the good agreement between the thin disk model and the results of the 3D-problem solution in the case of the dissipative acoustic instability. The eigenfrequencies of oscillations obtained from Eq. (29) at $k_z = 0$ using Eq. (30) and the solution of Eqs. (24), (25) practically coincide in the range $kh \lesssim 3$ (see Fig. 1a,b). Appreciable differences occur when $kh \gtrsim 3$. We emphasize that in this region the formal condition of applicability of the MTD ($kh \ll 1$) is obviously broken. In Fig. 1e,f, the dependences of the group velocity $v_g = \mathrm{Re}\,(\partial\omega/\partial k)$ and $\mathrm{Im}\,(\partial\omega/\partial k)$ on the wave number k for the 3D-model and the 2D-model at $\Gamma = \Gamma_1$ and $\Gamma = \Gamma_2$ are shown. In the case of small values of n, a difference occur at large kh as Fig. 1a,b shows. The reason is that the characteristic scale of inhomogeneity in the vertical direction increases at smaller n. The growth rate and the phase velocity of perturbations in the framework of the 3D-model are smaller than for the thin disk model (2D-model) in the short-wavelength region. This effect is caused by the inhomogeneous distribution of the equilibrium quantities in z-direction and by the transverse gravitational force. When the parameter α increases the growth rates grow linearly. The value of the wave number k, at which the differences between the models appear, does not depend on α. The considered low-frequency mode exists in both the 2D-model and in the 3D-model, and it depends weakly on the z-structure. We therefore call it the *fundamental mode*. When $\alpha \ll 1$, the expression for frequency is easily obtained from Eq. (29) as:

$$\omega = \pm\sqrt{æ^2 + k^2 C_s^2} + i\alpha \frac{\Omega C_s^2 k^2}{æ^2 + C_s^2 k^2}\,.$$

This approximation is sufficiently exact. Our results shows that the MTD adequately describes the dynamics of perturbations with characteristic spatial scales $\lambda \geq 2h$ in the disk plane.

The differential rotation ($l_\Omega = -\dfrac{d\ln\Omega}{d\ln r} > 0$) and the dependence of dynamic viscosity η on the thermodynamic parameters (for example, the dependence on density and temperature) are responsible for the instability. A simple model can demonstrate this. The last term in Eq. (3) (i.e., containing $\sigma_{r\varphi}$) is a source of instability of the acoustic modes. The expression $\eta = \sigma\nu$ (σ is the surface density, ν is the kinematic viscosity) is used in the 2D-model. For simplicity we assume here that ν is constant. Then a perturbation of the surface density $\tilde\sigma$ generates a perturbation $\tilde\eta$ of the dynamic viscosity. Now let us consider how a viscous force, which is caused by $\tilde\eta$, leads to an amplification of the amplitude of a sound wave in the plane of the disk. The evolution equation for the surface density, without including Coriolis's force, is $\dfrac{\partial}{\partial t}\left[\dfrac{\partial^2\tilde\sigma}{\partial t^2} - C_s^2\dfrac{\partial^2\tilde\sigma}{\partial r^2}\right] = A\dfrac{\partial^2\tilde\sigma}{\partial r^2}$, where $A = 2\Omega^2 l_\Omega \nu_0 > 0$. This equation is linear and the appropriate dispersion relation is $\omega[\omega^2 - C_s^2 k^2] = \mathrm{i}Ak^2$, which means that acoustic waves are unstable with a growth rate $\Im(\omega) \simeq \dfrac{\mathrm{i}}{2}\dfrac{A}{C_s^2}$ and entropy oscillations damping ($\Im(\omega) \simeq -\mathrm{i}A/C_s^2 < 0$). These estimates can only be used in the short-wavelength limit, because we have neglected the epicyclic oscillations.

4.2. Fundamental AS-mode

We now consider the low-frequency bending oscillations for which Eq. (27) is fulfilled. In the long-wavelength limit both symmetric and antisymmetric perturbations have the same frequency $\omega^2 \simeq æ^2 = \Omega_k^2$ in a Keplerian disk. This result can easily be obtained from Eq. (29) when $k = 0$. In the adiabatic model two other branches of oscillations ($\omega^2 = (C_s k_z)^2$) always damp at $k > 0$ and we do not examine them. The dispersion behaviour of S- and AS-modes is very similar (Fig. 1a–d, solid lines). For the fundamental bending mode it is possible to set $k_z h \simeq \pi/2$ in Eq. (29). The dispersion curves on Fig. 1c,d are shown as dotted lines for this case. The exact solution in the 3D-model exhibits differences from the result derived from Eq. (29). In the range $kh \lesssim 0.1$ these differences are caused by the following factors: 1) the 3D-model includes radial inhomogeneities of the equilibrium quantities; 2) the Eq. (29) does not take into account the vertical inhomogeneity of the disk. The result obtained is very unexpected and significant, since we used Eq. (29) beyond the limit of the 2D-model.

The physical reason for the AS-mode instability is similar to that for the S-oscillations.

4.3. High-frequency S- and AS-modes

Besides the fundamental S-mode, unstable harmonics can be generated. These harmonics differ from each other by the number of nodes of the perturbed pressure across the disk plane. The following estimate of the effective wave number in the z-direction can be made:

$$k_z h \simeq \pi j\,(\mathrm{S-mode}) \qquad k_z h \simeq \pi(j+1/2)\,(\mathrm{AS-mode})\,,$$

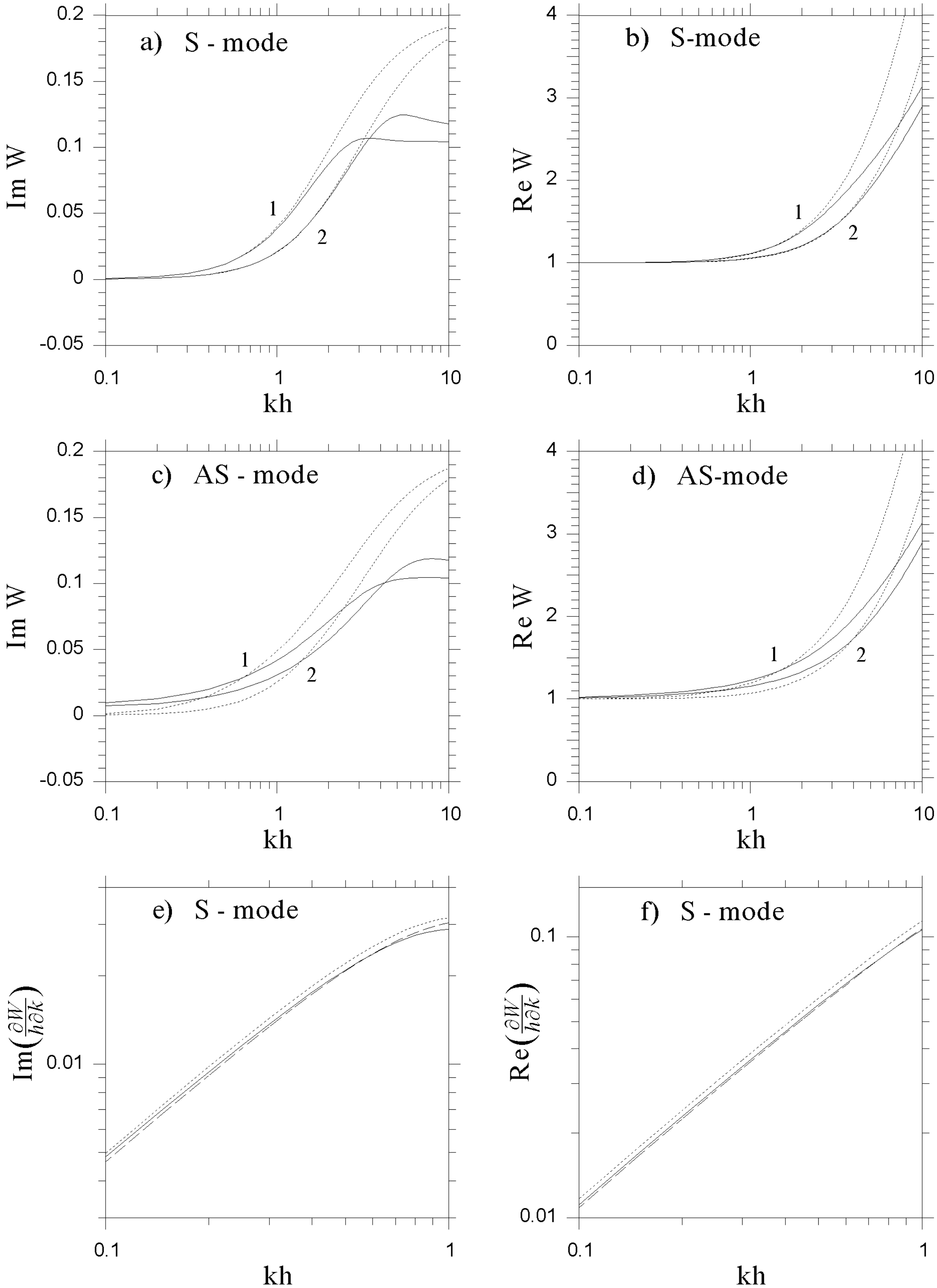

Fig. 1a–f. The dependence of the dimensionless eigenfrequency W on the dimensionless wave number kh for different values of n ($1-n=5/3$, $2-n=1.2$) (**a-d**). The dependence of $\partial W/h\partial k$ on kh for the S-mode (**e,f**). The solid lines correspond to the 3D-model, dotted lines to the 2D-model at $\Gamma=\Gamma_1$, long-dashed lines to the 2D-model at $\Gamma=\Gamma_2$.

where j is the number of the harmonic. The fundamental ($j=0$) and reflective ($j>0$) harmonics exist for both symmetric (S-) and antisymmetric (AS-) modes.

The dependences of the eigen-frequency ω on the radial wave number k for the fundamental mode $j=0$ and the first

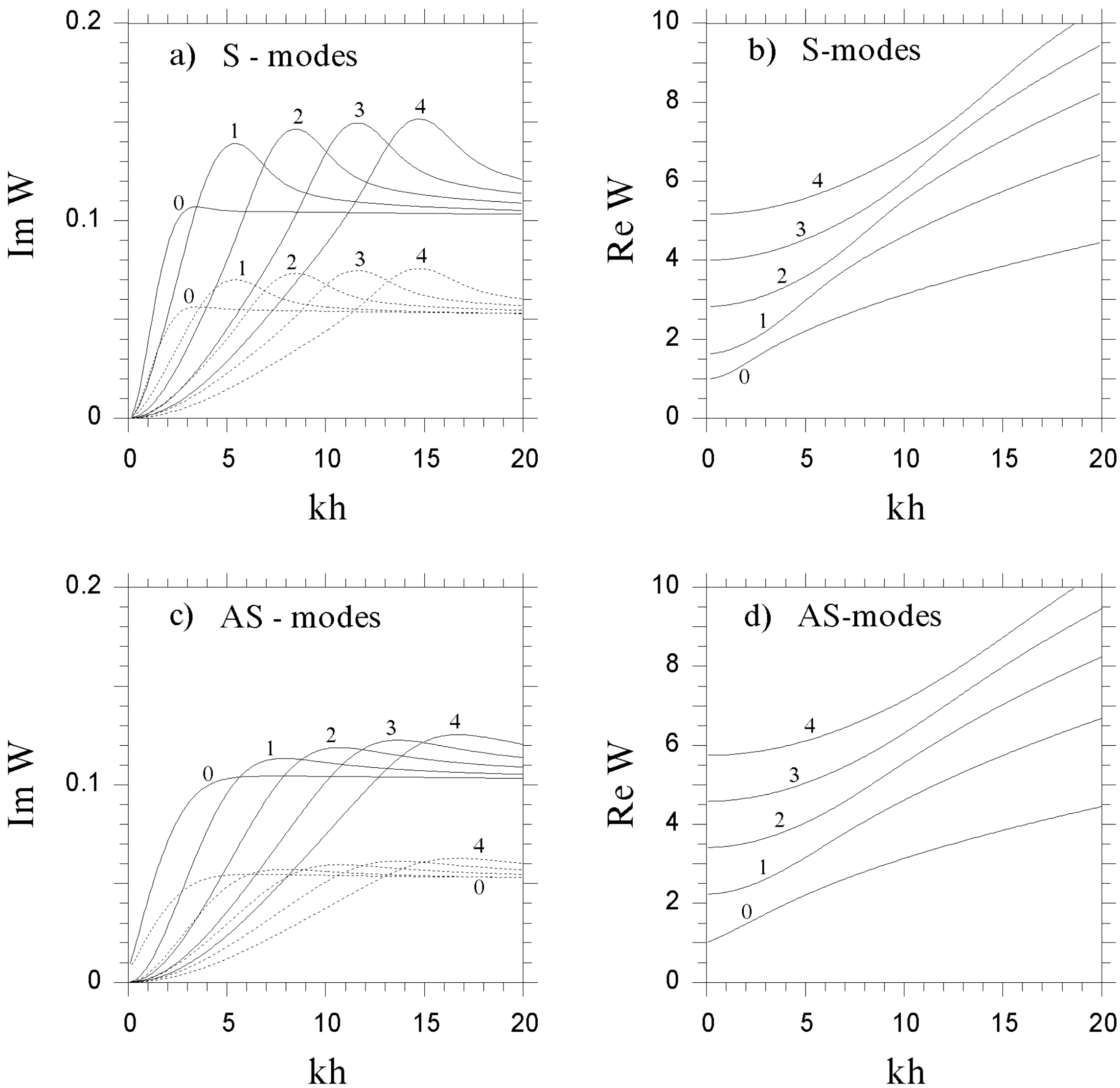

Fig. 2a–d. The dependence of W on kh for S- and AS-modes for various values of α. The solid line corresponds to $\alpha = 0.2$, the dotted line to $\alpha = 0.1$. The harmonic numbers j are indicated.

four reflective harmonics $j = 1, 2, 3, 4$ are displayed in Fig. 2a–d. The pinch-oscillations and the bending modes are both unstable ($\Im(\omega) > 0$). The imaginary part of the frequency grows with k and reaches a maximum at some value. The maximum of $\Im(\omega)$ moves to the region of shorter wavelengths when the harmonic number j grows, and the value increases with reduction of the characteristic scale of perturbations in z-direction. As indicated by Fig. 2a–d, the growth rate increases with α, while the perturbed phase velocity $\Re(\omega)/k$ does not depend on α. It should be noted that for very short-wave perturbations ($kh \gtrsim 10$), the damping of oscillations due to the presence of a perturbed velocity gradient in the viscous stress tensor can be of decisive importance (Khoperskov & Khrapov 1995). This factor can be essential for small-scale waves (as on r- and on z-coordinate) and it can result in complete stabilization.

The vertical structure of the disk in the 3D-model is determined primarily by the parameters γ and n. When $n = \gamma$, the entropy does not vary along the z-coordinate, i.e. the disk is stable against convective z-motions. When $n > \gamma$ we have $\partial S_0/\partial z < 0$ and the conditions for convective instability are fulfilled; in the opposite case $n < \gamma$, $\partial S_0/\partial z > 0$, so convection cannot appear. In Fig. 3a–d, the dispersion curves for various values of n ($n > \gamma$, $n < \gamma$ and $n = \gamma$) are shown for the fundamental modes ($j = 0$) and for the third reflective harmonic ($j = 3$). The phase velocity of the fundamental S- and AS-modes weakly depends on the parameter n (see Fig. 3b,d). The adiabatic sound speed in the disk, C_s grows with n (see Eq. (14)), and since for the high-frequency reflective harmonics $k_z \sim |d\hat{\xi}/\hat{\xi}dz| \neq 0$, the phase velocity of the perturbations with $j > 0$ also increases (see Fig. 3b,d).

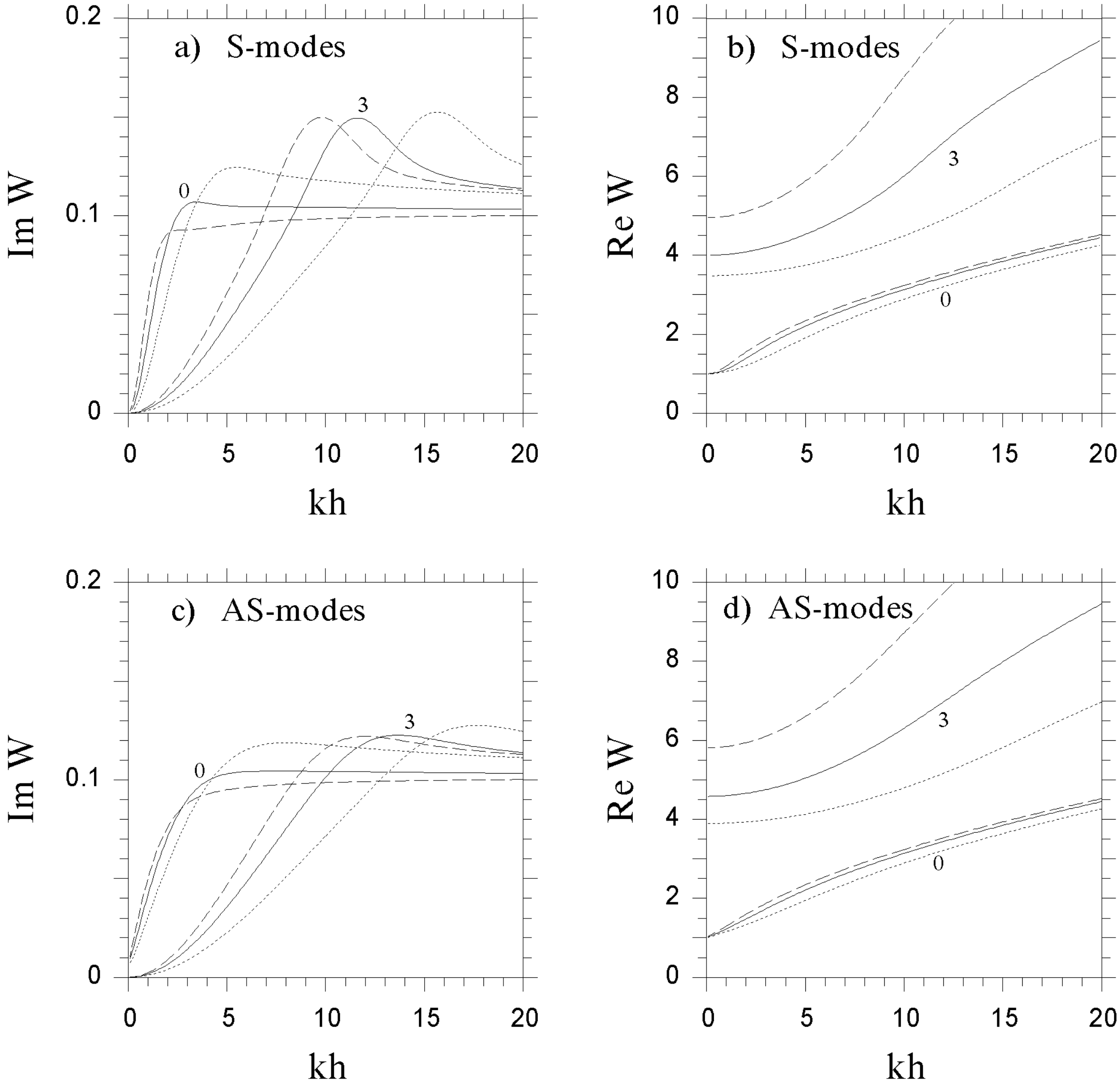

Fig. 3a–d. The dependence of W on kh for the fundamental mode ($j = 0$) and for the high-frequency one with $j = 3$ for various values of n. Solid line: $n = 5/3$, dotted line: $n = 1.2$, dashed line: $n = 10$. The harmonic numbers j are specified.

The dispersion properties of the acoustic perturbations do not depend on the values of the parameters l_P and l_ρ in the range $kh > 0.2$. The reason for this is that the radial gradients of equilibrium pressure and density give only a small contribution to the equilibrium balance in the case of a thin disk.

5. Discussion and conclusions

Acoustic waves in a differentially rotating gaseous disk can play an important role for understanding the turbulent viscosity in accretion disks. The amplitude of small-scale (in r and in z) waves grows most rapidly. Such instabilities do not destroy the initial flow at a nonlinear stage, but can effectively make the disk matter turbulent, and in turn, the turbulent viscosity generates unstable sound modes. A self-consistent regime with turbulent viscosity arises. Beside the dissipation mechanism analysed above, the development of global resonant Papaloizou-Pringle modes (Papaloizou & Pringle 1987; Savonije & Heemskerk 1990), the resonant amplification of acoustic oscillations in the regime of double-flow accretion (Mustsevoj & Khoperskov 1991) and in the case of disk accretion onto a magnetized compact object (Hoperskov et al. 1993) may be important for understanding the turbulent viscosity. It is remarkable that in all cases the characteristic time scale does not exceed the dynamical time scale ($\tau = 1/\Im(\omega) \sim \Omega^{-1}$), and that all values are of the same order.

We have demonstrated the possibility of unstable high-frequency acoustic waves in a differentially rotating gaseous disk. They exist in the system for a limited period, and the presence of a positive growth rate does not mean that the perturbations reach the nonlinear stage. The perturbations leave the disk with a speed $\partial\omega/\partial k \sim C_s kh/\sqrt{1+k^2h^2} < C_s$ and the characteristic life-time in the disk is $\sim r/h \sim 10^2$ disk rotation periods. In view of the derived estimation $\Im(\omega) \sim 0.1\Omega$ for $kh > 1$ we have obtained the growth rate of wave amplitude

($\exp\{0.1r/h\}$). Nonaxisymmetric perturbations may stay in the system for a longer time and reach a non-linear stage.

The main conclusions of this study are:

1. The linear wave dynamics of a thin disk model is compared to an exact solution, which takes into account the vertical structure. A small quantitative difference in the dispersion properties occur for a wavelength $\lambda < 2\pi h$, and even if $\lambda > 2\pi h$ the dispersion properties remain qualitatively similar. At $kh \leq 1$ the eigenfrequencies differ less than on 5%.
2. In addition to the fundamental dissipational unstable sound mode in MTD, an arbitrary number of high-frequency unstable harmonics of the pinch-oscillations is found. The different harmonics have a different vertical structure. The waves with small scales along the z-coordinate have a maximum growth rate at short wavelengths in the r-direction. The differential rotation and the variable dynamic viscosity cause instability of all oscillation branches, i.e., a dissipational mechanism lead to the growth of perturbations.
3. Taking into account the vertical structure of the disk we studied new bending modes, which cannot be investigated in the context of MTD. The bending oscillations, as well as the pinch-wave, are unstable. The physical mechanism of instability and the dispersion properties are similar to those of the pinch-oscillations. The significant feature of all these unstable acoustic modes is the fact that the spatial scales of the perturbations differ from each other, but that their characteristic growth rates have all the same order of magnitude.

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