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ASTRONOMY AND ASTROPHYSICS

# Phase mixing of Alfvén waves in a stratified and open atmosphere

**I. De Moortel**[1], **A.W. Hood**[1], **J. Ireland**[2], **and T.D. Arber**[1]

[1] School of Mathematical and Computational Sciences, University of St Andrews, North Haugh, St Andrews, Fife KY16 9SS, Scotland
[2] NASA/Goddard Space Flight Center, Greenbelt, MD 20711, USA

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**Abstract.** Phase mixing was introduced by Heyvaerts and Priest (1983) as a mechanism for heating the plasma in the open magnetic field regions of coronal holes. Here the basic process is modified to include a stratified atmosphere in which the density decreases with height. We present an analytical solution in the case of zero dissipation and use a numerical code in the non-zero dissipation case to describe the effect of stratification on phase mixing. The exponential damping behaviour derived by Heyvaerts and Priest is largely confirmed in the non stratified limit. However, it is shown that the decrease in density lengthens the oscillation wavelengths and thereby reduces the generation of transverse gradients. Furthermore we found that in a stratified atmosphere the perturbed magnetic field and velocity behave quite differently depending on whether we consider resistivity or viscosity. Ohmic heating is spread out over a greater height range in a stratified medium whereas viscous heating is not strongly influenced by the stratification.

**Key words:** Magnetohydrodynamics (MHD) – waves – Sun: corona

## 1. Introduction

The coronal heating mechanism remains one of the major unsolved problems in solar physics. As the energy flux carried by acoustic waves is too small, the possibility of heating by MHD waves has been investigated intensively since the magnetic structure of the corona must play an important role. However, a heating theory based on waves faces a double problem. Firstly, the waves have to transport a sufficient energy flux and secondly, they have to dissipate efficiently in order to deposit the right amount of energy at the right place. A prime candidate for transferring energy up to chromospheric and coronal levels is a flux of Alfvén waves. Since the difficulty of damping Alfvén waves has been recognised, various effects on the propagation of these waves have been studied. Heyvaerts and Priest (1983) proposed a simple but promising idea for the behaviour of Alfvén waves when the local Alfvén speed varies across the magnetic field lines. They suggested damping of Alfvén waves due to phase mixing could be a possible source of coronal heating. Since then, the propagation and damping of shear Alfvén waves in an inhomogeneous medium has been studied in more detail (Ireland 1997; Cally 1991; Browning 1984; Nocera 1983) by relaxing the Heyvaerts and Priest limits of weak damping and strong phase mixing. Recently, Hood et al. (1997a, 1997b) have found analytical, self-similar solutions describing phase mixing of Alfvén waves in both open (coronal holes) and closed (coronal loops) magnetic configurations. Possible observational evidence of coronal heating by phase mixing is discussed by Ireland (1996). The propagation of Alfvén waves in stratified (stellar) atmospheres has been studied in the context of the acceleration of stellar winds, without taking into account the inhomogeneity of the plasma in the horizontal direction (Moore et al. 1991, 1992; Lou & Rosner 1994). The acceleration is thought to be due to the reflection of the Alfvén waves from the vertical inhomogeneity in the Alfvén speed, caused by the stratification of the stellar atmosphere. Recently, Ruderman et al. (1998) considered phase mixing of Alfvén waves in planar two-dimensional open magnetic configurations, using a WKB method. However, the validity of the WKB technique requires a particular relationship between the magnetic Reynolds number, the wavelength of the basic Alfvén wave and the coronal pressure scale height. WKB solutions are presented in Sect. 3.

In this paper we aim to study the effect of both vertical and horizontal density stratifications on the phase mixing of Alfvén waves in an open magnetic atmosphere. We restrict ourself to a study of travelling waves, generated by photospheric motions and disturbances and propagating outwards from the Sun without total reflection. The inhomogeneity of the atmosphere in the vertical direction has important consequences for the efficiency of phase mixing.

## 2. Stratified atmosphere, zero dissipation

### 2.1. Equilibrium and linearised MHD equations

The basic equilibrium that we are considering in order to explain the effect of gravity on phase mixing consists of a uniform vertical magnetic field with an inhomogeneous density that generates an inhomogeneous Alfvén speed. If we assume that the background Alfvén speed (only) has gradients in the x-direction, then Alfvén waves on neighbouring field lines, driven with the

*Send offprint requests to*: ineke@mcs.st-and.ac.uk

same frequency, will have different wavelengths. This will cause them to become out of phase as they propagate up in height and, therefore, large gradients will build up. In this way, short lengthscales are created which means dissipation eventually becomes important and allows the energy in the wave to dissipate and heat the plasma. The question is to see how this picture is modified when gravitational stratification is considered.

Assuming a low-$\beta$-plasma and an isothermal atmosphere, i.e. $T_0$ uniform, the equilibrium is expanded in powers of $\beta$. Following Del Zanna et al. (1997), the leading order solution is a vertical, uniform field, $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. At order $\beta$ the vertical magnetohydrostatic force balance equations reduce to

$$\frac{\partial p_0}{\partial z} = -\rho_0 g\,.$$

Therefore,

$$p_0 = p_0(x)e^{-z/H}\,,$$

and

$$\rho_0 = \rho_0(x)e^{-z/H}\,,$$

where $H = \frac{\mathcal{R}T}{\tilde{\mu}g}$ is the pressure scale height. The horizontal force balance determines the finite $\beta$ correction to the magnetic field (Del Zanna et al., 1997). This differs from Ruderman et al. (1998) who expand the equilibrium in powers of the horizontal lengthscales to the vertical scale height, $\frac{x_0}{H}$. To obtain our equilibrium from theirs requires $\beta = O\left(\frac{x_0}{H}\right)$.

Assuming a time dependence of the form $\exp(i\Omega t)$ for both the perturbed magnetic field $\mathbf{B}_1 = b(x,z)e^{i\Omega t}\hat{\mathbf{y}}$ and the velocity $\mathbf{v} = v(x,z)e^{i\Omega t}\hat{\mathbf{y}}$, the linearised MHD equations become:

$$i\Omega\rho_0 v = \frac{B_0}{\mu}\frac{\partial b}{\partial z} + \rho_0\nu\nabla^2 v\,,$$

and

$$i\Omega b = B_0\frac{\partial v}{\partial z} + \eta\nabla^2 b\,,$$

where the magnetic diffusivity $\eta$ and the dynamic viscosity $\rho_0\nu$ only depend on the temperature (Priest 1982) and in the isothermal atmosphere are assumed constant. These equations can be combined to give either

$$b + \frac{\partial}{\partial z}\left(\frac{v_A^2(x,z)}{\Omega^2}\frac{\partial b}{\partial z}\right) + i\Lambda^2\nabla^2 b = 0\,, \tag{1}$$

or

$$\frac{\Omega^2}{v_A^2(x,z)}v + \frac{\partial^2 v}{\partial z^2} + i\Lambda^2\nabla^2 v = 0\,, \tag{2}$$

where $v_A^2 = B_0^2 e^{z/H}/\mu\rho_0(x)$ and $\Lambda^2$ determines the importance of the damping terms. In this paper, for simplicity, we consider either the ohmic heating, and solve Eq. (1) with $\Lambda^2 = \eta/\Omega$, or the viscous heating, and solve Eq. (2) with $\Lambda^2 = \rho_0\nu\mu\Omega/B_0^2$. Including both $\eta$ and $\nu$ at the same time means that, for example, Eq. (2) has an additional higher order term $-\left(\eta\rho_0\nu\mu/B_0^2\right)\nabla^4 v$. This term adds computational complexity and is usually neglected under the assumption that both $\eta$ and $\rho_0\nu$ are small. However, a multiple scales analysis shows that it is important over a height that is proportional to $(\rho_0\nu\eta)^{-1/5}$. If we assume $\rho_0\nu \approx \eta$, we obtain $(\eta)^{-2/5}$. The usual phase mixing length is proportional to $\eta^{-1/3}$ and while asymptotically it is true that we can neglect the extra term, this is not so clear for the values used in the numerical code.

It is convenient to use dimensionless variables for the numerical solution to Eqs. (1) and (2). Thus, we select the equilibrium density profile as

$$\rho_0 = \rho_{00}\frac{e^{-z/H}}{1+\delta\cos(m\pi x/L)}. \tag{3}$$

Now we set $x = x_0\overline{x}$, $z = z_0\overline{z}$ and

$$v_A^2 = v_{A0}^2\overline{v}_A^2 = v_{A0}^2\left(1+\delta\cos\left(\frac{m\pi\overline{x}x_0}{L}\right)\right)e^{z_0\overline{z}/H},$$

with $v_{A0}^2 = B_0^2/\mu\rho_{00}$. Thus, Eq. (1) becomes

$$b + \frac{v_{A0}^2}{\Omega^2 z_0^2}\frac{\partial}{\partial\overline{z}}\left(\overline{v}_A^2\frac{\partial b}{\partial\overline{z}}\right) + i\frac{\eta}{\Omega x_0^2}\left(\frac{\partial^2 b}{\partial\overline{x}^2} + \frac{x_0^2}{z_0^2}\frac{\partial^2 b}{\partial\overline{z}^2}\right) = 0\,. \tag{4}$$

So set $x_0 = L$, where $L$ is the typical width of the coronal hole, and $m$ is the number of density inhomogeneities inside the hole. This could be, for example, the number of coronal plumes. Then set $z_0 = v_{A0}/\Omega$ so that the vertical lengths are related to the wavelength, $2\pi z_0$, of the ideal Alfvén wave. $\overline{H}$ is the ratio of the pressure scale height $H$ to $z_0$ and $\overline{\Lambda}^2 = \eta/\Omega x_0^2$. A similar procedure is used for Eq. (2). From now on we drop the barred variables and emphasise that we are working in terms of dimensionless variables.

The boundary conditions are chosen as

$$v = 0, \qquad x = 0 \quad \text{and} \quad x = 1, \tag{5}$$

$$v = \sin\pi x, \qquad z = 0, \tag{6}$$

on the photospheric base and an outward propagating wave on the upper boundary. To obtain this upper boundary condition we assume that the density remains constant with height and dissipation is negligible outside the computational box so that $v_A^2 = k(x)e^{d/H}$. Then (2), in dimensionless form, becomes

$$\frac{\partial^2 v}{\partial z^2} + k^2(x)e^{-d/H}v = 0\,.$$

The solution corresponding to an outward propagating wave is

$$v \sim \exp(-ik(x)e^{-d/2H}z)\,.$$

Matching $v$ and $\frac{\partial v}{\partial z}$ onto the solution inside the computational box gives

$$\frac{\partial v}{\partial z} = -ik(x)e^{-z/2H}v, \qquad z = d, \tag{7}$$

where $k(x) = (1+\delta\cos(m\pi x))^{-1/2}$. When dissipation is included and the height of the numerical box, i.e. $d$, is taken sufficiently large, the waves are damped and the actual choice of the upper boundary condition is unimportant.

### 2.2. No dissipation

If we neglect dissipation, i.e. $\Lambda^2 = 0$, the solution for the magnetic field perturbations is

$$b = e^{-z/2H}\left\{C_1(x)J_1(\phi) + C_2(x)Y_1(\phi)\right\}, \tag{8}$$

and the velocity is

$$v = D_1(x)J_0(\phi) + D_2(x)Y_0(\phi), \tag{9}$$

where $\phi = 2Hk(x)\exp(-z/2H)$ and $J$ and $Y$ are Bessel functions of order either 0 or 1. The 'constants' $C_1(x), C_2(x), D_1(x)$ and $D_2(x)$ are chosen to satisfy the boundary conditions. The amplitude of the perturbed magnetic field decreases with height, whereas the velocity increases with height.

The ideal MHD solution can be used to investigate the behaviour of the current, $j$, and the vorticity, $\omega$. This allows one to see the regions where phase mixing causes the transverse gradients to build up and so obtain the regions where dissipation will become important.

We are illustrating the effect of stratification through the model Alfvén speed profile $v_A^2 = v_{A0}^2(1 + \delta\cos(m\pi x))e^{z/H}$, where $\delta$ regulates the magnitude of the equilibrium density variations.

The ideal MHD solution may be approximated by a simple WKB solution of the form

$$v = e^{z/4H}\sin\pi x\exp\left(-i2Hk(x)(1 - e^{-z/2H})\right), \tag{10}$$

corresponding to an outward propagating wave. Since we have analytic solutions in the case of no dissipation, the value of $d$ can be taken arbitrarily large and the outward propagating wave is easily identified. Solution (10) agrees with boundary condition (7) in the limit of large H. This approximate solution shows clearly that stratification increases both the wavelength and the amplitude and agrees with the analytical solution (9) that is shown in Figs. 1 and 2.

As the Alfvén speed $v_A \sim e^{z/2H}$, the wavelength $\lambda \sim e^{z/2H}$ as well so the wavelengths will indeed get longer when the scale height $H$ gets smaller. At first sight it appears that the heating of coronal holes by the phase mixing of Alfvén waves will be less efficient than in a unstratified medium since it will take longer for waves on neighbouring field lines to get out of phase and hence, to create the necessary short lengthscales for dissipation to become important! In addition, from Fig. 2, we see that the amplitude of the perturbed magnetic field decreases with height in a stratified atmosphere. However, the amplitude of the perturbed velocity on the other hand increases with height. This suggests that in a stratified atmosphere, viscous heating might be the dominant heating mechanism. In a stratified atmosphere, where the density falls off with height, the amplitude of the velocity $v$ will increase with height. However, because the Poynting flux, which is proportional to $v.b$, is approximately constant (Wright 1998), this implies that the amplitude of the magnetic field $b$ will decrease with height.

Although we are considering the zero dissipation case for the moment, so that we can not strictly talk about ohmic or viscous heating, it is still important to know where the heating

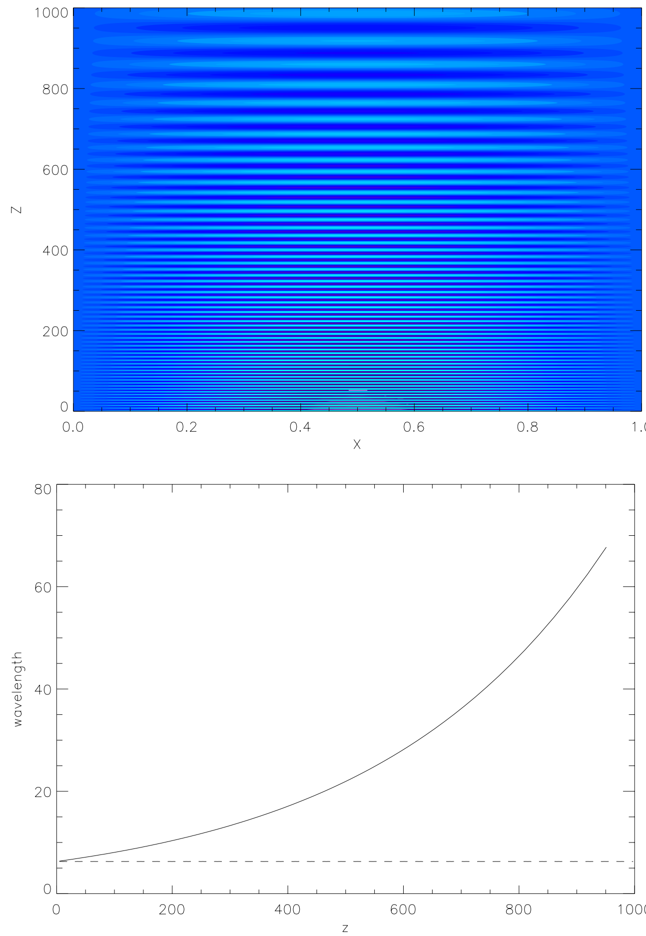


**Fig. 1.** (*top*) A contour plot of the perturbed magnetic field in an stratified plasma, $H = 200$ with $\delta = 0.0$. (*bottom*) The behaviour of the wavelength with height (at $x = 0.5$) with $\delta = 0.0$. The solid line is the solution for a stratified plasma, i.e. $H = 200$, the dashed line corresponds to an unstratified plasma, i.e. $H = \infty$.

would take place. To obtain an estimate of this, we consider the squares of the current density, i.e. $j^2$, and the vorticity, i.e. $\omega^2$, which indicate where the heating would occur if dissipation was included. Fig. 3 shows that $j^2$ initially increases with height but soon reaches a maximum and then the exponential decay of $b$ takes over. So Ohmic heating is only important if $\eta$ is sufficiently large. On the other hand, $\omega^2$ continues to increase with height suggesting, yet again, that viscous heating will be more important than ohmic heating.

From Figs. 4 and 5 we see that, when there is no phase mixing, i.e. $\delta = 0$, both the current density $j^2$ and the vorticity $\omega^2$ remain constant in an unstratified atmosphere, as gradients are not building up. However, when we include stratification, the current density and the vorticity show a very different behaviour. Due to the exponential decay of the magnetic field, the current density drops off when we include stratification. On the other hand, we see that the vorticity builds up in a stratified atmosphere when $x \neq 0.5$ due to the increasing x-gradients of the velocity. However, when $x = 0.5$ and $\delta = 0$, the vorticity decreases with height, as $\frac{\partial v}{\partial x} = 0$ and the only contribution to the vorticity will

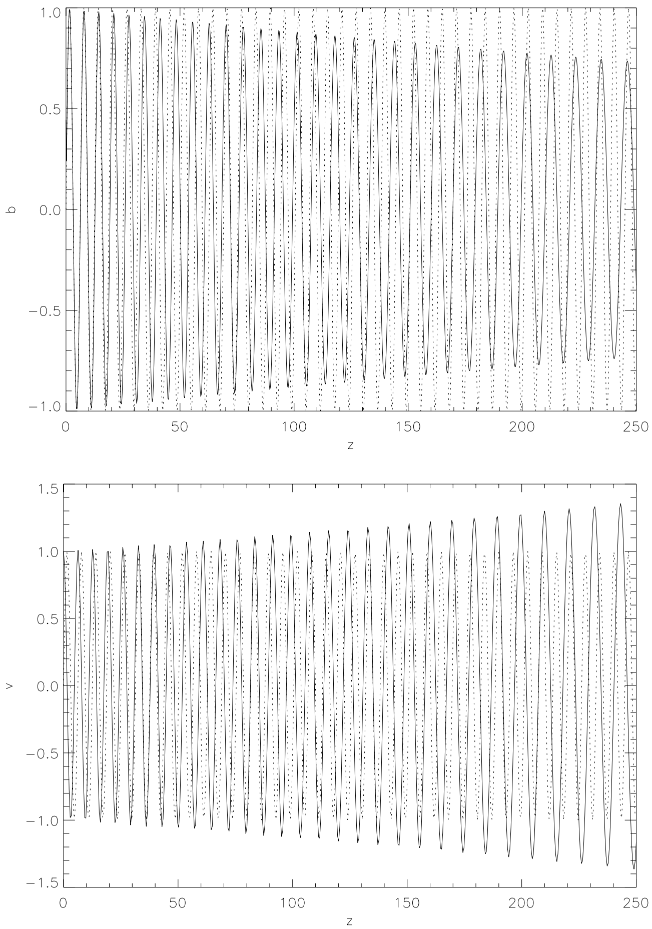


**Fig. 2.** A cross-section of the magnetic field (*top*) and the velocity (*bottom*) in a stratified plasma for $H = 200$ at $x = 0.5$. The dotted line is the corresponding solution for an unstratified plasma, $H = \infty$.

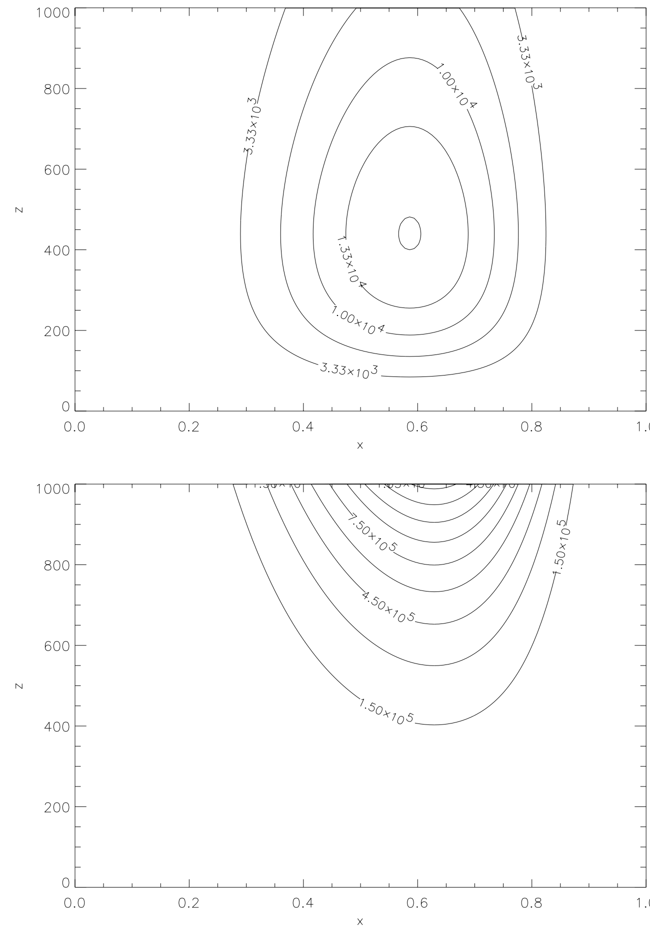


**Fig. 3.** A contour plot of $j^2$ (*top*) and $\omega^2$ (*bottom*) in a stratified plasma, $H = 200$ with $\delta = 0.5$.

**Table 1.** $\int_0^d j^2 dz$, at $x = 0.5$ for different values of the scale height $H$ and the phase mixing strength $\delta$.

| $\Lambda^2 = 0.0$ | $\delta = 0.0$ | $\delta = 0.1$ | $\delta = 0.5$ |
|---|---|---|---|
| $d = 1000$ | | | |
| $\mathrm{H} = 10^6$ | 999.3 | $6.32 \times 10^6$ | $8.53 \times 10^6$ |
| $\mathrm{H} = 10^3$ | 517.9 | $3.45 \times 10^6$ | $6.83 \times 10^6$ |
| $\mathrm{H} = 200$ | 134.7 | $4.05 \times 10^5$ | $3.64 \times 10^6$ |

**Table 2.** $\int_0^d \omega^2 dz$, at $x = 0.5$ for different values of the scale height $H$ and the phase mixing strength $\delta$.

| $\Lambda^2 = 0.0$ | $\delta = 0.0$ | $\delta = 0.1$ | $\delta = 0.5$ |
|---|---|---|---|
| $d = 1000$ | | | |
| $\mathrm{H} = 10^6$ | 999.8 | $6.32 \times 10^6$ | $8.53 \times 10^6$ |
| $\mathrm{H} = 10^3$ | 786.9 | $6.78 \times 10^6$ | $1.13 \times 10^7$ |
| $\mathrm{H} = 200$ | 371.2 | $1.09 \times 10^7$ | $5.46 \times 10^7$ |

come from the z-derivatives of the velocity which do not build up. When we include phase mixing, i.e. $\delta \neq 0$, we see that the current density and the vorticity build up strongly as gradients in the horizontal direction of both the magnetic field and the velocity are now increasing due to phase mixing. In the stratified case, however, $j^2$ decays away after an initial increase, due to the decrease of the magnetic field caused by stratification. Unlike the current density, the vorticity is enhanced when stratification is included and continues to increase with height due to the increase in the velocity amplitude.

From Table 1, we note that $\int_0^d j^2 dz$ decreases when the plasma becomes more stratified but increases when the phase mixing strength $\delta$ increases, i.e. when the plasma is more inhomogeneous. From Table 2, $\int_0^d \omega^2 dz$ behaves in a similar manner to the current density when $\delta = 0$. But when horizontal variations are included, $\int_0^d \omega^2 dz$ increases when the stratification is enhanced.

### 2.3. Energy

Combining the equation of motion and the induction equation gives the energy equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 v^2 + \frac{b^2}{2\mu}\right) = \frac{B_0}{\mu}\left(v\frac{\partial b}{\partial z} + b\frac{\partial v}{\partial z}\right) +$$

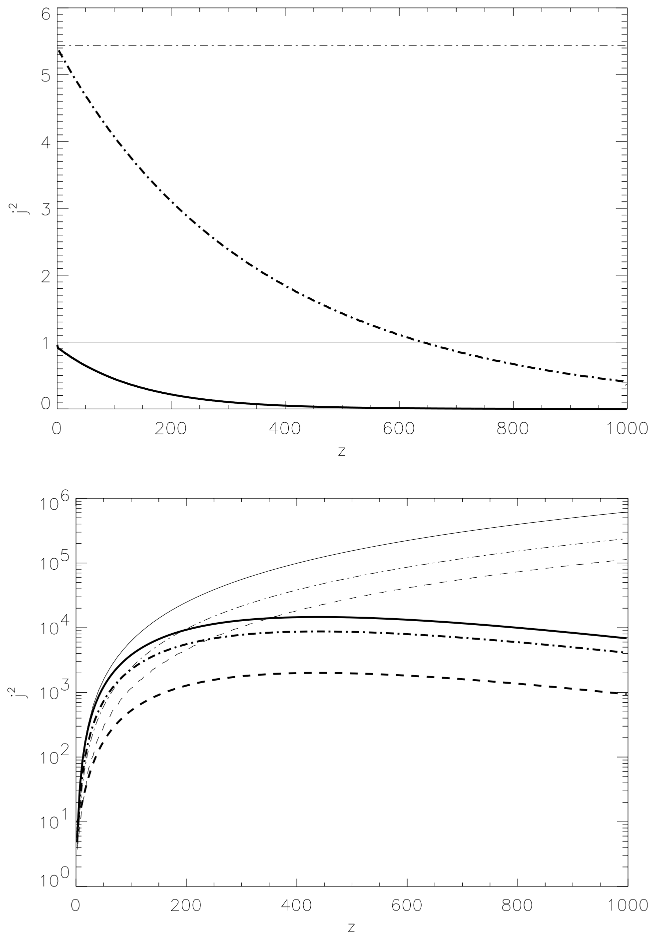


**Fig. 4.** A cross-section of the current density $j^2$ in a stratified plasma for $H = 200$ with $\delta = 0$ (*top*) and $\delta = 0.5$ (*bottom*). The thin lines represent the solution in an unstratified plasma,i.e. $H = \infty$. Cross-sections at $x = 0.5$ (solid line), $x = 0.25$ (dashed line) and $x = 0.75$ (dot-dashed line)

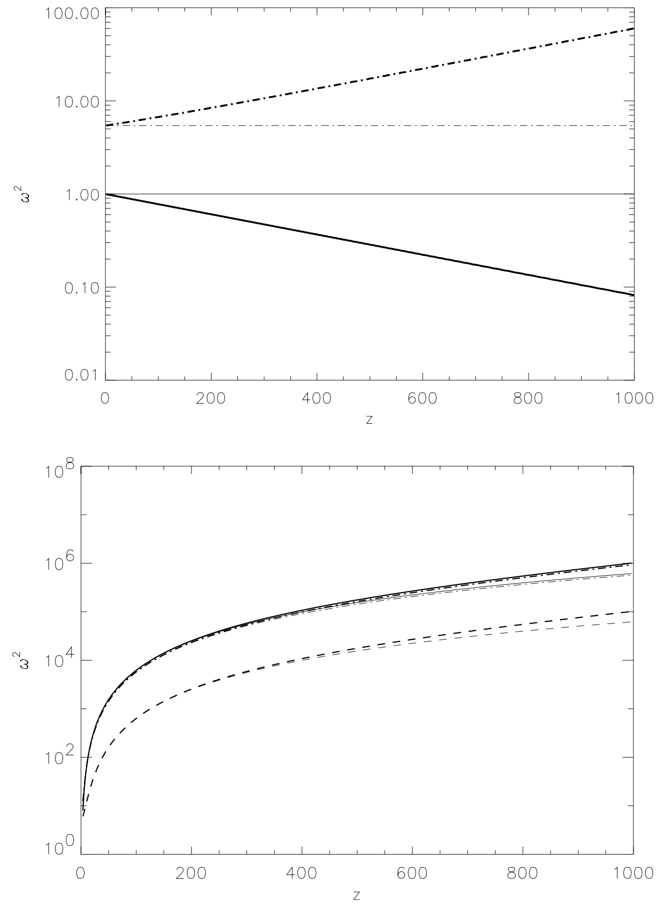


**Fig. 5.** A cross-section of the vorticity $\omega^2$ in a stratified plasma for $H = 200$ with $\delta = 0$ (*top*) and $\delta = 0.5$ (*bottom*). The thin lines represent the solution in an unstratified plasma,i.e. $H = \infty$. Cross-sections at $x = 0.5$ (solid line), $x = 0.25$ (dashed line) and $x = 0.75$ (dot-dashed line)

$$\rho_0 \nu v \nabla^2 v + \frac{\eta}{\mu} b \nabla^2 b \,. \qquad (11)$$

Integrating over the volume of the plasma and applying the boundary conditions (6), (7) and (5) gives

$$\frac{\partial}{\partial t}\int E dx dz = \frac{B_0}{\mu}\int (v.b)_{z=\infty} dx - \frac{B_0}{\mu}\int (v.b)_{z=0} dx$$
$$-\rho_0 \nu \int \omega^2 dx dz - \frac{1}{\sigma}\int j^2 dx dz \,, \quad (12)$$

where $E = \frac{1}{2}\rho_0(x,z)v^2 + \frac{1}{2\mu}b^2$ is the total energy. Averaging over a period in time, it is clear that the difference between the energy flowing through the base and out through the upper boundary is dissipated as either viscous heating or ohmic heating. With dissipation, all the wave energy is dissipated before the upper boundary so that all the energy propagating through the photospheric boundary goes into the ohmic and viscous heating terms. The ideal solution can be used to obtain how the time averaged Poynting flux through the photospheric boundary varies with the (dimensionless) scale height. The total energy is also used as a check on the numerical solutions to the phase mixing equations.

## 3. Stratified atmosphere, non-zero dissipation

In this section, $\Lambda^2 \neq 0$ and numerical results to (1) and (2) are presented. The numerical code used to obtain these results uses a $V$ pass multi-grid iteration technique to solve the coupled complex equations. Smoothing is performed by the Gauss-Seidel line relaxation method on all grids except the coursest. On this grid the solution can be obtained immediately from simple tridiagonal matrix inversion as long as the number of points in the $z$ direction is larger than that in the $x$ direction on the finest grid. The boundary conditions at $z = 0$ and $z = d$ are taken to be Eqs. (6) and (7) on the finest grid but zero gradient on all sub-grids. Iteration is stopped when the ratio of the $l_1$-*norm* of the residual to the $l_1$-*norm* of the function is less than $10^{-8}$. With this procedure it is found that the code always converges within 5 iterations of the full $V$ cycle. All the results in this section are obtained by using 65 points in the $x$ direction and 1025

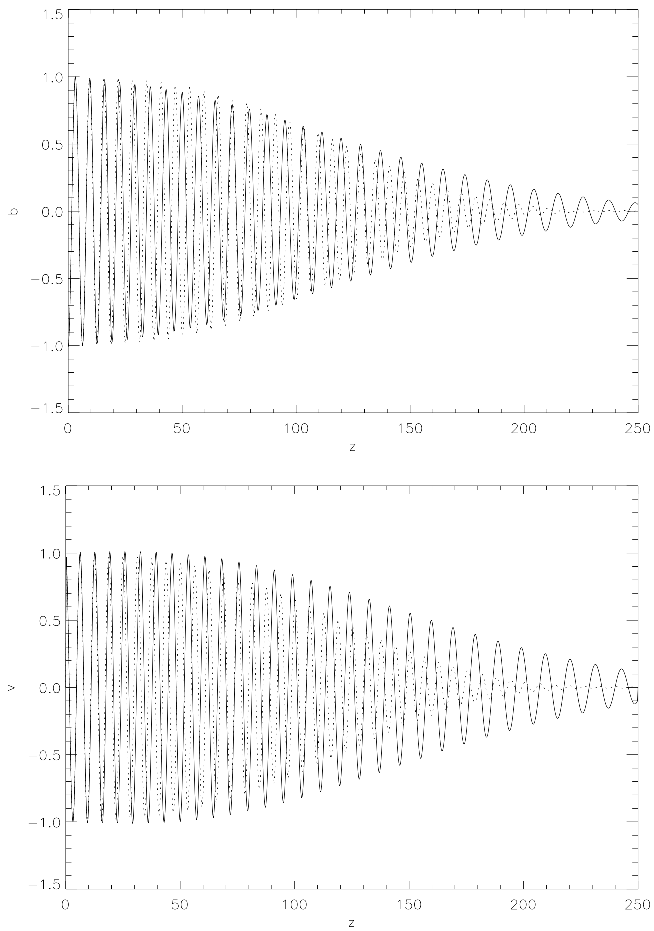


**Fig. 6.** A cross-section of the perturbed magnetic field (*top*) and the perturberd velocity (*bottom*) in a stratified atmosphere ($H = 200$) at $x = 0.5$ for the ohmic dissipation $\Lambda^2 = 10^{-4}$ and the phase mixing strength $\delta = 0.1$. The dotted line is the corresponding solution for an unstratified plasma, i.e. $H = \infty$.

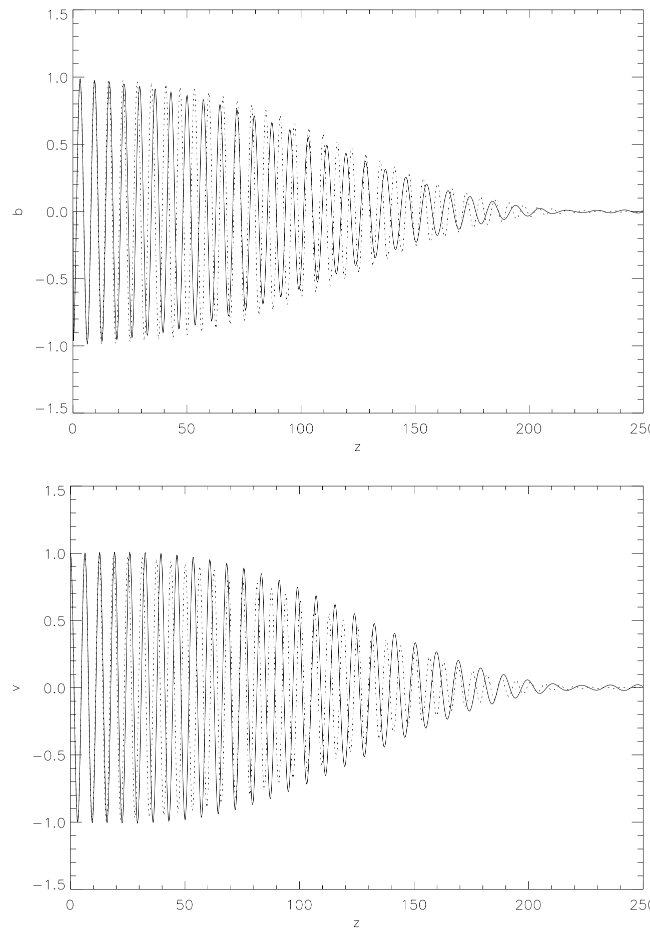


**Fig. 7.** A cross-section of the perturbed magnetic field (*top*) and the perturberd velocity (*bottom*) in a stratified atmosphere ($H = 200$) at $x = 0.5$ for the viscous dissipation $\Lambda^2 = 10^{-4}$ and the phase mixing strength $\delta = 0.1$. The dotted line is the corresponding solution for an unstratified plasma, i.e. $H = \infty$.

points in the $z$ direction as the multi-grid technique requires an approximately square grid to aid convergence.

As in the case of zero dissipation, we can conclude from Figs. 6 and 7 that stratification increases the wavelengths so that we can expect phase mixing to be less efficient when the plasma is stratified. On the other hand, in the case of zero dissipation, the amplitude of the velocity increases with height and this will enhance the viscous heating of the plasma. The effects of stratification in the plasma are far less important when phase mixing is stronger. Indeed, when the plasma is highly inhomogeneous, the horizontal gradients will build up very quickly implying that dissipation is important at much lower heights. Therefore, the waves will have significantly decayed before the effect of stratification can come into play. For example, when $\delta = 0.5$ and $\Lambda^2 = 10^{-3}$, the solution for $H = \infty$ and $H = 200$ are very similar. Due to the strong damping caused by phase mixing, the wave amplitudes have decayed completely before the effect of gravity can be noticed. Therefore, we will concentrate on the weak phase mixing case to examine the effects of stratification. If we look closely at Figs. 6 and 7 we see a slight initial increase in the velocity in the stratified plasma.This is the remnant of the velocity amplitude increase noted in the no dissipation case. We also see that both the perturbed magnetic field and the velocity decay faster in a stratified atmosphere when we consider viscous dissipation instead of ohmic dissipation. We notice in Figs. 6 and 7 that the amplitudes of the velocity $v$ and the magnetic field $b$ have not decayed completely. However, by comparing with numerical results obtained when the height of the numerical box, $d$, is taken sufficiently large so that $v$ and $b$ have decayed completely, we only found minor differences between the two cases. We therefore choose to use $d$ at lower heights as this gives better numerical resolution. To see the effect of gravity and dissipation it is useful to examine some approximate solutions.

When we consider resistivity, the WKB solutions including dissipation are, following the approach of Ruderman et al. (1998),

$$v = \sin \pi x \quad e^{-ik(x)Z} \exp\left(\frac{z}{4H} - \frac{\Lambda^2}{6} k'(x)^2 k(x) Z^3\right), \quad (13)$$

and,

$$b = \frac{-\sin \pi x \quad e^{-ik(x)Z}}{\left(\frac{i}{4H}e^{z/2H} + k(x)\right)} \exp\left(\frac{-z}{4H} - \frac{\Lambda^2}{6}k'(x)^2 k(x) Z^3\right), \quad (14)$$

where $Z = 2H(1-\exp(-z/2H))$ and $\Lambda^2 = \eta/\Omega x_0^2$ for ohmic dissipation.

When we consider viscosity, the WKB solutions become

$$v = \sin \pi x \, e^{-ik(x)Z} \exp\left(\frac{z}{4H} + \frac{1}{2}\Lambda^2 k'(x)^2 k(x)(2H)^3 \left(\frac{z}{H} - 2\sinh\left(\frac{z}{2H}\right)\right)\right), \quad (15)$$

and,

$$b = \frac{-\sin \pi x}{\left(\frac{i}{4H}e^{z/2H} + k(x)\right)} e^{-ik(x)Z} \exp\left(\frac{-z}{4H} + \frac{1}{2}\Lambda^2 k'(x)^2 k(x)(2H)^3 \left(\frac{z}{H} - 2\sinh\left(\frac{z}{2H}\right)\right)\right), \quad (16)$$

where $\Lambda^2 = \rho_0\nu\mu\Omega z_0^2/x_0^2 B_0^2$ for viscous dissipation. To obtain these WKB solutions, we neglected the last term on the left-hand side of Eq. (4). This term is not important for phase mixing as we expect the gradients in the x-direction to be much larger than the gradients in the z-direction. However, the numerical code used to obtain the results in this paper solves the full Eq. (4), without neglecting any terms. Note that the solutions for the perturbed magnetic field and the perturbed velocity are different depending on whether we include resistivity or dynamic viscosity in the system. When we consider viscous dissipation, the waves will be damped faster and therefore heat will be deposited into the system at lower heights. Furthermore, we see that the perturbed magnetic field will decay faster than the velocity in both cases. These differences are confirmed by the results shown in Figs. 6 and 7. We can also remark that solutions (13) and (15) converge to the Heyvaerts and Priest solution $v \sim e^{-ik(x)z} \exp\left(-\frac{\Lambda^2}{6}k'(x)^2 k(x) z^3\right)$ when $H$ tends to infinity, i.e. when we consider an unstratified atmosphere. Finally, note that the WKB solutions only satisfy the upper boundary condition (7) to leading order. Nonetheless, the WKB and numerical solutions give very good agreement for $\frac{d}{H}$ of order unity.

The contour plots of $j^2$ (Fig. 8) indicates already that the ohmic heating is spread out over a wider area when the scale height is smaller, i.e. when stratification is larger. But by comparing Fig. 8 and 9, we also see that in the stratified case, the maxima of the vorticity become larger than the current density. But, unlike the current density, the vorticity is not spread out wider than in the unstratified atmosphere. Note that the WKB solution suggests that the velocity amplitude will eventually increase at sufficiently large heights. However, the exact analytical solutions, for no dissipation, presented in the previous section show that the growth should asymptotically be linear with height from the behaviour of the $Y_0$ Bessel function. So the WKB solution is only valid for heights that satisfy the approximate condition $z < 2H \log 2H$. This is satisfied for all of our numerical solutions of the full set of equations. A similar effect can be found in the Ruderman et al. (1998) solution for a uniform magnetic field and an exponentially decreasing density.

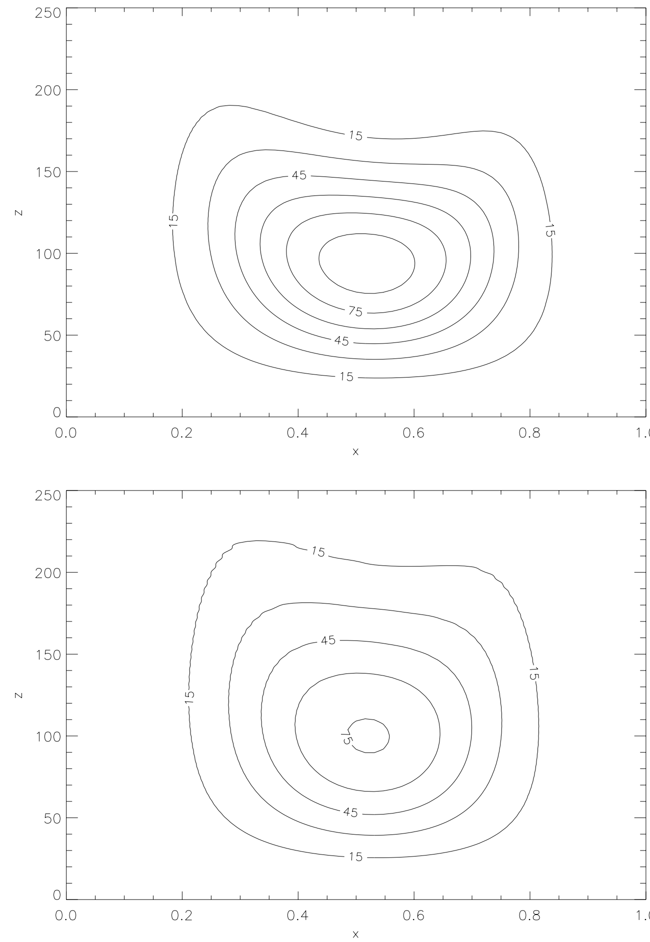


**Fig. 8.** A contour plot of $j^2$ for the damping $\Lambda^2 = 10^{-4}$ with $\delta = 0.1$ in an unstratified plasma, $H = \infty$ (*top*) and in a stratified plasma, $H = 200$ (*bottom*).

From the cross sections (Fig. 10) we see that the maximum of the current density is not only broader but also less high when the plasma is stratified. Because wavelengths are increased by a stratified plasma, the process of phase mixing will be slower than in an unstratified atmosphere as gradients in the horizontal direction will build up slower. And thus, for a similar height, less energy will have been transferred into heat through ohmic dissipation in the stratified case than in the unstratified case. We see, however, that this does not happen with the vorticity. The vorticity is only spread out very slightly due to the lengthening of the wavelengths and the maximum of the vorticity is almost the same as in the unstratified atmosphere. This different behaviour is due to the initial increase in the amplitude of the perturbed velocity and the fact that we considered the *dynamic* viscosity $\rho_0\nu$ to be constant, rather than the kinematic viscosity $\nu$.

By calculating the total amount of ohmic heating, $\eta \int_0^1 \int_0^d j^2 dz dx$ and the total viscous heating, $\rho_0\nu \int_0^1 \int_0^d \omega^2 dz dx$, we find that these are more or less independent of the resistivity and viscosity, which confirms the

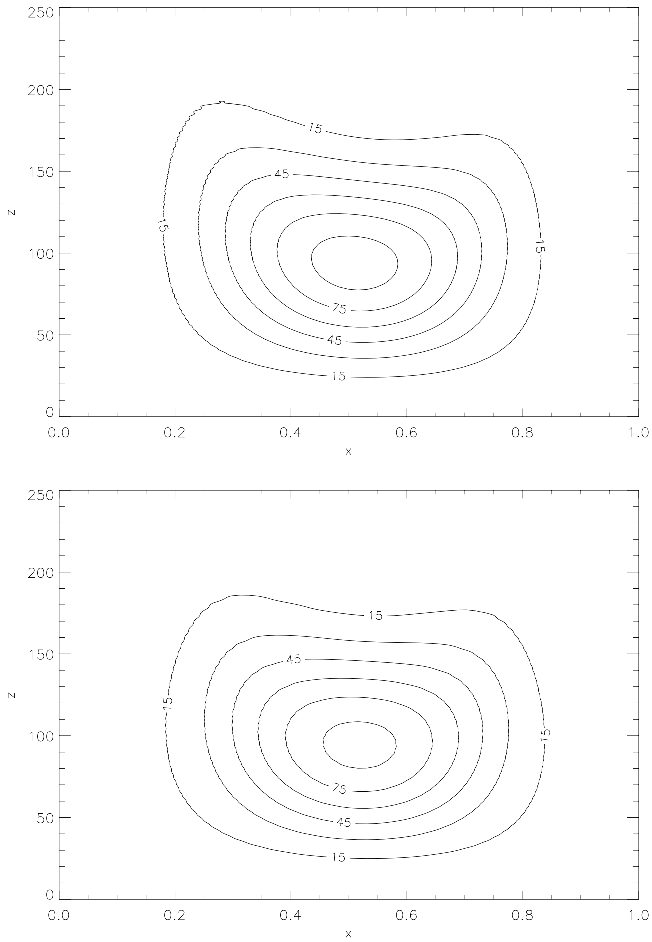


**Fig. 9.** A contour plot of $\omega^2$ for the damping $\Lambda^2 = 10^{-4}$ with $\delta = 0.1$ in an unstratified plasma, $H = \infty$ (*top*) and in a stratified plasma, $H = 200$ (*bottom*).

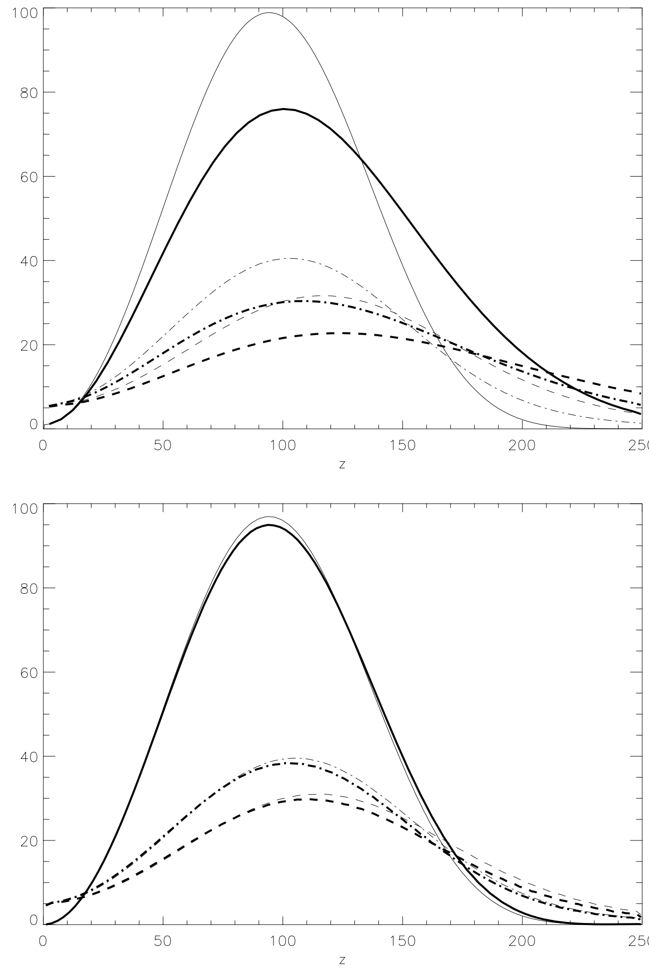


**Fig. 10.** A cross-section of $j^2$ (*top*) and $\omega^2$ (*bottom*) for the damping $\Lambda^2 = 10^{-4}$ in a stratified atmosphere for $H = 200$ with $\delta = 0.1$. The thin lines represent the solution in an unstratified plasma, i.e. $H = \infty$. Cross-sections at $x = 0.5$ (solid line), $x = 0.25$ (dashed line) and $x = 0.75$ (dot-dashed line)

result obtained by Hood et al. (1997a). This result also provides us with a check on the accuracy of the numerical code as we expect the total ohmic and viscous heating to be constant when $d$ is taken large enough so that wave is completely damped. These calculations also confirmed that the Poynting flux at the base, which does not vary with the scale height, is balanced by either the ohmic or viscous heating when the amplitudes of the perturbed magnetic field or velocity have been damped. This indicates that the total ohmic and viscous heating stays constant as the scale height decreases or, that stratification does not seem to have an effect on the total ohmic and viscous heating.

So we find that in a stratified atmosphere the ohmic heating is spread out over a wider height range and the maximum does not build up as high as in the unstratified atmosphere. But despite these facts, the total amount of ohmic heat transferred to the plasma is independent of the resistivity and does not seem to be influenced very much by the stratification as long as $d$ is sufficiently large enough. The viscous heating however is not influenced at all by the stratification. This shows that in a stratified medium, the viscous heating will dominate the ohmic heating at lower heights, under the assumption that the dynamic viscosity $\rho_o \nu$ is constant. However, the total amount of heat deposited into the plasma by either mechanism will be the same.

We now want to examine how the energy is transferred down to shorter lengthscales, for an Alfvén wave propagating up into the corona, both in a stratified and an unstratified atmosphere. We can do this by decomposing the numerical solution into orthogonal functions (Cally 1991; Ireland 1997). So the magnetic field is expanded as a Fourier sine series as

$$b(x,z) = \sum_{n=1}^{\infty} B_n(z) \sin(n\pi x) \; . \tag{17}$$

The q$^{th}$ mode is then found to be:

$$B_q(z) = 2\int_0^1 b(x,z)\sin(q\pi x)\,dx \, ,$$

and therefore the magnetic energy in the q$^{th}$ mode is given by $M_q = \dfrac{1}{2} B_q(z) B_q^*(z)$ . Similarly, the kinetic energy in the q$^{th}$ mode is given by $E_q = \dfrac{1}{2}\rho_0 V_q(z) V_q^*(z)$ .

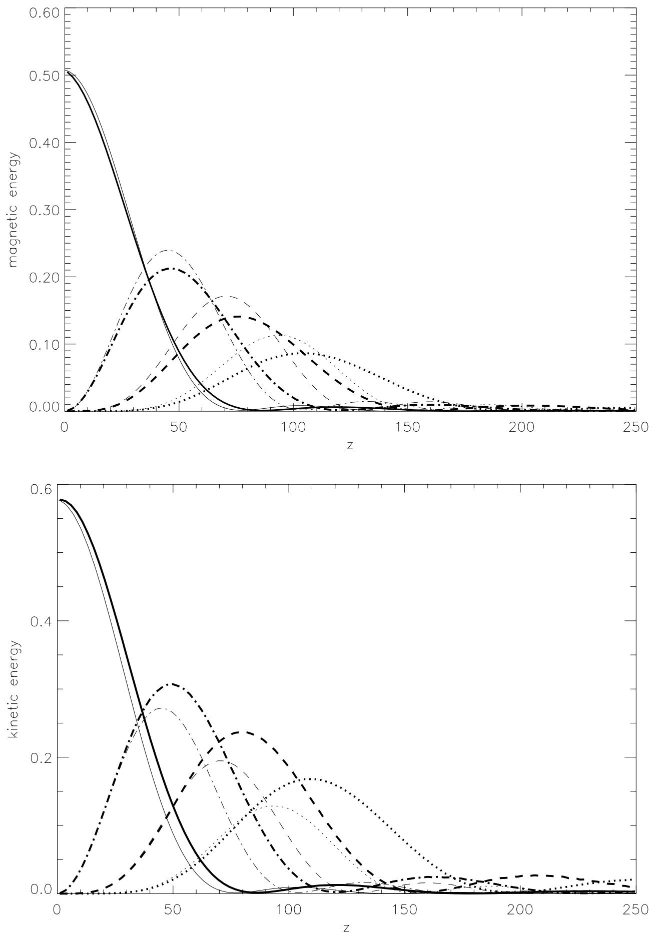


**Fig. 11.** The lengthscale excitation of the Fourier modes $q = 1$ (solid line), $q = 2$ (dashed line), $q = 3$ (dot-dashed line), $q = 4$ (dotted line) in a stratified plasma, $H = 200$, with $\Lambda^2 = 10^{-4}$ and $\delta = 0.1$. The thin lines represent the solutions in an unstratified plasma, $H = \infty$.

Fig. 11 examines the excitation of the first four Fourier modes in the plasma. As the waves propagate higher up into the atmosphere, shorter lengthscales are being created and phase mixing carries significant amounts of energy down to smaller lengthscales where dissipation can be important. We see however that the behaviour of the kinetic and the magnetic modes in the stratified and the unstratified plasma is quite different. The kinetic modes build up higher when stratification is put in while the magnetic modes build up higher in an unstratified plasma.

The horizontal integrals of the magnetic and kinetic energies are given by $\int_0^1 \frac{B^2}{2} dx$ and $\int_0^1 \frac{1}{2}\rho_0 v^2 dx$. The total energy is the sum of these magnetic and kinetic energies. From Fig. 12 and Fig. 13 we see that the behaviour of the total energy in a stratified medium changes, depending on whether we consider ohmic or viscous heating. When ohmic dissipation is considered, both the magnetic and the kinetic energy initially drop off faster in a stratified atmosphere. This can be understood by looking at Figs. 6 and 7 which shows that the magnetic field drops off faster in a stratified atmosphere at lower heights when ohmic dissipation is considered. A similar behaviour is found for $\sqrt{\rho_0}v$ which explains the kinetic energy. When we consider viscous dissipation, the magnetic and kinetic energy both drop off quicker in the stratified atmosphere at all heights. From Figs. 6 and 7 we see that the magnetic field decays faster in a stratified atmosphere than in an unstratified atmosphere at all heights. The same is true for $\sqrt{\rho_0}v$.

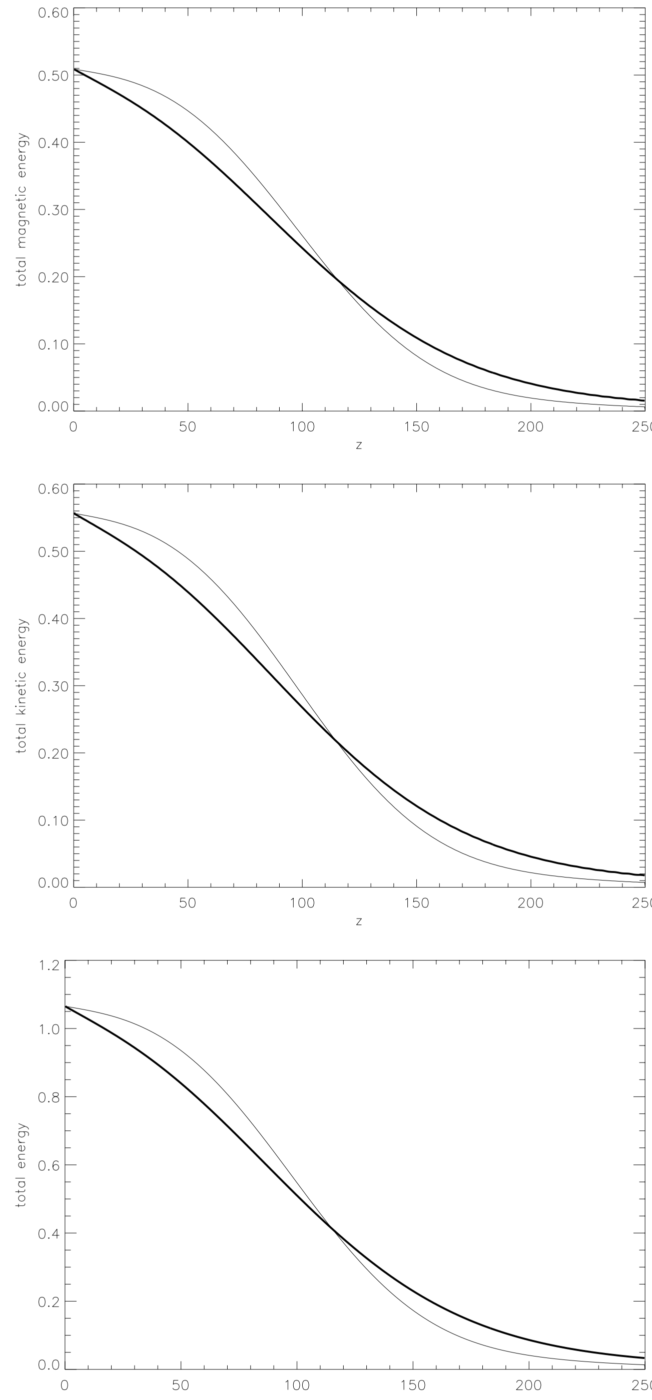


**Fig. 12.** Evolution of the magnetic energy (*top*), the kinetic energy (*middle*) and the total energy (*bottom*) for an Alfvén wave in a stratified plasma ($H = 200$) with $\Lambda^2 = 10^{-4}$ and $\delta = 0.1$ when ohmic heating is considered. The thin lines represent the solutions in an unstratified plasma.

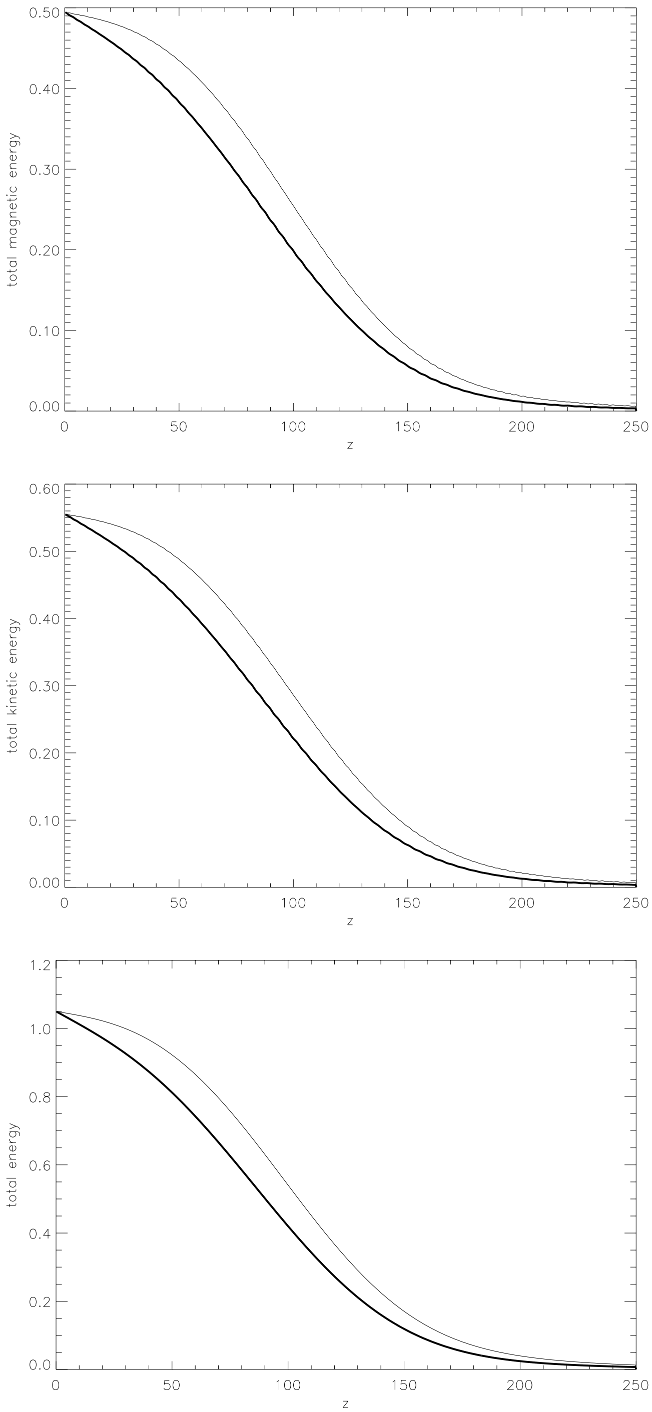


**Fig. 13.** Evolution of the magnetic energy (*top*), the kinetic energy (*middle*) and the total energy (*bottom*) for an Alfvén wave in a stratified plasma ($H = 200$) with $\Lambda^2 = 10^{-4}$ and $\delta = 0.1$ when viscous heating is considered. The thin lines represent the solutions in an unstratified plasma.

## 4. Applications

In this paper we have studied the effect of stratification on phase mixing in an open atmosphere and we now put the obtained results into typical solar conditions. By considering the WKB solutions (14) for the magnetic field in a stratified atmosphere, we can obtain the following expression for the height $z_{max}$ where most heat would be deposited,

$$z_{max} = \frac{v_{A0}}{\Omega} \left\{ \left( \frac{2}{\Lambda^2 k'^2 k} \right)^{1/3} + \frac{v_{A0}}{6H\Omega} \left( \frac{2}{\Lambda^2 k'^2 k} \right)^{2/3} \right\}, \quad (18)$$

where $\Lambda^2 = \frac{\eta}{\Omega L^2}$ and $H = \frac{\mathcal{R}T}{\tilde{\mu} g}$. We have to remark here that we are working with dimensional variables in this section. Values obtained from expression (18) agree with the results shown in Fig. 8. However, results for large heights are uncertain. Nonetheless it is informative to investigate how $z_{max}$ depends on the phyiscal parameters of the system. As the value of the resistivity in the solar corona is not exactly known, we consider $\eta$ as a free parameter and use expression (18) to examine where the maximum of the ohmic heating would occur for a range of driving frequencies and temperatures. From expression (18) we see that the height of this maximum is influenced by several factors. When we increase the phase mixing parameter $\delta$, we see that the maximum will be situated at a lower height. If we, for example, assume that the plasma density inside a coronal plume is a factor 4 higher than the surrounding plasma, we find $\delta = 0.6$. If we further assume that the average coronal hole has a width of $10^8$m, and that there are about 15 plumes in a coronal hole, we find $L \sim 7 \times 10^6$m. We notice that the height of maximum ohmic heating would be reduced by reducing the background Alfvén speed. In the section we assume the background Alfvén velocity $v_{A0} \sim 500\,\mathrm{km\,s^{-1}}$ in a coronal hole (Woo 1996).

From Fig. 14, we see how the height of maximum ohmic heating varies with temperature T and cyclic frequency $\Omega = \frac{1}{P}$ where P is the period, for different values of the resistivity. We consider a temperature range from $10^5$ K up to $5 \times 10^6$ K and a frequency range which represents a period from 30 sec to 600 sec. When studying the effects of different temperatures we considered two different cases, a 5-minute oscillation and a 1-minute oscillation. When studying the effect of different frequencies, we took T=$1.5 \times 10^6$ K. From expression (18) we know that the height of maximum ohmic heating will decrease when $\eta$ is increased, which is confirmed by the results shown in Fig. 14. Furthermore we see that for a certain value of the resistivity, the height at which most heat will be deposited decreases when temperatures increase. For the 5-minute oscillation, we see that even for temperatures as high as $5 \times 10^6$ K and $\eta = 10^8$, the maximum will still occur at almost 10 solar radii. However, if we consider a driver with a shorter period, e.g 1 minute, we see that the maximum can be as low as a few solar radii. From Fig. 14 we see that for a fixed value of the resistivity, the height of maximum heating increases when the period is decreased. We find that for T=$1.5 \times 10^6$ K, a 1 or 2-minute oscillation will deposit most heat within a few solar radii and could therefore be a candidate for heating the coronal holes, while e.g. a 5-minute oscillation might be a way to deposit heat in the solar wind. However, the results shown in Fig. 14 and expression (18) show clearly that stratification increases the height at which the maximum of the ohmic heating occurs.

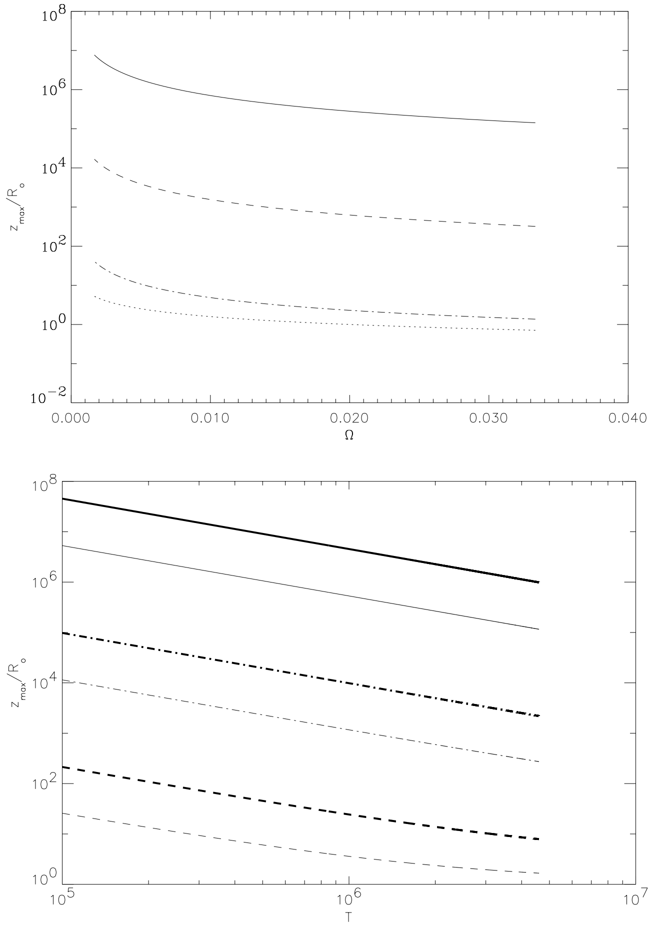


**Fig. 14.** The height in solar radii at which the maximum of the ohmic heating is situated for different values of the resistivity, $\eta = 1$ (solid line), $\eta = 10^4$ (dot-dashed line), $\eta = 10^8$ (dashed line), for a range of driving frequencies (*top*) and temperatures (*bottom*). In the temperature-plot, the thick lines represent a driver with a 5-minute period while the thin lines represent a driver with a 1-minute period. In the frequency-plot, the dotted line represents the solutio sphere for $\eta = 10^8$.

## 5. Conclusions

From this study of the effect of stratification on phase mixing in an open field region, we conclude that wavelengths lengthen when Alfvén waves propagate through a stratified plasma. So the main result is that a vertical stratification of the density makes phase mixing by ohmic heating less efficient as a coronal heating mechanism. In a stratified atmosphere, it will take longer to build up sufficiently small lengthscales for dissipation to play a role. Therefore, the ohmic heating is spread out over a wider height range and the maxima do not build up as high as in an unstratified atmosphere. However, the total ohmic heating deposited into the plasma is not effect by the stratification. We found that for viscous heating there was only a very slight difference between the stratified and the unstratified cases and the maximum builds up to a similar value in both cases. Just as the total ohmic heating, the total viscous heating does not seem to be influenced by the stratification. We can therefore conclude that in a stratified atmosphere, the heat will be deposited higher up than in an unstratified atmosphere and that the viscous heating will be the dominant component in the heating process at lower heights. Finally we can remark that when the inhomogeneity in the horizontal direction in the plasma is sufficiently large (i.e. for large values of $\delta$), so that phase mixing is strong, stratification is unimportant. Due to the rapid phase mixing, energy can be dissipated before the effects of stratification build up. However, in Sect. 4 we saw that, when considering likely coronal conditions, the effect of stratification on the efficiency of phase mixing in the solar corona would still be large as the height at which most heat would be deposited through ohmic dissipation was increased considerably by stratification for a range of driving frequencies and temperatures.

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