Astron. Astrophys. 346, 652–662 (1999)

ASTRONOMY AND ASTROPHYSICS

Escape with the formation of a binary in two-dimensional three-body problem. I

Navin Chandra and K.B. Bhatnagar

Centre for Fundamental Research in Space Dynamics and Celestial Mechanics IA/47C, Ashok Vihar, New Delhi-110 052, India

Received 10 September 1998 / Accepted 25 January 1999

Abstract. The escape with the formation of a binary in three-body problem is studied in a series of two papers. This paper deals with the systematic regularity of escape with the formation of a binary with low perturbing velocities for equal masses in the evolution of stellar systems in a plane. The main results are: (a) For triple close approaches $I_m \geq \left(\dfrac{C^2}{2|E_t|}\right)^2$, (b) $a_\infty(3v_\infty^2 - 4E_t) = 2$, (c) $v_\infty = \left[\dfrac{2}{3}\left\{2E_t + (a_\infty)^{-1}\right\}\right]^{1/2}$, where I_m is the minimum moment of inertia of the system, **C** is the angular momentum, E_t the total energy, a_∞ the semimajor axis of the binary formed and v_∞ the escape velocity of the escaper. Our results also indicate that the conjecture of Szebehely (1977), viz. "The measure of escaping orbits is significantly higher than the measure of stable orbits" is likely to be true. Further our result regarding escape probability is in contrast to the result of Agekian's et al. (1969). The second paper deals with certain parameters of the participating bodies in 3D space.

Key words: celestial mechanics, stellar dynamics – stars: binaries: close

1. Introduction

In the general three-body systems, one encounters motions of different types under certain initial conditions. In fact, the main problem is the partition of the phase space of the initial conditions. According to Henon (1974), the region of phase space with bounded motion is mixed with escape.

The conjecture of Birkhoff (1922, 1927) and later reformulated by Szebehely & Peters (1967), Szebehely (1973) states that sufficiently triple close simultaneously asymmetric approach results with the formation of a binary and escape of the third body. The above statement is numerically confirmed by Agekian & Anosova (1967), Szebehely & Peters (1967), Szebehely (1974b), Chandra & Bhatnagar (1998a) and others. Closely related to this problem has also been investigated by Waldvogel (1976) and Marchal & Losco (1980).

According to Sundman (1912) for a triple collision the total angular momentum **C** must be equal to zero and for triple close approach, **C** should be sufficiently small.

Here, the subject is the general problem of three bodies and the equilateral Lagrangian solution in a symmetric rotating configuration. The masses of the participating bodies are equal. If the mean motion is such that the virial coefficient is unity, the distances between the bodies are constant. In Lagrangian solution, the symmetric configuration is never destroyed, therefore, escape does not occur and all motions are periodic, even when angular momentum **C** is small. It is equally important for unstable Lagrangian solutions.

In the equilateral configuration, if **C** is small and asymmetric changes of the initial conditions are introduced, it leads to escape instead of periodic orbits. Furthermore, if the total energy of the system is positive, instability always occurs and the system either explodes or a binary is formed and the third body escapes, but in the astronomically more important case of negative total energy, one or several triple close approaches proceed and an eventual escape, if it occurs at all. Therefore, the detailed behaviour during triple close approaches might be considered relevant to stellar dynamics.

In the present paper, our aim is to study which body is likely to escape and the other two forming a binary with the help of relative distances of the participating bodies when perturbation is used, and compare the results with the Agekian's escape probability.

According to the Szebehely (1971) and Agekian & Martinova's (1973) classification of the states of motion in the general problems of three bodies, the family presented in this paper belongs to class '0'.

2. Statement of the problem

Initially, three equal masses are considered to occupy the vertices of an equilateral triangle $P_1P_2P_3$ in a plane. These masses in the absence of any disturbance will move along the medians of the triangle and collide at the centroid of the triangle. Here, to avoid such a collision the masses at P_1, P_2 and P_3 are subjected to small perturbing velocities $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ in directions α_1, α_2 and α_3 respectively with P_1P_2.

We wish to study for what values of perturbing velocities $\mathbf{v}_i$ and α_i (i = 1, 2, 3), the asymmetry will result in systematic regularity of escape with the formation of a binary with the help of the relative distances of the participating bodies.

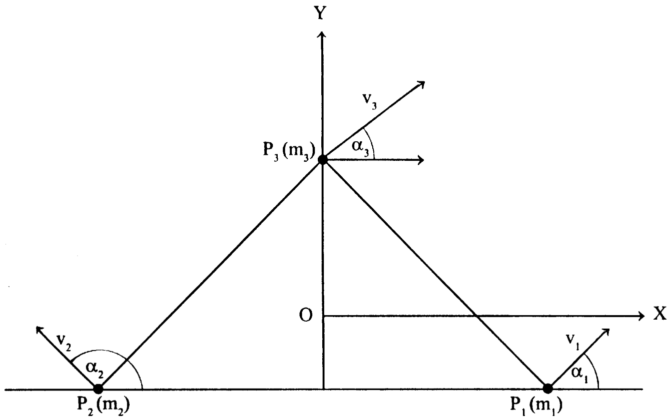

Fig. 1. Initial conditions.

3. Equations of motion

Let there be three equal masses $m_1 = m_2 = m_3$ each equal to unity occupy the vertices of an equilateral triangle $P_1P_2P_3$. At $t = 0$, the centre of mass, O, of the system is taken as the origin, x-axis parallel to one of the side P_1P_2 of the equilateral triangle $P_1P_2P_3$ and y-axis lying in the plane of the triangle. We choose the unit of distance such that at $t = 0, P_1P_2 = P_2P_3 = P_3P_1 = 1$ (Fig. 1).

The masses at P_1, P_2, P_3 are subjected to small perturbing velocities $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively such that $\mathbf{v}_i = v_i[\text{Cos}\,\alpha_i\, I + \text{Sin}\,\alpha_i\, J]$ (i = 1, 2, 3). At $t = 0$, the position (x_i, y_i) and velocities $(\dot{x}_i, \dot{y}_i)$, i = 1, 2, 3 of P_1, P_2, P_3 are given by

$$x_1 = 1/2, x_2 = -1/2, x_3 = 0, y_1 = y_2 = -1/(2\sqrt{3}),$$
$$y_3 = 1/\sqrt{3}, \dot{x}_i = v_i \,\text{Cos}\,\alpha_i, \dot{y}_i = v_i\, \text{Sin}\alpha_i, (i = 1, 2, 3)\,.$$

The equations of motion of P_i, having position vector $\mathbf{r}_i$ are given by

$$\ddot{Z}_k = G \sum_{i \neq k = 1}^{3} \frac{Z_i - Z_k}{|Z_i - Z_k|^3}\, m_i$$

where $Z_k = x_k + iy_k, i$ = 1, 2, 3 and G is the constant of gravitation.

By symmetry, the motion of P_i with zero perturbing velocity will take place along $\overline{P_iO}$ and triple collision will occur at O. When there is no perturbation and $\overline{F_{ij}}$ is the force between the ith and jth mass and the unit of time is so chosen that $G = 1$, then the equation of motion of each mass in terms of r, its distance from the origin is expressed as $\ddot{r} = -\dfrac{1}{\sqrt{3}r^2}$.

Taking $rd\tau = dt$ and solving, we obtain

$$r = \frac{1}{2\sqrt{3}}(1 + \text{ Cos}\,\sqrt{2}\tau)\,,$$
$$t = \frac{1}{2\sqrt{3}}(\tau + \frac{1}{\sqrt{2}}\,\text{Sin}\,\sqrt{2}\tau)\,,$$

This corresponds to the Kepler's equation. The colliding time is thus given by

$$\tau_c = \frac{\pi}{\sqrt{2}}\,,$$
$$\tau_c = \frac{\pi}{\sqrt{24}} = 0.6412749151\ldots$$

The introduction of perturbing velocities break the symmetry and collision can be avoided. The distances between the masses are functions of $\mathbf{v}_i$ and t, (i = 1, 2, 3).

Let r_{12}, r_{23}, r_{31} represent the lengths of the sides P_1P_2, P_2P_3, P_3P_1, respectively at time t. In subsequent motion the values of the distances up to first order of t are

$$r_{12} = 1 + (A_1 - B_2)t + \cdots,$$
$$r_{23} = r_{12} - A_1t + \frac{1}{2}A_2t + \frac{1}{2}A_3t + \cdots,$$
$$r_{31} = r_{12} - \frac{1}{2}B_1t + B_2t + \frac{1}{2}B_3t + \cdots,$$

where
$A_1 = v_1 \,\text{Cos}\,\alpha_1, A_2 = v_2(\text{Cos}\,\alpha_2 - \sqrt{3}\,\text{Sin}\,\alpha_2),\ A_3 = v_3(\text{Cos}\,\alpha_3 + \sqrt{3}\,\text{Sin}\,\alpha_3), B_1 = v_1(\text{Cos}\,\alpha_1 + \sqrt{3}\,\text{Sin}\,\alpha_1), B_2 = v_2\,\text{Cos}\,\alpha_2,\ B_3 = v_3(\text{Cos}\,\alpha_3 - \sqrt{3}\,\text{Sin}\,\alpha_3)$.

In the present study we require some more parameters at $t = 0$, which are given below:
(i) Moment of inertia I:

$$I(t) = 1 + \left\{v_1\left(\text{Cos}\,\alpha_1 - \frac{1}{\sqrt{3}}\,\text{Sin}\,\alpha_1\right) - v_2\left(\text{Cos}\,\alpha_2 + \frac{1}{\sqrt{3}}\,\text{Sin}\,\alpha_2\right) + \frac{2}{\sqrt{3}}v_3\,\text{Sin}\,\alpha_3\right\} t + (v_1^2 + v_2^2 + v_3^2 - 3)t^2 + O(t^3)\,,$$
$$I(0) = 1,$$
$$\dot{I}(0) = v_1\left(\text{Cos}\,\alpha_1 - \frac{1}{\sqrt{3}}\,\text{Sin}\,\alpha_1\right) - v_2\left(\text{Cos}\,\alpha_2 + \frac{1}{\sqrt{3}}\,\text{Sin}\,\alpha_2\right) + \frac{2}{\sqrt{3}}v_3\,\text{Sin}\,\alpha_3\,,$$
$$\ddot{I}(0) = 2\left(\sum_{i=1}^{3} v_i^2 - 3\right)\,.$$

(ii) Total energy E_t:

Kinetic energy $T(0) = \dfrac{1}{2}\left(\displaystyle\sum_{i=1}^{3} v_i^2\right)$.

Potential energy $V(0) = -3$.

Thus $E_t = \dfrac{1}{2}\left(\displaystyle\sum_{i=1}^{3} v_i^2 - 6\right)$.

(iii) Angular momentum **C**:

$$|\mathbf{C}| = \frac{1}{2\sqrt{3}}\Big[v_1(\text{Cos}\,\alpha_1 + \sqrt{3}\,\text{Sin}\,\alpha_1) - v_2(\text{Cos}\,\alpha_2 + \sqrt{3}\,\text{Sin}\,\alpha_2) + v_3\,\text{Cos}\,\alpha_3\Big]\,.$$

(iv) Virial Coefficient $Q : Q = \dfrac{2T}{|F|}$,

$$Q(0) = \frac{1}{3}\sum_{i=1}^{3} v_i^2.$$

Immediately after the motion starts, $\ddot{I} \gtrless 0$ for $\sum_{i=1}^{3} \mathbf{v}_i^2 \gtrless 3$.

From this inequality, it may be observed that

1. the curve $I = I(t)$ is convex or concave from below,
2. the total energy of the system is positive or negative,
3. the virial coefficient of the system is $>$ or $<$ unity.

For small values of v_i (i = 1, 2, 3), the total energy E_t and $\ddot{I}(0)$ are negative and virial coefficient of the system is less than unity. This is evidently true for $0 < \sum_{i=1}^{3} v_i^2 < 3$ and so the motion begins with contraction for all directions of the perturbing velocities in a plane.

Proceeding as in Szebehely (1974a), we can show that the minimum value of I, say I_m, for initial collapse satisfies

$$I_m \geq \left(\frac{c^2}{2|E_t|}\right)^2 .$$

It is also true for unequal masses as well. Moreover, this is more general than Szebehely's (1974b) result viz $I_m \geq v_0^4/64$ which can be deduced from our result.

Since the distances of the escaper from the two-body which form a binary will go on increasing and eventually will be greater than the distance between the binary, the body which is likely to escape must be opposite to the shortest distance between the participating bodies. Keeping this criterion in view, it may be observed from the relative distances of the participating bodies that when low values of perturbing velocities $\mathbf{v}_i$ (i = 1, 2, 3) are introduced the initial symmetry gets destroyed and that there is a possibility of a body to escape which is opposite to the smallest side provided escape conditions are satisfied.

4. Possible regions of escape

From relative distances of the participating bodies for small value of t, it may be possible to point out the body which is likely to escape with the formation of a binary. We may have $r_{23} \gtreqless r_{12}$ and $r_{31} \gtreqless r_{12}$. When we combine each of $r_{23} \gtreqless r_{12}$ with each of $r_{31} \gtreqless r_{12}$, for small values of v_i, we get the following cases.

Case (I): when $r_{23} > r_{12}$ and $r_{31} > r_{12}$.

It means r_{12} is the smallest and therefore m_3 may escape and m_1, m_2 form a binary.

Case (II): when $r_{23} > r_{12}$ and $r_{31} < r_{12}$.

Thus we have $r_{31} < r_{12} < r_{23} \Rightarrow m_2$ may escape and m_3, m_1 form a binary.

Case (III): when $r_{23} > r_{12}$ and $r_{31} = r_{12}$.

We cannot take a decision as it is not possible to point out the smallest distance.

Case (IV): when $r_{23} < r_{12}$ and $r_{31} > r_{12}$.

Thus we have $r_{23} < r_{12} < r_{31} \Rightarrow m_1$ may escape and m_2, m_3 form a binary.

Case (V): when $r_{23} < r_{12}$ and $r_{31} < r_{12}$.

Now there are three possibilities according as $r_{23} \gtreqless r_{31}$.

(a) When $r_{23} > r_{31}$, we have $r_{31} < r_{23} < r_{12} \Rightarrow m_2$ may escape and m_3, m_1 form a binary.
(b) When $r_{23} < r_{31}$, it means r_{23} is smallest and therefore m_1 may escape and m_2, m_3 form a binary.
(c) When $r_{23} = r_{31}$. We can not take a decision due to equality sign.

Case (VI): when $r_{23} < r_{12}$ and $r_{31} = r_{12}$.

Thus we get $r_{23} < r_{12} = r_{31} \Rightarrow m_1$ may escape and m_2, m_3 form a binary.

Case (VII): when $r_{23} = r_{12}$ and $r_{31} > r_{12}$.

Here, we can not take a decision due to equality sign.

Case (VIII): when $r_{23} = r_{12}$ and $r_{31} < r_{12}$.

It means $r_{31} < r_{12} = r_{23} \Rightarrow m_2$ may escape and m_3, m_1 form a binary.

Case (IX): when $r_{23} = r_{12}$ and $r_{31} = r_{12}$.

Here, we cannot take a decision due to equality sign.

The above results may change for higher values of t corresponding to certain directions of the perturbing velocities given to the participating bodies but escaper must be opposite to the smallest relative distance of the participating bodies.

From the above analysis, we conclude that out of the 9 cases, we may be able to take a decision only in 6 cases.

In the literature, many escape conditions exist. By Sundman (1912), it is sufficient for escape that

$$I_m \leq S_1^2,$$

where, I_m = minimum moment of inertia,

and $S_1 = \dfrac{I_c\sqrt{I_o}}{P + I_o + I_c}$.

Here, $I_o = a^2(g_1 + g_2)$,

$$P = \frac{1}{8|E_t|}(A + A' I_o^{1/4})^2,$$

$$I_c = \frac{C^2}{2|E_t|},$$

$$a = \frac{G(m_1m_2 + m_2m_3 + m_3m_1)}{|E_t|},$$

$$g_1 = \frac{m_1m_2}{\mu}\ g_2 = \frac{\mu m_3}{M}$$

$$|E_t| = T - F,$$

$$A = \sqrt{2g_1aG(m_1m_2 + m_2m_3 + m_3m_1)},$$

$$A' = 4\sqrt{2MGg_2^{3/2}},$$

$$C^2 = \left[\sum_{i=1}^{3} m_i(\mathbf{r}_i \times \dot{\mathbf{r}}_i)\right]^2, \mu = m_1 + m_2,$$

$$M = m_1 + m_2 + m_3,$$

$$T = \frac{1}{2}\sum m_i\dot{\mathbf{r}}_i^2, F = G\sum_{1\leq i<j\leq 3}\frac{m_im_j}{r_{ij}},$$

G = Constant of gravitation,

$\mathbf{r}_i, \dot{\mathbf{r}}_j$ = Position and velocity vectors of m_i.

According to the above escape condition, m_1, m_2 form a binary and m_3 escapes. It has to be modified according to the escaper.

In our case, the above condition is satisfied and escape does occur for sufficiently small values of $\mathbf{v}_i$ (i = 1, 2, 3). This follows from the fact that as $v_i \to 0^+, I_m \to 0^+$. The asymmetric triple close approaches after reaching I_m generates sufficiently large values of I and $\ddot{I}$ for escape with the formation of a binary. After attaining of I_m one of the three bodies that is opposite to the shortest distance escapes and the remaining two form a binary.

5. Escape velocity in terms of semimajor axis

The asymptotic hyperbolic escape velocity v_∞ relative to the centre of mass of the general gravitational three bodies and its asymptotic value of the semimajor axis of the binary a_∞, depends on the perturbing velocities $\mathbf{v}_i$ (i = 1, 2, 3). For these small values of v_i and any values of their directions α_i (i = 1, 2, 3), we have $a_\infty(3v_\infty^2 - 4E_t) = 2$.

To show it, first attention is directed to the double limit process involved in the above result. After the binary is formed, the distance r between the escaper and the centre of mass of the binary, $r \to \infty$, the velocity of the escaper, $v \to v_\infty$, and the semimajor axis of the binary 'a' approaches asymptotically to the value a_∞. In this limit process the original three-body problem approaches its partition into two two-body problems. The escaper and the centre of mass of the binary form a hyperbolic two-body problem and the members of the binary form another elliptic two-body problem.

The first limit process refers to the behaviour of the general gravitational three-body problem as the perturbing velocities approach zero and we consider the general form of the total energy

$$E_t = \frac{1}{2}\left(\sum_{i=1}^{3} v_i^2 - 6\right).$$

The second limit process refers after the binary is formed. The total energy, E_t, may be written as

$$E_t = E_e + E_b + E_{eb},$$

where E_e is the escape energy, E_b is the energy stored in the binary, and E_{eb} is the correction due to three-body effects. Equations for E_e and E_b may be written from two-body consideration as follows:

$$E_e = \frac{Mm_2}{2(m_1+m_2)}\,v^2 - \frac{G\,m_2(m_1+m_2)}{r},$$

$$E_b = -\frac{G\,m_1 m_3}{2a},$$

where m_1 and m_3 form the binary and M is the total mass of the participating bodies. In this limit process E_t is fixed,

$$E_{eb} \longrightarrow 0, E_b \to -\frac{G\,m_1 m_3}{2a_\infty} \text{ and } E_e \to \frac{Mm_2 v_\infty^2}{2(m_1+m_3)},$$

or $E_t = \dfrac{Mm_2}{2(m_1+m_3)}\,v_\infty^2 - \dfrac{Gm_1m_3}{2a_\infty}$.

Equating the above two forms of the total energy we have

$$a_\infty(3v_\infty^2 - 4E_t) = 2$$

or

$$v_\infty = \left[\frac{2}{3}\left\{2E_t + \frac{1}{a_\infty}\right\}\right]^{1/2}.$$

It is also true for unequal masses as well. Moreover this is more general than Szebehely's (1974a, b) result viz., $a_\infty(v_\infty^2 - v_o^2 + 4) = 2/3$ or $v_\infty = \left[\dfrac{2}{3a_\infty} + v_o^2 - 4\right]^{1/2}$, which can be deduced from our results.

6. Numerical analysis

The numerical integration is performed by two different computer programmes through Unix System to obtain consistent and reliable numerical results. We have chosen an upper bound for 12 places of decimal for the local truncation error in both the programmes.

The first numerical integration is performed by using Runge-Kutta-Fehlberg 7(8) method. The second programme is developed by Sverre Aarseth, IOA, Cambridge, with the Aarseth & Zare (1974) regularization applied simultaneously to the two smallest distances at all times. The programme is able to handle the values of v_i (i = 1, 2, 3) up to the order of 10^{-10}.

6.1. Actual regions of escape

As the purpose of this paper is to investigate triple close approaches with systematic regularity of escape with the formation of a binary, so it is restricted to low velocities of the participating bodies. It has been observed that keeping two velocities fixed, the maximum magnitude of the third velocity for which the escape occur with the formation of a binary depends upon its direction. It is denoted by $v_i(\alpha_i)$, (i = 1, 2, 3) (Fig. 2).

It is observed that the values of $v_1(\alpha_1)$ fluctuates between maximum and minimum values. When the magnitude of the velocity is greater than $v_1(\alpha_1)$ the escape changes into ejection or interplay. Here, escape means that escape occurs simultaneously with the formation of a binary whereas in the case of interplay, the bodies performed repeated close approaches and in the case of ejection the two bodies form a binary and the third body ejected with elliptic relative velocity (Szebehely 1973). The trend remains the same in all other cases.

Now we define the following families:

1. $\mathbf{v}_1$ varies and $\mathbf{v}_2, \mathbf{v}_3$ fixed,
2. $\mathbf{v}_2$ varies and $\mathbf{v}_1, \mathbf{v}_3$ fixed,
3. $\mathbf{v}_3$ varies and $\mathbf{v}_1, \mathbf{v}_2$ fixed.

There are two sub-cases in each of these three families,
1(a) varying magnitude v_1 and keeping direction α_1 fixed,
 (b) varying direction α_1 and keeping magnitude v_1 fixed.
Similarly for the cases (2) and (3).

We have studied these families with perturbing velocities varying from $\mathbf{v}_i = 10^{-10}$ to $v_i(\alpha_i)$ and for all values of α_i i.e. $0 \le \alpha_i \le 2\pi$ (i = 1, 2, 3).

In actual experiment escape with the formation of a binary does occur according to our analysis in Sect. 4.

As our main aim is to study in detail the effect of the perturbing velocities on various parameters, we have taken typical representative member of the cases 1(a) and 1(b) of the first family. For case 1(a), we have taken $v_2 = 10^{-3}$, $v_3 = 10^{-1}$ and v_1 varying between 10^{-1} and 10^{-10} with the grid of 10^{-1},

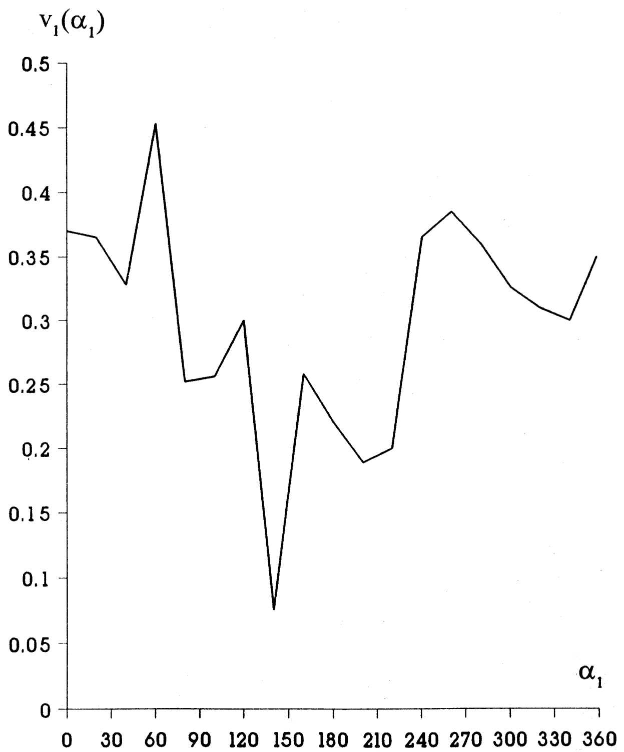

Fig. 2. Maximum initial velocity to escape with the formation of a binary ($0^\circ \leq \alpha_1 \leq 2\pi$, $v_2 = 10^{-2}$, $\alpha_2 = 30^\circ$, $v_3 = 10^{-1}$, $\alpha_3 = 50^\circ$).

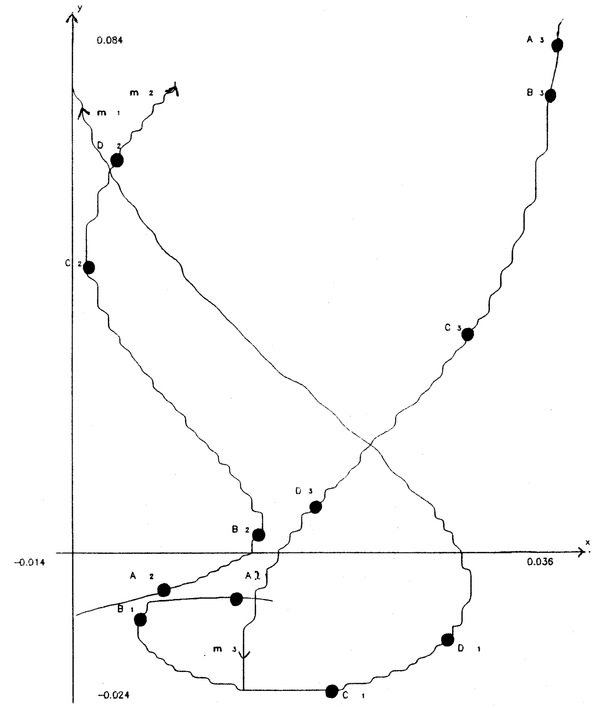

Fig. 3. Orbits near triple close approaches ($v_1 = 10^{-2}$, $\alpha_1 = 20^\circ$; $v_2 = 10^{-3}$, $\alpha_2 = 30^\circ$; $v_3 = 10^{-1}$, $\alpha_3 = 50^\circ$).

having directions $\alpha_1 = 20^\circ$, $\alpha_2 = 30^\circ$ and $\alpha_3 = 50^\circ$ (Table 1) and for case 1(b) $v_1 = 10^{-2}$, $v_2 = 10^{-3}$, $v_3 = 10^{-1}$, having directions $\alpha_2 = 30^\circ$, $\alpha_3 = 50^\circ$ and α_1 lying between 0 and 2π at an interval of 20° (Table 2).

In these tables, in the column of time, there are two values against each v_1 for Table 1 and each angle α_1 for Table 2. The first value gives the time t_b for the first close approach and the second t_m at the time of second close approach. Here, first close approach means the first minimum relative distance between the participating bodies and the second close approach means the smallest relative distance between the participating bodies when minimum moment of inertia is attained. Corresponding to these timings the other columns in the two tables give the values of the moment of inertia I_b and I_m, the distances r_1, r_2, r_3 from the centre of mass, the relative distances r_{12}, r_{23}, r_{31}, the magnitude of the velocities V_1, V_2, V_3 and the relative velocities v_{12}, v_{23}, v_{31} of the three participating bodies. We have also calculated absolute potential F, virial coefficient Q, and escape probability $\eta(= F/|E_t|)$ that is linked with the behaviour of escaper according to Agekian's et al. (1969) and Szebehely (1974b). The escaping body is mentioned in the last column.

In actual experiments we have observed that

1. two close approaches occur before the formation of a binary.
2. minimum smallest relative distance of the smallest relative distance among the relative distances of the participating bodies occur at the time of first close approach.
3. I_m occurs slightly latter than the first close approach.
4. the body which escapes is opposite to the minimum relative distance r_{ij} corresponding to the first close approach, though the actual escape occurs after I_m is attained. The distances of the escaper from the two bodies forming the binary go on increasing to infinity as t goes to infinity.
5. the two bodies, amongst the participating bodies having the first close approach, are different from the two bodies which have the second close approach.
6. the values of virial coefficient Q, F and η varies for each family. For case 1(a), of the first family these vary up to a certain value of v_1 after that they remain constant, and for case 1(b) they vary with α_1.

Similarly, we can deal with the other two families.

6.2. Orbits near triple close approaches

The orbits near triple close approaches for a typical representative member of a family (Case (I), Sect. 4) have been shown in Fig. 3.

The time t [also the transformed time $\tau = (t - 0.641) \times 10^6$] corresponding to the points of the above figure are listed in Table 3. The first close approach occurs at the point B and the second close approach (when I_m occurs) at the point C.

We observe that at the time of first close approach, the minimum moment of inertia does not occur. The reason is that when asymmetric initial conditions are introduced, just then all the three bodies begin their motion with a contraction towards the centre of mass and an interesting feature occurs between two bodies. These two bodies experience a first close approach while

Table 1. Timings t_b & t_m; I_b & I_m; $r_1, r_2, r_3; r_{12}, r_{23}, r_{31}; V_1, V_2, V_3; v_{12}, v_{23}, v_{31}; F, Q, \eta$ and escaper corresponding to v_1 ($\alpha_1 = 20°; v_2 = 10^{-3}, \alpha_2 = 30°; v_3 = 10^{-1}, \alpha_3 = 50°$).

$v_1 \downarrow$	Time (t)	M.I. (I)	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{r}_{12}$	$\mathbf{r}_{23}$	$\mathbf{r}_{31}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_{12}$	$\mathbf{V}_{23}$	$\mathbf{V}_{31}$	F	E	Q	Esc. Body
10^{-1}	0.665922	9.348E − 03	6.817E − 03	7.362E − 03	6.817E − 02	7.524E − 02	7.524E − 01	1.859E − 05	299	5	234	224	240	464	53818	17999	1.999	m_2
	0.670687	8.022E − 03	6.078E − 02	2.348E − 02	6.148E − 02	4.213E − 02	4.226E − 02	5.594E − 02	5	8	6	12	13	8	64	2	1.907	
10^{-2}	0.654100	6.913E − 03	9.454E − 03	1.457E − 02	8.131E − 02	6.316E − 03	8.835E − 01	8.799E − 02	10	15	5	25	20	6	181	60	1.967	m_3
	0.660517	4.679E − 03	3.307E − 02	3.981E − 02	4.473E − 02	6.767E − 02	3.482E − 02	5.470E − 02	4	7	8	7	14	10	62	21	1.903	
10^{-3}	0.651993	8.179E − 03	1.410E − 02	1.700E − 02	8.770E − 02	5.216E − 03	1.003E − 01	1.002E − 01	12	16	5	28	21	8	212	71	1.972	m_3
	0.659779	5.406E − 03	3.946E − 02	4.127E − 02	4.633E − 02	7.795E − 02	4.353E − 02	6.113E − 02	4	6	7	6	13	10	52	17	1.885	
10^{-4}	0.651784	8.312E − 03	1.460E − 02	1.729E − 02	8.832E − 02	5.120E − 03	1.014E − 01	1.014E − 01	12	16	4	28	21	8	215	72	1.972	m_3
	0.659599	5.485E − 03	3.982E − 02	4.088E − 02	4.720E − 02	7.825E − 02	4.450E − 02	6.255E − 02	4	6	7	6	12	10	51	17	1.883	
10^{-5}	0.651762	8.326E − 03	1.464E − 02	1.732E − 02	8.838E − 02	5.111E − 03	1.016E − 01	1.015E − 01	12	16	4	28	21	8	215	72	1.972	m_3
	0.659731	5.493E − 03	4.015E − 02	4.169E − 02	4.628E − 02	7.924E − 02	4.446E − 02	6.155E − 02	4	6	7	6	12	10	51	17	1.833	
10^{-6}	0.651760	8.327E − 03	1.465E − 02	1.733E − 02	8.839E − 02	5.110E − 03	1.016E − 01	1.015E − 01	12	16	4	28	21	8	215	72	1.972	m_3
	0.659727	5.493E − 03	4.015E − 02	4.168E − 02	4.630E − 02	7.923E − 02	4.447E − 02	6.158E − 02	4	6	7	6	12	10	51	17	1.883	
10^{-7}	0.651760	8.327E − 03	1.464E − 02	1.733E − 02	8.839E − 02	5.110E − 03	1.016E − 01	1.015E − 01	12	16	4	28	21	8	215	72	1.972	m_3
	0.659729	5.494E − 03	4.016E − 02	4.169E − 02	4.629E − 02	7.924E − 02	4.447E − 02	6.157E − 02	4	6	7	6	12	10	51	17	1.883	
10^{-8}	0.651760	8.327E − 03	1.464E − 02	1.733E − 02	8.839E − 02	5.110E − 03	1.016E − 01	1.015E − 01	12	16	4	28	21	8	215	72	1.972	m_3
	0.659729	5.494E − 03	4.016E − 02	4.169E − 02	4.630E − 02	7.924E − 02	4.447E − 02	6.157E − 02	4	6	7	6	12	10	51	17	1.883	
10^{-9}	0.651760	8.327E − 03	1.464E − 02	1.733E − 02	8.839E − 02	5.110E − 03	1016E − 01	1.015E − 01	12	16	4	28	21	8	215	72	1.972	m_3
	0.659727	5.494E − 03	4.015E − 02	4.168E − 02	4.631E − 02	7.923E − 02	4.447E − 02	6.158E − 02	4	6	7	6	12	10	51	17	1.883	
10^{-10}	0.651760	8.327E − 03	1.464E − 02	1.733E − 02	8.839E − 02	5.110E − 03	1.016E − 01	1.015E − 01	12	16	4	28	21	8	215	72	1.972	m_3
	0.659728	5.494E − 03	4.016E − 02	4.168E − 02	4.630E − 02	7.923E − 02	4.447E − 02	6.158E − 02	4	6	7	6	12	10	51	17	1.883	

the third body is delayed. At the time of this first close approach, the moment of inertia is not minimum, since the delayed third body called latecomer for all members of the family still moves towards the centre of mass along with another body. The latecomer comes closest to the centre of mass with the other two and experiences a second close approach. After the minimum moment of inertia is attained, the latecomer at the time of first close approach is always the escaping or ejected body and the other two that experience close approach at this time form a binary.

We have carried out experiments for a large values of $v_i (10^{-10} \leq v_i \leq v_i(\alpha_i))$ where α_i lies between 0 and 2π, and in each family, we see how a condition of complete collapse may be perturbed to obtain well-established families of asymmetric triple close approaches with systematic regularity of escape of the third body with the formation of a binary. Thus, we see that infinite escapes occur with the formation of a binary which indicate that the conjecture of Szebehely (1977), viz. "The measure of escaping orbits is significantly higher than the measure of stable orbits", is likely to be true.

6.3. *Comparison with Agekian's escape probability*

Agekian's et al. (1969) have defined the probability of escape

$$\eta = \frac{F}{|E_t|} \, .$$

(Strictly speaking η is not a probability as it is already >1).

The same definition is used by Szebehely (1974b). Both of them linked the behaviour of η with the escape and the formation of a binary. Since E_t, the total energy is fixed, so it depends on the absolute value of F which varies with $\mathbf{v}_i(\alpha_i)$. The details may be seen in Tables 1 and 2, where the first and second rows of each α_1 (or v_1) correspond to the first and second close approaches respectively. The point B (Table 3, see also Fig. 3) which corresponds to the first close approach, the absolute value of the potential F is maximum. At the point C (Table 3, see also Fig. 3) which corresponds to the second close approach and at this point the absolute value of the potential F is another maximum but smaller than at the point B. Consequently $F_B > F_C$ i.e. $\eta_B > \eta_C$. And therefore, the minimum moment of inertia is not associated with the maximum absolute value of the potential. This fact agrees with Agekian's et al. (1969) observations that F may be governed by one single close binary approach while I is governed by the closeness of all three bodies which is indeed for escape and which is an indication of a forthcoming escape. But on the other hand, we have also some cases where minimum moment of inertia I_m is associated with maximum absolute value of the potential F and in such case $F_C > F_B$ i.e. $\eta_C > \eta_B$. For example, when case (II) (Sect. 4; $v_1 = 10^{-3}$, $\alpha_1 = 50°$; $v_2 = 10^{-2}$, $\alpha_2 = 30°$; $v_3 = 10^{-1}$, $\alpha_3 = 10°$) is taken, we have $\eta_B = 27$ and $\eta_C = 4004$. It means $\eta_C > \eta_B$. Though this does not indicate the forthcoming escape but in actual experiment escape does occur with the formation of a binary.

Thus we conclude that the value of η that links the behaviour of escape according to Agekian's et al. (1969) does not necessarily indicate the forthcoming escape for all perturbing velocities. Hence our result is in contrast with Agekian's et al. Here the reason is that the value of F is governed by one single close binary approach after I_m is attained and not before it, whereas I_m is governed by the closeness of all the three bodies. So we cannot connect F with I_m.

7. Characteristics of the families

The following characteristics of the earlier mentioned families are observed for $10^{-10} \leq v_i \leq v_i(\alpha_i), 0 \leq \alpha_i \leq 2\pi$.

Table 2. Timings t_b & t_m; I_b & I_m; r_1, r_2, r_3; r_{12}, r_{23}, r_{31}; V_1, V_2, V_3; v_{12}, v_{23}, v_{31}; F, Q, η and escaper corresponding to α_1 ($v_1 = 10^{-2}$; $v_2 = 10^{-3}, \alpha_2 = 30°$; $v_3 = 10^{-1}, \alpha_3 = 50°$).

$\alpha_1 \downarrow$	**Time (t)**	**M.I. (I)**	$\mathbf{r}_1$	$\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{r}_{12}$	$\mathbf{r}_{23}$	$\mathbf{r}_{31}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_{12}$	$\mathbf{V}_{23}$	$\mathbf{V}_{31}$	**F**	**E**	**Q**	**Esc. Body**
0^0	0.654230	7.207E − 03	1.185E − 02	1.580E − 02	8.257E − 02	5.216E − 03	9.129E − 02	9.112E − 02	12	16	5	28	21	7	214	71	1.972	m_3
	0.660900	4.882E − 03	3.676E − 02	3.804E − 02	4.573E − 02	7.019E − 02	3.846E − 02	5.656E − 02	4	6	8	7	13	10	58	19	1.897	
20^0	0.654100	6.913E − 03	9.454E − 03	1.457E − 02	8.131E − 02	6.316E − 03	8.835E − 02	8.799E − 02	10	15	5	25	20	6	181	60	1.967	m_3
	0.660517	4.679E − 03	3.307E − 02	3.981E − 02	4.473E − 02	6.767E − 02	3.482E − 02	5.470E − 02	4	7	8	7	14	10	62	21	1.903	
40^0	0.653711	6.768E − 03	7.576E − 03	1.359E − 02	8.079E − 02	7.355E − 03	8.682E − 02	8.641E − 02	9	14	5	23	19	5	159	53	1.962	m_3
	0.659866	4.558E − 03	2.967E − 02	4.048E − 02	4.515E − 02	6.523E − 02	3.240E − 02	5.498E − 02	4	7	8	7	14	11	65	22	1.908	
60^0	0.653098	6.796E − 03	6.626E − 03	1.293E − 02	8.114E − 02	8.127E − 03	8.702E − 02	8.667E − 02	9	14	5	22	18	5	146	49	1.959	m_3
	0.659364	4.539E − 03	2.800E − 02	4.221E − 02	4.443E − 02	6.566E − 02	3.144E − 02	5.460E − 02	3	7	8	7	15	10	65	22	1.908	
80^0	0.652308	7.015E − 03	6.828E − 03	1.281E − 02	8.249E − 02	8.474E − 03	8.930E − 02	8.876E − 02	9	13	5	22	18	5	140	47	1.958	m_3
	0.658750	4.632E − 03	2.773E − 02	4.270E − 02	4.527E − 02	6.670E − 02	3.206E − 02	5.655E − 02	3	7	8	7	14	10	64	21	1.906	
100^0	0.651455	7.393E − 03	8.294E − 03	1.319E − 02	8.456E − 02	8.342E − 03	9.290E − 02	9.254E − 02	9	13	5	22	18	5	141	47	1.958	m_3
	0.658290	4.835E − 03	2.926E − 02	4.337E − 02	4.580E − 02	6.961E − 02	3.422E − 02	2.891E − 02	3	7	8	7	14	10	61	20	1.901	
120^0	0.650619	7.915E − 03	1.047E − 02	1.433E − 02	8.718E − 02	7.791E − 03	9.775E − 02	9.739E − 02	9	14	5	23	18	5	149	50	1.960	m_3
	0.658127	5.132E − 03	3.233E − 02	4.477E − 02	4.564E − 02	7.451E − 02	3.762E − 02	6.063E − 02	3	6	8	6	13	10	57	19	1.894	
140^0	0.649923	8.506E − 03	1.308E − 02	1.597E − 02	8.989E − 02	6.953E − 03	1.029E − 01	1.027E − 01	10	14	4	24	19	6	163	54	1.963	m_3
	0.657997	5.492E − 03	3.585E − 02	4.497E − 02	4.673E − 02	7.883E − 02	4.184E − 02	6.370E − 02	3	6	7	6	13	10	53	18	1.886	
160^0	0.649451	9.092E − 03	1.569E − 02	1.792E − 02	9.233E − 02	5.985E − 03	1.078E − 01	1.076E − 01	11	15	4	26	19	7	186	62	1.968	m_3
	0.658138	5.865E − 03	3.964E − 02	4.535E − 02	4.730E − 02	8.344E − 02	4.632E − 02	6.583E − 02	3	6	7	6	12	9	49	16	1.877	
180^0	0.649274	9.586E − 03	1.804E − 02	1.980E − 02	9.417E − 02	5.025E − 03	1.117E − 01	1.116E − 01	12	16	4	28	20	8	217	72	1.972	m_3
	0.658346	6.198E − 03	4.292E − 02	4.476E − 02	4.850E − 02	8.658E − 02	5.065E − 02	6.811E − 02	3	5	7	6	12	9	46	15	1.870	
200^0	0.649418	9.913E − 03	1.993E − 02	2.129E − 02	9.520E − 02	4.178E − 03	1.143E − 01	1.143E − 01	13	18	4	31	22	9	257	86	1.977	m_3
	0.653810	6.437E − 03	4.574E − 02	4.433E − 02	4.878E − 02	8.920E − 02	5.408E − 02	6.873E − 02	4	5	7	6	11	9	44	15	1.865	
220^0	0.649857	1.003E − 02	2.110E − 02	2.232E − 02	9.534E − 02	3.506E − 03	1.153E − 01	1.153E − 01	15	19	4	34	23	11	303	101	1.980	m_3
	0.659597	6.549E − 03	4.816E − 02	4.450E − 02	4.742E − 02	9.175E − 02	5.601E − 02	6.694E − 02	4	5	7	6	11	9	44	15	1.864	
240^0	0.650535	9.933E − 03	2.155E − 02	2.271E − 02	9.462E − 02	3.041E − 03	1.146E − 01	1.146E − 01	16	20	4	36	24	12	346	116	1.983	m_3
	0.660104	6.523E − 03	4.912E − 02	4.277E − 02	4.775E − 02	9.104E − 02	5.692E − 02	6.649E − 02	4	5	7	6	11	9	44	15	1.863	
260^0	0.651361	9.638E − 03	2.126E − 02	2.245E − 02	9.317E − 02	2.795E − 03	1.124E − 01	1.124E − 01	17	21	4	38	25	13	375	125	1.984	m_3
	0.660698	6.371E − 03	4.939E − 02	4.136E − 02	4.712E − 02	8.972E − 02	5.617E − 02	6.464E − 02	4	5	7	6	11	9	44	15	1.865	
280^0	0.652226	9.197E − 03	2.029E − 02	2.164E − 02	9.120E − 02	2.778E − 03	1.090E − 01	1.090E − 01	17	21	4	38	25	13	378	126	1.984	m_3
	0.660973	6.122E − 03	4.829E − 02	3.896E − 02	4.767E − 02	8.600E − 02	5.439E − 02	6.388E − 02	4	5	7	6	11	10	46	15	1.870	
300^0	0.653027	8.672E − 03	1.875E − 02	2.035E − 02	8.892E − 02	3.002E − 03	1.047E − 01	1.047E − 01	16	20	4	37	25	12	352	118	1.983	m_3
	0.661206	5.814E − 03	4.650E − 02	3.741E − 02	4.746E − 02	8.212E − 02	5.126E − 02	6.223E − 02	4	5	7	6	12	10	48	16	1.874	
320^0	0.653665	8.132E − 03	1.669E − 02	1.887E − 02	8.659E − 02	3.487E − 03	9.999E − 01	9.993E − 02	15	19	4	24	24	11	307	102	1.980	m_3
	0.661344	5.484E − 03	4.403E − 02	3.704E − 02	4.662E − 02	7.835E − 02	1.714E − 02	5.989E − 02	4	6	7	6	12	10	51	17	1.883	
340^0	0.654079	7.628E − 02	1.433E − 02	1.729E − 02	8.440E − 02	4.237E − 03	9.538E − 01	9.524E − 02	13	18	5	31	22	9	257	86	1.977	m_3
	0.661159	5.116E − 03	4.052E − 02	3.685E − 02	4.654E − 02	7.373E − 02	4.281E − 02	5.855E − 02	4	6	7	7	13	10	54	18	1.888	

Table 3. Time t and transformed time τ corresponding to points on the orbits in Fig. 3

Case ↓ (Sect. 4)	**Point → Time ↓**	$\mathbf{A}_i$	$\mathbf{B}_i$	$\mathbf{C}_i$	$\mathbf{D}_i$
Case (I)	t	0.652480	0.654100	0.660517	0.663747
	τ	11480	13100	19517	22747

Besides the observation of Sect. 6.1, we further observe that

1. For all members of a family, the general shape of the orbits is the same so long so the escaper remains the same.
2. We know that

$$F = G \sum_{1 \le i < j \le 3} \frac{m_i m_j}{r_{ij}}\,,$$

for $G = 1$, and $m_1 = m_2 = m_3$ (F is positive)

$$= \sum_{1 \le i < j \le 3} \frac{1}{r_{ij}}\,.$$

When a body escapes, its distances from the other two bodies tend to ∞ and F varies according to $\frac{1}{r_{ij}}$, where i and j do not take up the values corresponding to the suffix of the escaper. This also means that F is minimum or maximum according as the distance between the binary is maximum or minimum. Thus we conclude that the value of F is governed by one single close binary approach after I_m is attained and not before it.
3. When the perturbing velocities have the same magnitude and direction, by symmetry the relative distances between the participating bodies remain equal. But if directions of the perturbing velocities are not equal (keeping the magnitude of the velocities equal), we are able to say which body amongst them is going to escape with the formation of a binary.
4. The quantity $\sqrt{\frac{I_m}{3}}$ (the total mass is three), which is used as the radius of gyration by Birkhoff (1927) and Sundman (1912) instead of moment of inertia, always hold the following inequalities

$$a_\infty \geq \sqrt{\frac{I_m}{3}} \quad \geq \quad \text{smallest relative distance at the time of second close approach.}$$

and $\quad a_\infty(1-e) \quad \geq \quad$ smallest relative distance at the time of first close approach.

5. The magnitude of velocity of the latecomer (escaper) after the time of first close approach increases and the escaper moves towards the centre of mass.
6. The latecomer (escaper) must pass close to the centre of mass after the first close approach.
7. In each family, the body for which the perturbing velocity is varied never escapes.
8. The semimajor axis a_∞ and its eccentricity e for case (I) (Sect. 4) of the first family with their sub- cases are shown in Figs. 4a and 4b.
9. Proceeding as in Szebehely (1974a), it can be shown that

$$a_\infty(3v_\infty^2 - 4E_t) = 2$$

or $\quad v_\infty = \left[\frac{2}{3}\left\{2E_t + \frac{1}{a_\infty}\right\}\right]^{1/2}$ for all families.

The relation is in good agreement with the numerical results. We have shown the value of v_∞ for both the sub cases of the first family for case (I) (Sect. 4) in Figs. 5a and 5b.

It may be noted from the above Figs. 4a and 4b that for the family 1(a), when v_1 varies from 10^{-10} to 10^{-1} and other parameters are fixed, the semimajor axis a_∞ decreases as v_1 increases up to a certain value and then a rapid fall. Further, eccentricity e decreases slowly with v_1 up to a certain value and then there is sharp decrease and after that it increases rapidly. And for the family 1(b) when α_1 varies from $0°$ to 2π and other parameters are kept fixed as above, the semimajor axis a_∞ and its eccentricity e fluctuate. These trends are the same for all other families. Further, it may also be noted in Figs. 5a and 5b that for the first family 1(a) the escape velocity v_∞ decreases slowly as v_1 increases up to a certain value and after that it increases rapidly. And for the family 1(b), escape velocity v_∞ decreases as α_1 increases up to a certain value after that it increases rapidly.

10. The minimum moment of inertia I_m and the difference Δt between t_m and $t_c (\Delta t = t_m - t_c)$ of the families are characterised. For the family 1(a) for case (I) (Sect. 4), I_m decreases slowly up to a certain value as v_1 increases after that it decreases sharply and then increases rapidly where as Δt increases slowly up to a certain value with v_1, after that it increases rapidly (Fig. 6a). For the family 1(b) Δt and I_m fluctuate (Fig. 6b). The trends for I_m and Δt are the same for all other families.

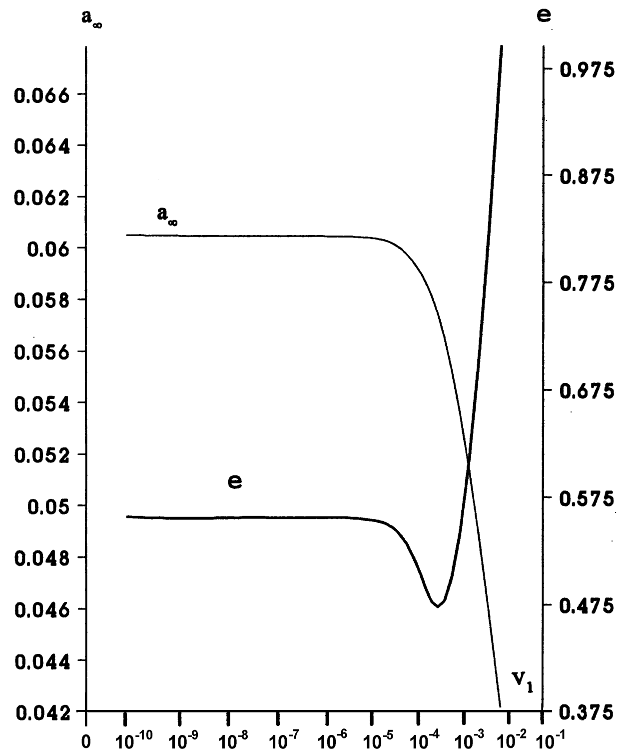

Fig. 4a. Semimajor axis a_∞ and eccentricity e Vs v_1 ($\alpha_1 = 20°$; $v_2 = 10^{-3}$, $\alpha_2 = 30°$; $v_3 = 10^{-1}$, $\alpha_3 = 50°$).

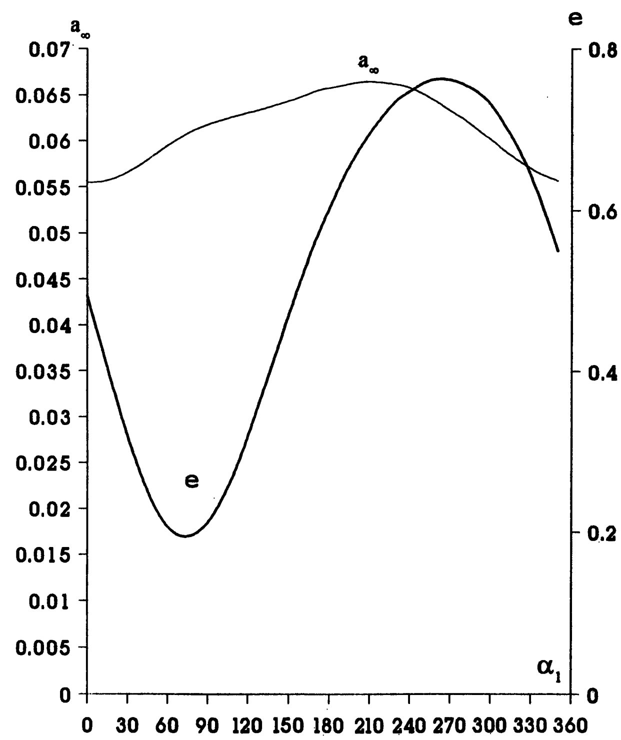

Fig. 4b. Semimajor axis a_∞ and eccentricity e Vs α_1 ($v_1 = 10^{-2}$; $v_2 = 10^{-3}$, $\alpha_2 = 30°$; $v_3 = 10^{-1}$, $\alpha_3 = 50°$).

8. Applications

Early references regarding individual orbits or statistically established families exist in the literature (Agekian & Anosova 1967, Szebehely & Peters 1967 and others). Nahon (1973), Waldvogel (1977), Szebehely (1974a, b, 1979), Chandra & Bhatnagar (1998a) have also investigated dynamical methods

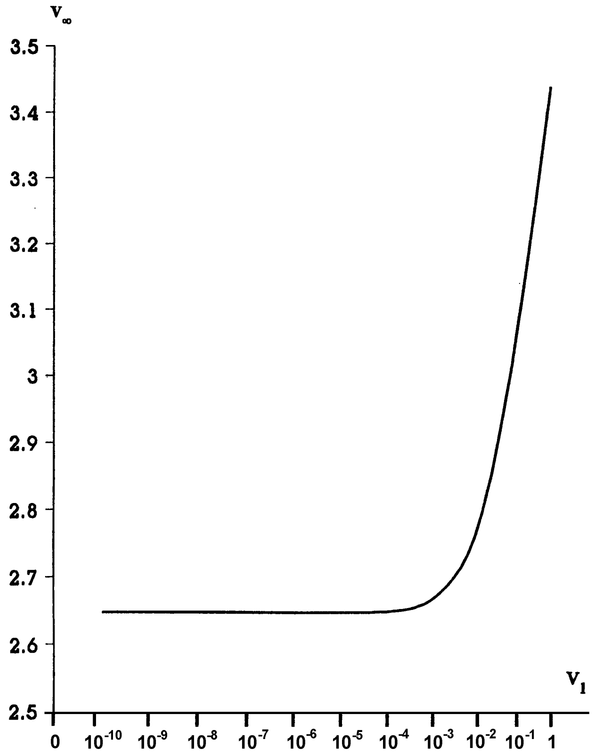

Fig. 5a. Escape velocity v_∞ Vs $v_1(\alpha_1 = 20°; v_2 = 10^{-3}, \alpha_2 = 30°; v_3 = 10^{-1}, \alpha_3 = 50°)$.

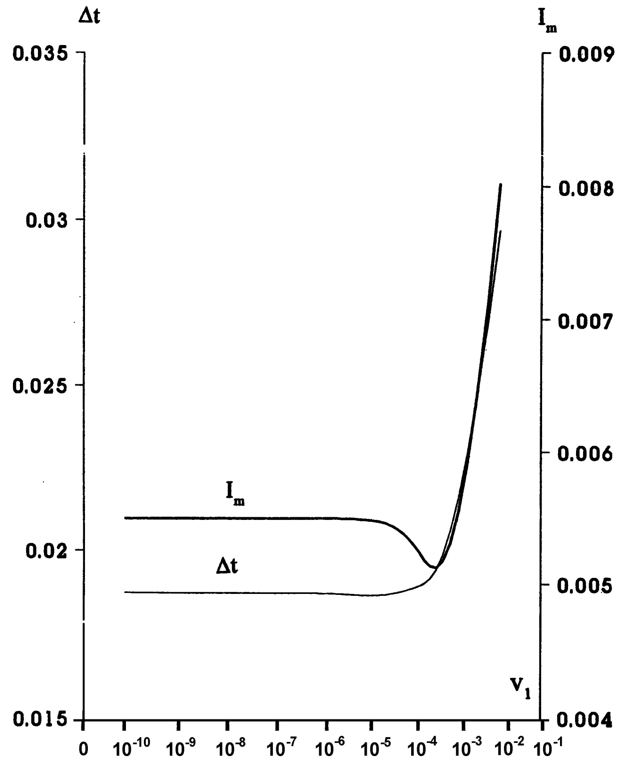

Fig. 6a. Time to triple close approach Δt and minimum moment of inertia I_m Vs $v_1(\alpha_1 = 20°; v_2 = 10^{-3}, \alpha_2 = 30°; v_3 = 10^{-1}, \alpha_3 = 50°)$.

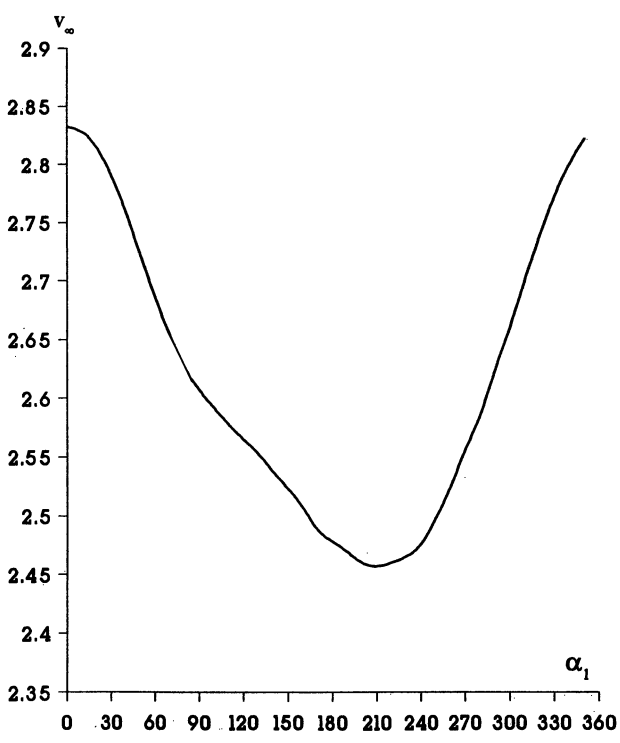

Fig. 5b. Escape velocity v_∞ Vs $\alpha_1(v_1 = 10^{-2}; v_2 = 10^{-3}, \alpha_2 = 30°; v_3 = 10^{-1}, \alpha_3 = 50°)$.

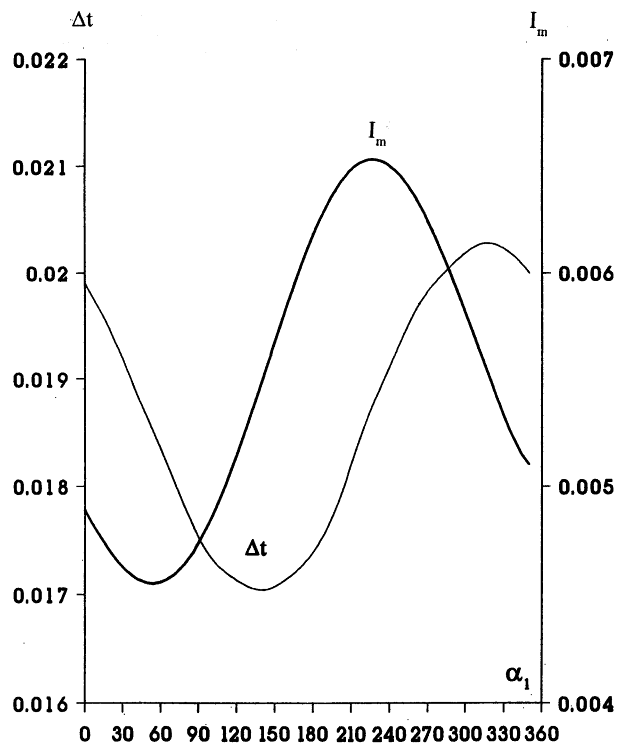

Fig. 6b. Time to triple close approach Δt and minimum moment of inertia I_m Vs $\alpha_1(v_1 = 10^{-2}; v_2 = 10^{-3}, \alpha_2 = 30°; v_3 = 10^{-1}, \alpha_3 = 50°)$.

that is closely related to this problem. The present study is based on a family rather on individual orbits. So it allows the continuous adjustment and matching of physical parameters.

For physical consideration, we shall consider the relative distances of the bodies at the time which has the minimum smallest relative distance between the first and the second close approaches. To simplify the notation in the sequel, we denote asymptotic value of the semimajor axis a_∞ as a, the relative distances r_{12}, r_{23} and r_{31} of the participating bodies as r, r', r'' respectively at the time of first or second close approach (which has the minimum smallest relative distance). The magnitude of relative velocity between the bodies at the time of having first close approach as κ and the hyperbolic escape velocity v_∞ of the escaper as v.

In actual experiment, we have found in the range of v_i i.e. $10^{-10} \le v_i \le 10^{-3}$, ($i$ = 1, 2, 3) for the same escaper of each of the families that

(a) the minimum smallest relative distance between the bodies at the time of first or second close approach (which has the minimum smallest relative distance) is directly proportional to the semimajor axis of the binary 'a' i.e. $\frac{a}{r}$ = const. = ν (say). Here r is the minimum of r, r', r''.
(b) $\nu r(3v^2 + 4E_t)$ = 2,
(c) $\frac{\kappa}{v}$ = const. = ξ (say).

These results mean that if in the non-dimensional system used, a minimum relative distance r at the time of first close approach is selected, we may compute a, v and κ. The members of the family of escape orbits corresponding to these preselected r is associated with a given value of the parameters v_i and α_i. For $0 < v_i \le 10^{-3}$, the above mentioned formulas (a)-(c) are applicable with sufficient accuracy.

The system reaches minimum size or rather minimum moment of inertia in approximately $\pi/\sqrt{24}$ = 0.6413 non- dimensional time-units.

All results of numerical aspects of triple close approaches are presented in non-dimensional form to facilitate any applications. The non-dimensional parameter is $GM_m T_m^2 = L_m^3$ in the numerical investigation. Here, G is the constant of gravity and M_m, T_m, L_m are the units of mass, time and length and the subscript m refers to 'model'. Consequently, the unit of time $T_m = L_m^{3/2}/\sqrt{(GM_m)}$ and the unit of velocity is $V_m = \sqrt{(GM_m)}/\sqrt{L_m}$. Here, note that in the following discussion lower case letters will denote the dimensionless quantities, while capitals will be reserved for dimensional quantities. The relations between these symbols are $R = rL_m$; $V = vV_m$ and $T = tT_m$. In addition, it is expedient to introduce the symbols $\rho = R_b/R_\odot$, where R_b is the distance between the binary. Moreover, as well as $\mu = M_m/M_\odot$ and $\lambda = R/2R_b$. The theory so developed can be applied to any desired model without difficulties. We have, therefore, considered the following four astronomical models as in Szebehely (1979).

Model (I): The first model consists of three Sun-like stars. For characteristic units, $M_m = M_\odot$ placed at the apices of an equilateral triangle at L_m = 1 pc apart. The unit of time T_m = 1.5 $\times 10^7$ yr and the unit of velocity V_m = 0.0655 Km/Sec. Here, $\mu = \rho$ = 1 and $T_c = 10^7$ yr.

Model (II): The second model consists of three 40 Eridani-B-like stars with $M_m = 0.43 M_\odot$ and $R_b = 0.016 R_\odot$. The units of time $T_m = 2.27 \times 10^7$ yr and $V_m = 0.043$ Km/Sec. Here, μ = 0.43, ρ = 0.016 and $T_c = 1.45 \times 10^7$yr.

Model (III): The third model consists of three white dwarfs of solar masses (i.e. $M_m = M_\odot$) which placed at 1 pc distances. The radius of the participating bodies is $R_b = 0.0068 R_\odot$, following Chandrasekhar's estimate. The unit of time $T_m = 1.5 \times 10^7$ yr and the unit of velocity is $V_m = 0.0655$ Km/Sec. In this case much closer approaches are allowed without significant tidal effects. Here μ = 1, ρ = 0.0068, $T_c = 10^7$yr.

Model (IV): This model consists of three neutron stars of solar masses and placed at 1 pc distances. The radius of participating bodies is R_b = 10 Km. The units of time $T_m = 1.5 \times 10^7$ yr and velocity V_m = 0.0655 Km/Sec. Here, μ = 1, ρ = 1.44 $\times 10^5$ and $T_c = 10^7$yr.

Model (V) of Szebehely (1979) involving galaxies cannot be taken as galaxies are not point masses.

Here, our main aim is to find the principal characteristics of the models. Such characteristics are the velocity of escape v of the escaper, semi-major axis a of the binary formed, its eccentricity e, and the time necessary to reach maximum contraction. The time necessary from the beginning of the motion to the occurrence of the escape in all cases is approximately $t_c = 0.64$ time units. Therefore, the escape times are determined by the time units applicable to the various cases. Here, we have taken typical representative member of both the sub-cases 1(a) and 1(b) of the first family. For this, we have observed from the Tables 1 and 2 that the values of earlier mentioned formulas (a)-(c) are constant (i.e. $\nu = 12, \xi$ = 11) for the same escaper for both the sub-cases of the first family as mentioned earlier. Here, smallest relative distance at the time having the first close approach is r_{12} and escaper is m_3. The method of selection of the proper member of the families are based on the choice of minimum distance from physical considerations. This minimum distance R during a realistic, physically possible motion must be considerably longer than $2R_b$. Since $R = 2\lambda R_b$, the bodies touch when $\lambda = 1$. For the purposes of this paper, we will select the following three values for λ/λ_1 :

$\frac{\lambda}{\lambda_1}$ = 10 (which is admittedly rather lower), 100 (corresponding to 1AU if $R_b = R_\odot$) and 1000.

At this point some simple results are offered as follows:

1. the semimajor axis of the binary formed is $A = \nu R = 2\nu R_b$,
2. the velocity of escape is

$$V\text{esp} = \left[\frac{1}{3\nu}\left(\frac{\mu}{\lambda\rho}\right)\left(\frac{GM_\odot}{R_\odot}\right) + \frac{2}{3}\left(V_m^2 \sum_{i=1}^{3} v_i^2\right) - \right.$$

Table 4. Applications to models of three-body systems

Model	λ	R	A	$\mathbf{V}_{esc}$ Km/Sec	$\mathbf{V}_{max}$ Km/Sec
(I)	10	$20\,R_\odot = 0.09\,AU$	$23 \times 10\,R_\odot = 1.1\,AU$	23	253
	10^2	$2 \times 10^2 R_\odot = 0.9\,AU$	$24 \times 10^2 R_\odot = 11\,AU$	7.3	80.3
	10^3	$2 \times 10^3 R_\odot = 9\,AU$	$24 \times 10^3 R_\odot = 110\,AU$	2.3	25.3
(II)	10	$0.32\,R_\odot = 1.5 \times 10^{-3} AU$	$3.84\,R_\odot = 1.8 \times 10^{-2} AU$	19.25	1311.75
	10^2	$3.2\,R_\odot = 1.5 \times 10^{-2} AU$	$38.4\,R_\odot = 1.8 \times 10^{-1} AU$	37.7	414.7
	10^3	$32\,R_\odot = 1.5 \times 10^{-1} AU$	$384\,R_\odot = 1.8\,AU$	11.93	131.23
(III)	10	$0.136\,R_\odot = 6.3 \times 10^{-4} AU$	$1.632\,R_\odot = 7.6 \times 10^{-3} AU$	729	3069
	10^2	$1.36 R_\odot = 6.3 \times 10^{-3} AU$	$16.32\,R_\odot = 7.6 \times 10^{-2} AU$	88.2	970.2
	10^3	$13.6\,R_\odot = 6.3 \times 10^{-2} AU$	$116.2 R_\odot = 7.6\,AU$	27.9	306.9
(IV)	10	$2 \times 10^2 Km = 1.3 \times 10^{-6} AU$	$2.4 \times 10^3 Km = 1.6 \times 10^{-5} AU$	6,062	66682
	10^2	$2 \times 10^3 Km = 1.3 \times 10^{-5} AU$	$2.4 \times 10^4 Km = 1.6 \times 10^{-4} AU$	1,917	21087
	10^3	$2 \times 10^4 Km = 1.3 \times 10^{-4} AU$	$2.4 \times 10^5 Km = 1.6 \times 10^{-3} AU$	606	6666

$$\frac{4G^2 M_m}{3}\left(\frac{1}{R}+\frac{1}{R'}+\frac{1}{R''}\right)\Bigg]^{1/2},$$

3. the magnitude of highest relative velocity i.e. $V_{\text{max}} = \xi V_{\text{esp}}$.

The results regarding the four models are given in Table 4. The first column gives the model. The second column gives various values of λ taken, as mentioned above. The third and fourth columns give R and A in astronomical units and in the fifth column, we have given magnitude of escape velocity V_{esc} determined from the formula mentioned in (ii) above and in the last column, we have mentioned magnitude of maximum relative velocity V_{max} by using formula given in (iii) above.

From the above table, we observe that the highest values of V_{esc} and V_{max} among all the models occur in the model (IV) that is for the model of three neutron stars. The highest value of V_{max} corresponds to approximately 19% of the velocity of light.

Acknowledgements. We are thankful to the University Grants Commission for funding this project. We are also thankful to all members of Delhi University Computer Centre, Delhi for assisting us in the computational work. Our thanks are also due to *'Centre For Fundamental Research in Space Dynamics and Celestial Mechanics'* for providing all facilities to carry out the present work. We wish to thank Dr. Sverre Aarseth, Institute of Astronomy, University of Cambridge, UK, for providing his computational programme and guide line to complete the work.

References

Aarseth S., Zare K., 1974, Celest. Mech. 10, 217
Agekian T.A., Anosova Zh. P., 1967, AZh 44, 1261
Agekian T.A., Anosona Zh.P., Bezgubova V.N., 1969, Astrofizika 5, 637
Agekian T.A., Martinova A.I., 1973, Univ. of Leningrad Publ. No. 1, 122
Birkhoff G.D., 1922, Bull. Nat. Res. Council 4, 1
Birkhoff G.D., 1927, Dynamical Systems. Am. Math. Soc. Publ., Providence, R.I.
Chandra N., Bhatnagar K.B., 1998, Submitted for publication to Ap&SS
Henon M., 1974, Celest. Mech. 10
Marchal C., Losco L., 1980, A&A 84, 1
Nahon F., 1973, Celest. Mech. 8, 169
Sundman K.F., 1912, Acta. Math. 36, 105
Szebehely V., Peters F., 1967, AJ 72, 876
Szebehely V., 1971, Celest. Mech. 4, 116
Szebehely V., 1973, In: Tapley B., Szebehely V. (eds.) Recent Advances in Dynamical Astronomy. D. Reidel, Dordrecht, p. 175
Szebehely V., 1974a, 1974b, AJ 79, 981 (paper I) and 1449 (paper II).
Szebehely V., 1977, In: Franzz O.G., Pismis P. (eds.) Proc. IAU Coll. 331, Univ. of Mexico Press Publ. 3, 145
Szebehely V., 1979, A&A 78, 349
Waldvogel J., 1976, Celest. Mech. 14, 287
Waldvogel J., 1977, Bull. de la Classe des Sciences, Academie Royal de Belgique, 63, 34