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Stellar evolution with rotation IV: von Zeipel's theorem and anisotropic losses of mass and angular momentum

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Abstract. The von Zeipel theorem is generalised to account for differential rotation in the case of a "shellular" rotation law (cf. Zahn 1992). We write this law in the form $\Omega = \Omega(r)$, a simplification which does not apply to fast rotation. We find that von Zeipel's relation contains a small additional term, generally further increasing the radiative flux at the pole and decreasing it at the equator. We also examine the local Eddington factor in rotating stars and notice some significant differences with respect to current expressions.

We examine the latitudinal dependence of the mass loss rates $\dot{M}(\vartheta)$ in rotating stars and find two main source of wind anisotropies: 1) the "$g_{\rm eff}$" effect which enhances the polar ejection; 2) the "opacity effect" (or "κ–effect"), which favours equatorial ejection. In O–stars the $g_{\rm eff}$ effect is expected to largely dominate. In B– and later type stars the opacity effect should favour equatorial ejection and the formation of equatorial rings. We also examine the behaviour of the wind density and notice a strong enhancement at the equator of B– and later type stars. Possible relations with the polar ejections and the skirt of η Carinae and with the inner and outer rings of SN 1987 A are mentioned. If $\dot{M}(\vartheta)$ has sharp extrema due to some peaks in the opacity law, non equatorial and symmetrical rings may be produced.

We also show that the global mass loss rate of a star at a given location in the HR diagram is rapidly increasing with rotation, which is in good agreement with the numerical models by Friend & Abbott (1986).

Anisotropic stellar winds remove selectively the angular momentum. For example, winds passing through polar caps in O–stars remove very little angular momentum, an excess of angular momentum is thus retained and rapidly redistributed by horizontal turbulence. These excesses may lead some Wolf–Rayet stars, those resulting directly from O–stars, to be fast spinning objects, while we predict that the WR–stars which have passed through the red supergiant phase will have lower rotation velocities on the average. We also show how anisotropic ejection can be treated in numerical models by properly modifying the outer boundary conditions for the transport of angular momentum. Finally, in an Appendix the equation of the surface for stars with shellular rotation is discussed.

Key words: stars: evolution – stars: interiors – stars: rotation – stars: winds, outflows

1. Introduction

Rotation has several different effects on stellar evolution. Particularly during the H– and He–burning phases, we may consider: 1. The change of the internal hydrostatic equilibrium. 2. The transport of chemicals and angular momentum by shears in differentially rotating stars, by meridional circulation and by horizontal turbulence. 3. The various effects of rotation at the stellar surface, in particular the effects on the mass loss rates. In the first three papers of this series, effects 1 and 2 had been considered in detail (cf. Meynet & Maeder 1997; Maeder 1997; Maeder & Zahn 1998). Other developments on rotational effects during evolution were also made recently by Heger & Langer (1996) and by Langer (1997, 1998).

In this work we concentrate on the effects of rotation at the stellar surface. The relation between the local flux and the effective gravity at the surface of a uniformly (or cylindrically) rotating star (von Zeipel 1924) is quite an important one because it allows us to know how the flux and $T_{\rm eff}$ are varying over the surface of rotating stars with radiative envelope. Here we shall examine a generalisation of von Zeipel's theorem for the interesting case of differential rotation known as "shellular rotation" proposed by Zahn (1992). The basic hypothesis for shellular rotation is that the differential rotation in the radiative zone of a non-magnetic star is giving rise to anisotropic turbulence, much stronger in the horizontal directions than in the vertical one, due to stratification. This geostrophic turbulence tends to reduce or suppress the horizontal shears, thus the latitudinal differential rotation is small with respect to the radial differential rotation. According to Zahn (1992 and private comm.), shellular rotation applies to fast as well as to slow rotators. On a surface level, the angular velocity Ω may be written

$$\Omega\,(r,\,\vartheta) = \overline{\Omega}(r) + \widehat{\Omega}\,(r,\,\vartheta)\,,$$

with $\widehat{\Omega} \ll \overline{\Omega}$, the horizontal average being taken as

$$\overline{\Omega} \;=\frac{\int \Omega \sin^3 \vartheta d\vartheta}{\int \sin^3 \vartheta d\vartheta}.$$

Indeed the above way of writing $\Omega(r,\vartheta)$ implies some limitations. Ω constant in horizontal directions does not mean that Ω is the same at a constant distance r to the stellar center. There is an angle ϵ (cf. Appendix) between the radial direction and the direction of gravity. As an example in a Roche model (cf. expr. A8–A10) at break–up velocity, at a colatitude $\vartheta = 45°$ the angle ϵ is about $13°$; for lower rotation it rapidly decreases essentially like Ω^2. Thus,the fact of taking $\Omega \simeq \Omega(r)$ as in most previous papers about shellular rotation implies that one does not consider the case of a too fast rotation. And by fast rotation, we mean cases where large deviations from spherical symmetry would occur as a result of the centrifugal force and of the possibly anisotropic effects of radiation on a rotating star.

We also want to know how the mass loss rates $\dot{M}$ are varying with rotation, a problem of great concern for stellar evolution. For a star of given mass and metallicity Z we are likely to expect some scatter in the global $\dot{M}$–rates according to rotation.

Another issue is the latitudinal dependence of the $\dot{M}$–rates at the surface of a rotating star. For example, Owocki & Gayley (1997; see also Owocki et al. 1998) have suggested that the mass ejection is minimum at the equator. This point is of key importance regarding the loss or gain of angular momentum by the residual star and also for the structure of the outer nebulae. Amazingly, we may expect a progressive concentration of angular momentum in hot stars which lose most of their mass through polar regions. Whether this is leading to fast spinning WR stars is an interesting possibility to be examined carefully. Also, anisotropic mass loss may considerably influence the shape of the nebulae surrounding massive stars, and as a matter of fact many nebulae show large deviations from sphericity.

Sect. 2 deals with the von Zeipel theorem and its generalisations. In Sect. 3 we examine the concept of the Eddington flux in rotating stars. In Sect. 4 the latitudinal variations of the mass loss rates at the surface of a rotating star are examined, as well as the change of the global $\dot{M}$–rates with rotation. In Sect. 5 we examine the growth or decline of angular momentum for non–spherical stellar winds and express new boundary conditions. Further perspectives are considered in Sect. 6.

2. The von Zeipel theorem revisited

Von Zeipel (1924) found that the radiative flux $\boldsymbol{F}$ of an equipotential in a uniformly rotating star is proportional to the local effective gravity $\boldsymbol{g}_{\mathrm{eff}}$. The flux is

$$\boldsymbol{F} = \chi \nabla T \tag{2.1}$$

with $\chi = 4\,ac\,T^3/(3\kappa\rho)$. In a uniformly rotating star the equipotentials and isobars coincide (barotropic case; cf. also Appendix), and using the equation of hydrostatic equilibrium in a rotating star one has

$$\boldsymbol{F} = \rho\chi\frac{dT}{dP}\frac{\nabla P}{\rho} = -\rho\chi\frac{dT}{dP}\boldsymbol{g}_{\mathrm{eff}} \tag{2.2}$$

From equilibrium relations (cf. Zahn 1992), one obtains

$$\rho\chi\frac{dT}{dP} = \frac{L(P)}{4\pi G M_\star(P)} \tag{2.3}$$

with

$$M_\star = M(1-\frac{\Omega^2}{2\pi G\rho_m})$$

where ρ_m is the average density inside the considered surface level. This gives

$$\boldsymbol{F} = -\frac{L(P)}{4\pi G M_\star(P)}\,\boldsymbol{g}_{\mathrm{eff}} \tag{2.4}$$

which is von Zeipel's relation. Thus, knowing the local effective gravity at a colatitude ϑ on the surface of a rotating star allows us to obtain the local radiative flux, and thus the local T_{eff}

$$T_{\mathrm{eff}}(\vartheta) \sim g_{\mathrm{eff}}^{1/4}(\vartheta) \tag{2.5}$$

We now consider the more general case of a non–uniformly rotating star with a "shellular" rotation law of the form $\Omega \simeq \Omega(r)$, subject to the above mentioned limitations. The star is baroclinic and the properties of the stellar surface differ from the case of uniform rotation (cf. Appendix). All quantities are developed linearly around their average on an isobar (cf. Zahn 1992), for example

$$\begin{aligned} T(P,\vartheta) &= \bar{T}(P) + \tilde{T}(P)P_2(\cos\vartheta) \\ \nabla T(P,\vartheta) &= \nabla\bar{T} + \nabla\tilde{T}P_2(\cos\vartheta) + \tilde{T}\nabla P_2(\cos\vartheta) \end{aligned} \tag{2.6}$$

Calling $f(\vartheta) = P_2(\cos\vartheta)$, we obtain for the radiative flux

$$\begin{aligned} \boldsymbol{F} &= (\bar{\chi}+\tilde{\chi}f(\vartheta))\left\{(\bar{\rho}+\tilde{\rho}f(\vartheta))\left[\frac{d\bar{T}}{dP}+\frac{d\tilde{T}}{dP}f(\vartheta)\right]\right. \\ &\quad \left.\frac{\nabla P}{\bar{\rho}} + \tilde{T}\nabla f(\vartheta)\right\} \end{aligned} \tag{2.7}$$

Keeping only first order perturbations, we get

$$\begin{aligned} \boldsymbol{F} &= \left(\bar{\rho}\bar{\chi}\frac{d\bar{T}}{dP}\right)\frac{\nabla P}{\bar{\rho}} + \left[\bar{\rho}\tilde{\chi}\frac{d\bar{T}}{dP} + \tilde{\rho}\bar{\chi}\frac{d\bar{T}}{dP} + \bar{\rho}\bar{\chi}\frac{d\tilde{T}}{dP}\right] \\ &\quad \frac{\nabla P}{\bar{\rho}} f(\vartheta) + \bar{\chi}\tilde{T}\nabla f(\vartheta) \end{aligned} \tag{2.8}$$

If we ignore the horizontal fluctuations we find again the usual von Zeipel relation. We call

$$\Theta = \frac{\tilde{\rho}}{\bar{\rho}} \quad \text{and} \quad \Lambda = \frac{\tilde{\mu}}{\bar{\mu}} \tag{2.9}$$

and from the equation of state applied to horizontal fluctuations, one has

$$\delta\frac{\tilde{T}}{\bar{T}} = \varphi\Lambda - \Theta \tag{2.10}$$

with the thermodynamic coefficients $\delta = -(\partial\ln\rho/\partial\ln T)_{P\mu}$ and $\varphi = (\partial\ln\rho/\partial\ln\mu)_{PT}$. From the hydrostatic equation, the density fluctuations are given by

$$\Theta = \frac{1}{3}\frac{r^2}{\bar{g}}\frac{d\Omega^2}{dr} \tag{2.11}$$

and for a stationary situation (Maeder & Zahn 1998) the horizontal μ–fluctuations are related to the vertical μ–gradient by

$$\Lambda = -\frac{1}{6}\frac{d\ln\mu}{d\ln r}\frac{U(r)}{|U(r)|} \tag{2.12}$$

Simple developments using the equation of state also give

$$\frac{\tilde{\chi}}{\bar{\chi}} = -\chi_T\frac{\Theta}{\delta} + (\chi_\mu + \frac{\varphi}{\delta}\chi_T)\Lambda \tag{2.13}$$

and

$$\frac{d\tilde{T}}{d\bar{T}} = \frac{\bar{T}}{\delta}\left(\varphi\frac{d\Lambda}{d\bar{T}} - \frac{d\Theta}{d\bar{T}}\right) = \frac{H_T}{\delta}\left(\frac{d\Theta}{dr} - \varphi\frac{d\Lambda}{dr}\right) \tag{2.14}$$

with $H_T = -\bar{T}(dr/d\bar{T})$. With these expressions the flux becomes

$$\boldsymbol{F} = \frac{-L(P)}{4\pi GM_\star(P)}\left\{\left[1 + \left[\left(1-\frac{\chi_T}{\delta}\right)\Theta + \left(\chi_\mu + \frac{\varphi}{\delta}\chi_T\right)\Lambda + \frac{H_T}{\delta}\left(\frac{d\Theta}{dr} - \varphi\frac{d\Lambda}{dr}\right)\right]f(\vartheta)\right] \boldsymbol{g_{\rm eff}} - \frac{3}{\delta}\frac{H_T}{r}\bar{g}(\Theta - \varphi\Lambda)\cos\vartheta\sin\vartheta\,[\boldsymbol{\vartheta}]\right\} \tag{2.15}$$

The last term only contributes to the horizontal flux, it is an order of magnitude smaller than the other terms in (2.15) because the horizontal fluctuations are multiplied by H_T/r, which is very small at the stellar surface. Thus we may neglect this term.

At many stages during stellar evolution there is no chemical gradient just beneath the stellar surface; thus, according to expression (2.12), we may take the approximation $\Lambda = 0$ and get

$$\boldsymbol{F} = \frac{-L(P)}{4\pi GM_\star(P)}\boldsymbol{g_{\rm eff}}(1 + \zeta(\vartheta)) \tag{2.16}$$

with

$$\zeta(\vartheta) = \left[\left(1-\frac{\chi_T}{\delta}\right)\Theta + \frac{H_T}{\delta}\frac{d\Theta}{dr}\right]f(\vartheta) \tag{2.17}$$

In this case the relation between the flux and effective gravity only depends on the vertical gradient of Ω. This approximation may not always be true, for example in some LBV and WR stars where there is a steep chemical gradient near the surface. In such cases we may expect variations of the horizontal composition on the horizontal surface and thus a non zero Λ.

2.1. The particular case of no mass loss

In the particular cases where there is no transport of angular momentum through the stellar surface, and thus in general no mass loss, expression (2.17) further simplifies. In such cases, the outer boundary condition is $d\Omega/dr = 0$ (cf. Talon, 1997; Talon et al. 1997), i.e. $\Theta = 0$ and thus we have the following simplification

$$\zeta(\vartheta) = \frac{H_T}{\delta}\frac{d\Theta}{dr}f(\vartheta) \tag{2.18}$$

In this case, close to the stellar surface Θ increases from negative values (since $d\Omega/dr$ is negative in most of the envelope) to zero at the surface and therefore $\frac{d\Theta}{dr}$ is positive at the stellar surface. Numerical models by G. Meynet show that $\frac{H_T}{\delta}\frac{d\Theta}{dr}$ may be of the order of 10^{-2}. Since $f(\vartheta)$ is equal to 1.0 at the pole and -0.5 at the equator, we see that the term $\zeta(\vartheta)$ in expression (2.18) is adding its effect to that of $g_{\rm eff}$ to enhance the radiative flux at the pole and to decrease it at the equator. For the general case (2.17), the result depends very much on the rotation law at the outer boundary.

3. The concept of Eddington flux in a rotating star

In a non–rotating star the Eddington factor Γ is the ratio of the luminosity L to the limiting luminosity, i.e.

$$\Gamma = \frac{L\kappa}{4\pi cGM} \tag{3.19}$$

where in general it is appropriate to take for κ the total Rosseland mean opacity rather than the electron scattering opacity only (but see also Sect. 4). Some authors consider this form of the Eddington factor also in rotating stars (cf. Lamers 1997; Langer 1997). Since in rotating stars the effective gravity and the flux are varying over the stellar surface, we need to consider a local $\Gamma(\vartheta)$ taken as the ratio of the stellar flux to the (local) limiting flux, i.e.

$$\Gamma(\vartheta) = \frac{F(\vartheta)}{F_{\rm lim}(\vartheta)} \tag{3.20}$$

Let us call $\boldsymbol{g_{\rm tot}}$ the sum of the gravitational, centrifugal and radiative accelerations

$$\boldsymbol{g_{\rm tot}} = \boldsymbol{g_{\rm eff}} + \boldsymbol{g_{\rm rad}} = \boldsymbol{g_{\rm grav}} + \boldsymbol{g_{\rm rot}} + \boldsymbol{g_{\rm rad}} \tag{3.21}$$

where

$$\boldsymbol{g_{\rm rad}} = \frac{-1}{\rho}\boldsymbol{\nabla}P_{\rm rad} = \frac{\kappa\boldsymbol{F}}{c} \tag{3.22}$$

with κ the total Rosseland mean opacity. According to expression (2.16), $\boldsymbol{g_{rad}}$ and $\boldsymbol{g_{\rm eff}}$ have the same direction, which introduces a major simplification.

The local Eddington limit in a rotating star is to be defined by the limit of a vanishing total gravity $\boldsymbol{g_{tot}} = 0$, this implies a limiting flux

$$\boldsymbol{F_{\rm lim}} = \frac{-c}{\kappa}\boldsymbol{g_{\rm eff}} \tag{3.23}$$

The expression for $\boldsymbol{g_{\rm eff}}$ depends on the assumptions made for the shape of the stellar surface (see Appendix). We emphasize that we do not assume that in a rotating star the total gravity should be zero, but $\boldsymbol{g_{tot}} = 0$ is the condition to impose to find the Eddingtong limiting flux. The vector $\boldsymbol{g}_{\mathbf{eff}}$ has both radial and tangential components, its modulus is

$$g_{\rm eff} \;=\; \left[\left(\frac{-GM}{r^2}+\Omega^2 r\,\sin^2\vartheta\right)^2 \right.$$
$$\left. +\Omega^2 r^2 \sin^2\vartheta\cos^2\vartheta\right]^{1/2} \quad (3.24)$$

while the radial component is merely

$$g_{\rm eff,r} = -\frac{GM}{r^2}\left(1-\frac{\Omega^2 r^3}{GM}\sin^2\vartheta\right) \quad (3.25)$$

The appropriate $\Omega(r)$ at the considered r–value on the stellar surface has to be considered. There is an angle ϵ between the radial direction and the direction of $\boldsymbol{g}_{\mathbf{eff}}$, (cf. Appendix) and as stated in the introduction this approach is not valid for fast rotation. From (3.23) and (3.24) we may obtain the expression of the limiting flux.

At the surface of a rotating star the Eddington factor is a local concept at colatitude ϑ, and from (2.16) and (3.23) we can express the ratio (3.20)

$$\Gamma(\vartheta) = \frac{\kappa(\vartheta)L(P)}{4\pi c GM_\star(P)}\left[1+\zeta(\vartheta)\right] \quad (3.26)$$

The luminosity $L(P)$ on an isobar is just the total stellar luminosity, since there is no stockage of luminosity in the outer layers and $M_\star(P)$ is given by expression (2.3). There is no direct dependence of Γ on $g_{\rm eff}$, since both the effective flux and the limiting flux have the same dependence on it. There is an explicit dependence of Γ on ϑ through $\zeta(\vartheta)$ and $\kappa(\vartheta)$. As seen above, $\zeta(\vartheta)$ is positive near the pole (for $\vartheta < 54°$) and negative towards the equator; thus the term ζ favours a higher Eddington factor in polar regions.

Let us now consider the main term, which is the only one in the barotropic case

$$\Gamma(\vartheta) = \frac{\kappa(\vartheta)L}{4\pi c GM_\star} \quad (3.27)$$

where $\kappa(\vartheta)$ is the total local opacity. For a constant opacity (e.g. electron scattering), the Eddington factor would be the same over the whole stellar surface. For an opacity law $\kappa(\rho,\,T)$ like Kramer's opacity, that increases with decreasing temperature, κ will be higher near the equator and Γ is thus closer to 1.0 at the equator than at the pole.

Another important difference between (3.27) and (3.19) is the term $M_\star = M(1-\frac{\Omega^2}{2\pi G\rho m})$, which indicates an explicit dependence of $\Gamma(\vartheta)$ on Ω, with a higher $\Gamma(\vartheta)$–value for larger rotation, while in (3.19) the dependence of Γ on rotation is missing. Thus, the appropriate coupling of the radiation flux with rotation leads to three differences between the usual Γ factor (3.19) and the factor $\Gamma(\vartheta)$ given by (3.26) for rotating stars: 1) the fact that the local opacities $\kappa(\vartheta)$ have to be considered; 2) the explicit dependence of $\Gamma(\vartheta)$ on rotation; 3) the presence of the term $\zeta(\vartheta)$.

4. Mass loss rates in rotating stars

The interaction of mass loss and rotation is complex and there are many aspects which may be considered: 1). The latitudinal dependence of the mass loss rates for a given rotation. 2) The change of the global mass loss rate with rotation. 3). The selective way with which the anisotropic mass loss removes the angular momentum and thus modifies the internal rotation law. 4) Anisotropic mass loss may significantly contribute to the driving or to the damping of meridional circulation. The first two points will be studied in this section, the third and the fourth ones in the next section.

4.1. Latitudinal variations of the mass loss rates

In order to obtain the distribution of the mass loss rates over the surface of a rotating star, we locally apply the theory of radiative winds as a function of the local gravity $g_{tot}(\vartheta)$ and of the local flux $F(\vartheta)$. Limits to this approach could come from collective effects in wind acceleration such as pulsation, running waves etc. Here we derive the basic expressions which will be used in numerical models later on. According to the radiative wind theory (cf. Castor et al. 1975; Pauldrach et al. 1986; Kudritzki et al. 1989; Puls et al. 1996) the mass loss fluxes expressed with the above notations are scaling like

$$\dot M(\vartheta) \sim (k\alpha)^{1/\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1-\alpha}{\alpha}} F(\vartheta)^{1/\alpha} g_{tot}^{1-\frac{1}{\alpha}}(\vartheta) \quad (4.28)$$

where k and α are the force multiplier parameters, k representing the number of lines with strengths above a critical value and α the slope of the line strength distribution. The values of k and α are changing with $T_{\rm eff}$ (cf. Pauldrach et al. 1986); for $T_{\rm eff}$ = 50'000, 40'000, 30'000 and 20'000 K one has respectively k= 0.124, 0.124, 0.17, 0.32, and α = 0.64, 0.64, 0.59, 0.565. Because the total opacity at a given optical depth is expressed with the force multiplier parameters in terms of the electron scattering opacity $\kappa_{\rm es}$, one has to be careful to use, in the expressions below, the Eddington factor expressed with $\kappa_{\rm es}$, i.e.

$$\Gamma(\vartheta) = \frac{\kappa_{\rm es}L(P)}{4\pi c GM_\star(P)}\left[1+\zeta(\vartheta)\right] \quad (4.29)$$

This means that here the only dependence of $\Gamma(\vartheta)$ on colatitude ϑ is coming through $\zeta(\vartheta)$, while the opacity variations will come through the dependence of k and α on ϑ.

Now, taking into account expressions (2.16) for the radiative flux $F(\vartheta)$ and (3.28) for the total gravity g_{tot}, we obtain for the mass flux

$$\dot M(\vartheta) \sim (k\alpha)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1-\alpha}{\alpha}}\left[\frac{L(P)}{4\pi GM_\star(P)}\right]^{\frac{1}{\alpha}}$$
$$\frac{g_{\rm eff}\,(1+\zeta(\vartheta))^{\frac{1}{\alpha}}}{(1-\Gamma(\vartheta))^{\frac{1}{\alpha}-1}} \quad (4.30)$$

This relation expresses the dependence of the mass loss rates on colatitude ϑ. If α and k are constant in latitude (as normally expected in O–stars), we see that $\dot M(\vartheta)$ mainly depends on $g_{\rm eff}$

(cf. also Owocki et al. 1996, 1998). This means that in a rotating hot star the mass loss rates by unit surface are much larger over the polar caps than at the equator. The terms $\zeta(\vartheta)$ and $\Gamma(\vartheta)$ in (4.30) slightly reinforce the polar mass loss.

As mentioned above k and α vary with $T_{\rm eff}$, these variations being larger for lower $T_{\rm eff}$, in particular for B and later type stars. This means that over the surface of a rotating star k and α also vary. The term $(k\alpha)^{\frac{1}{2}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1-\alpha}{\alpha}}$ increases by a factor of three from $T_{\rm eff}$= 50'000 K to 20'000 K. The term in brackets in (4.30) is also larger for lower α–values. The term containing Γ increases with Γ, the growth being much faster in case of lower α–values. On the whole the following picture emerges: 1. In hot, rotating stars the mass flux is higher at the poles and lower at the equator (the respective surface areas must of course be accounted for in numerical models). Let us call this the "$g_{\rm eff}$–effect" in rotating stars. 2. Polar ejection is also present near the Γ–limit where the mass flux is strongly increased as shown by (4.30). 3. In B and later type stars, the enhanced polar ejection is progressively compensated by the effects of larger bound-free and line opacities (higher k and lower α), which then favour a larger mass flux in the cooler equatorial regions. We shall call this the "κ–effect" in rotating stars. 4. For B and later type stars near the Γ–limit the mass flux is strongly enhanced, particularly in the equatorial regions. 5. Opacity peaks may lead to matter ejection in the form of rings at intermediate latitudes.

A so–called bistability of stellar winds has been found by Lamers et al. (1995; cf. also Lamers 1997) in non–rotating stars: near a $T_{\rm eff}$ of 20'000 K and also close to 10'000 K, large and rather abrupt changes of the force multiplier parameters k and α modify the mass loss rates and the relations between v_∞ and $v_{\rm esc}$ (cf. Sect. 4.2). These important changes of k and α should also occur on the surface of B– and A–type rotating stars, as a result of the decrease of $T_{\rm eff}$ between the pole and the equator; furthermore they should produce some corresponding strong changes in the mass loss regime. Numerical models of rotating stars will examine these effects, which are described by expression (4.30; see also Sect. 4.2).

4.2. Latitudinal variations of the terminal velocities and wind densities on the surface of a rotating star

The terminal velocity v_∞ scales as (Puls et al. 1996)

$$v_\infty \simeq 2.24\,\frac{\alpha}{1-\alpha}v_{\rm esc} \tag{4.31}$$

where $v_{\rm esc}$ is the escape velocity. For a rotating star these quantities must be considered at a colatitude ϑ on the Roche model. Assuming that the ejected particles keep their angular momentum, we easily obtain

$$v_\infty(\vartheta) \simeq 3.168\frac{\alpha}{1-\alpha}\left(\frac{GM}{r}\right)^{1/2}(1-\Gamma(\vartheta))^{1/2} \left[1-\frac{\Omega^2 r^3}{GM}\frac{\sin^2\vartheta}{2}\right]^{\frac{1}{2}} \tag{4.32}$$

The value of v_∞ is obviously higher at the pole than at the equator. The difference is even reinforced if α decreases from the pole to the equator.

If we wish to modelize the wind around a rotating star (cf. Owocki et al. 1994, 1996, 1998) it is also necessary to know the latitudinal variation of the wind density $\rho(\vartheta)$ at a distance $R(\vartheta)$, where v_∞ is reached. For $R(\vartheta)$ we take a constant multiple of the values $r(\vartheta)$ of the Roche model. Thus one has

$$\rho(\vartheta) = \frac{\dot M(\vartheta)}{v_\infty(\vartheta)}\frac{r^2(\vartheta)}{R^2(\vartheta)} = {\rm const}\frac{\dot M(\vartheta)}{v_\infty(\vartheta)} \tag{4.33}$$

We recall that $\dot M(\vartheta)$ is the mass flux at the stellar surface. Using (4.30), (4.31) and (4.33) we obtain

$$\rho(\vartheta) \;=\; {\rm const}(k\alpha)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{\alpha}}\left[\frac{L(P)}{4\pi GM_\star(P)}\right]^{\frac{1}{\alpha}} \frac{g_{\rm eff}\,(1+\zeta(\vartheta))^{\frac{1}{\alpha}}}{[1-\Gamma(\vartheta)]^{\left(\frac{1}{\alpha}-\frac{1}{2}\right)}} \left(\frac{GM}{r}\right)^{1/2}\left[1-\frac{\Omega^2 r^3}{GM}\frac{\sin^2\vartheta}{2}\right]^{\frac{1}{2}} \tag{4.34}$$

If α is constant over the stellar surface of the rotating stars (like in O–stars), the wind density is then higher at the pole than at the equator. If the values of α and k vary over the stellar surface (which can occur in rotating B– and A–type stars, particularly if the bistability limit is crossed), the wind density will then become much larger at the equator according to expr. (4.34). We notice that these equatorial enhancements of the wind density will be somewhat larger than those of the mass flux (expr. 4.30), because a decrease of α contributes to a decrease of the terminal velocity.

4.3. Possible relations with η Carinae and SN 1987 A

We now examine the possible relations of the above models with some recent observations. The HST pictures of η Carinae show two broad polar ejections and an equatorial skirt (cf. Ebbets et al. 1997; Davidson 1997). The star η Carinae is clearly a hot star close to the Γ–limit. Among the various explanations possible for the observed geometry of the ejections from η Carinae, we may point out the possibility that polar ejections result from the "$g_{\rm eff}$–effect" in (4.30) while the equatorial skirt is more likely to stem from the "κ–effect".

The complex structure around SN 1987 A consists of a bright elliptical inner ring and of two outer rings moved away from the central ring (cf. Crotts et al. 1989; Burrows et al. 1995). Currently the two outer rings are interpreted as real rings and not as rings due to the limb brightening of an hour glass shell (cf. Burrows et al. 1995). Their location and CNO composition suggest (cf. Panagia et al. 1996) that they were ejected at an earlier stage of evolution than the inner ring, perhaps when the SN progenitor was a blue supergiant. The bright inner ring is generally associated to the red supergiant stage in view of its composition, location and the timescales involved (cf. Crotts et al. 1995; Panagia et al. 1995, 1996; Burrows et al. 1995). We notice that an equatorial ring could be consistent with the "κ–effect" in cool stars while symmetrical outer rings would better correspond with peaks in the function $\dot{M}(\vartheta)$ due for example to some opacity peaks.

We must really wonder about the possibility of sharp extrema of the functions $\dot{M}(\vartheta)$ and $\rho(\vartheta)$ given by (4.30) and (4.34). This appears as a likely possibility. Indeed, if for some ϑ, the $T_{\rm eff}$ is such that there is a peak or a discontinuity in the opacity, $\dot{M}(\vartheta)$ will also show corresponding features at this colatitude. The net result will be the formation of rings. As a matter of fact, the very strong variations of α (cf. Lamers et al. 1995), which changes abruptly at $T_{\rm eff}$= 20'000 and 10'000 K, may produce steep enhancements in $\dot{M}(\vartheta)$ and $\rho(\vartheta)$ and lead to the formation of symmetrical rings in nebulae. Future numerical models may tell us what are the features in the observed anisotropic nebulae which can be accounted for by the above rotational effects.

4.4. Change of the global mass loss rates with rotation

A problem currently met in the computation of stellar models with rotation is to know, for a star at a given location in the HR diagram, how much higher the mass loss rates would be if this star is rotating. We may integrate numerically the expression of $\dot{M}(\vartheta)$ over the stellar surface, but often a simpler estimate may be useful. For such an estimate, we assume the star to have a fixed total stellar luminosity, a choice which is justified since the overall bolometric flux undergoes only very small changes with rotation, despite the large latitudinal variations (cf. Maeder & Peytremann 1970, 1972). Thus from (4.28) we obtain

$$\dot{M} \sim \bar{g}_{\rm eff}^{1-\frac{1}{\alpha}} \left[1-\bar{\Gamma}\right]^{1-\frac{1}{\alpha}} \tag{4.35}$$

where the average values $\bar{g}_{\rm eff}$ and $\bar{\Gamma}$ are taken at the root of $P_2(\cos\vartheta)=0$, i.e. for $\cos^2\vartheta = 1/3$. This gives the following ratio of the global mass loss rate for a rotating star to that of a non-rotating star at the same location in the HR diagram:

$$\frac{\dot{M}(\omega_0)}{\dot{M}(0)} = \frac{1}{\left[(1-\omega_0^2)\left(1-\frac{\kappa L}{4\pi c G M(1-\omega_0^2)}\right)\right]^{\frac{1-\alpha}{\alpha}}} \tag{4.36}$$

with the rotation parameter

$$\omega_0^2 = \frac{2}{3}\frac{\Omega^2 r_0^3}{GM} \tag{4.37}$$

The radius r_0 is also taken in $P_2(\cos\vartheta)=0$. The use of the parameter ω_0^2 is quite convenient since r_0 is the usual Eulerian coordinate of the interior models of rotating stars (cf. Meynet & Maeder 1997). The relations between r_0 and the polar or equatorial radii are given by the corresponding solutions of the surface equation (A7).

For stars far enough from the Eddington luminosity, one simply has

$$\frac{\dot{M}(\omega_0)}{\dot{M}(0)} = \frac{1}{(1-\omega_0^2)^{\frac{1-\alpha}{\alpha}}} \tag{4.38}$$

For O–stars $\frac{1-\alpha}{\alpha} = 0.56$, and we see that the behaviour predicted by (4.38) is in fact close to that found by Friend and Abbott (1986; cf. also Langer 1997). Far enough from the Eddington limit, expression (4.38) predicts the same mass loss rate as the complete form (4.36).

5. Change of the specific angular momentum as a result of anisotropic mass loss

The anisotropic mass loss demonstrated above may have major consequences for stellar evolution. For example, polar ejection, as in O-type stars, removes very little angular momentum, and in particular much less than if the mass loss rates would be the same over the stellar surface. This implies that the angular momentum not embarked by polar winds remains as an excess $\mathcal{L}_{\rm excess}$ in the outermost layers. This excess is rapidly redistributed within these layers by strong horizontal turbulence which operates on short timescales. In this respect, we recall that the adopted expression for shellular rotation implies that rotation should be small or moderate. Conversely equatorial mass loss will remove more angular momentum than the specific average. Curiously enough, equatorial mass loss by reducing the stellar rotation at the surface may also contribute to increase the degree of differential rotation and thus the effects of shear transport. Thus the interplay of these various physical effects will be a most interesting problem to further examine.

The above considerations imply that we must carefully rediscuss the surface boundary conditions applicable to the equation expressing the conservation of angular momentum in a rotating star, which is in Eulerian coordinates (cf. Zahn 1992).

$$\frac{\partial}{\partial t}\left(\rho r^2 \bar{\Omega}\right)_r = \frac{1}{5r^2}\frac{\partial}{\partial r}\left(\rho r^4 \bar{\Omega}\left[U(r)-5\dot{r}\right]\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho \nu r^4 \frac{\partial \bar{\Omega}}{\partial r}\right) \tag{5.39}$$

The first term in the second member expresses the transfer of angular momentum by the vertical component of the meridional circulation, the term $\dot{r}$ has been added to account for the effects of expansion or contraction (cf. Maeder & Zahn 1998), and the last term expresses the transport by shears. As well known (cf. Zahn 1992), this equation is of the 4th order, since $U(r)$ depends on the various derivatives of Ω up to the 3rd order. We have two boundary conditions at the inner edge of the radiative zone and two at the outer edge (cf. Talon, 1997; Talon et al. 1997). If we have no viscous, nor magnetic coupling at the stellar surface, the integration of Eq. (5.39) gives for an external shell of thickness $\Delta r = R - r$:

$$\frac{\partial}{\partial t}\left[\bar{\Omega}\int_r^R \rho r^4 dr\right] = -\frac{1}{5}\rho r^4 \bar{\Omega}\left[U(r)-5\dot{r}\right] + \mathcal{L}_{\rm excess} \tag{5.40}$$

This is expressed in Eulerian coordinates. In Lagrangian coordinates we would have, for a shell of mass ΔM

$$\Delta M \frac{d}{dt}\left(\bar{\Omega} r^2\right) = -\frac{4\pi}{5}\rho r^4 \Omega U(r) + 4\pi \mathcal{L}_{\rm excess} \tag{5.41}$$

Let us now express the excess of angular momentum $\mathcal{L}_{\rm excess}$ applied to the last remaining shell as a result of the upper inhomogeneous mass removal. This excess is the difference between the angular momentum $\mathcal{L}(\Omega)$ of the last shell and the angular momentum $\mathcal{L}_{\rm anis}(\Omega)$ anisotropically removed by mass loss. One has

$$\begin{aligned}\mathcal{L}_{\rm excess}(\Omega) &= \mathcal{L}(\Omega) - \mathcal{L}_{\rm anis}(\Omega) \\ &= [J(\Omega) - J_{\rm anis}(\Omega)]\,\Omega \\ &= J(\Omega)\Omega\left[1 - \frac{J_{\rm anis}(\Omega)}{J(\Omega)}\right] \end{aligned} \quad (5.42)$$

$J(\Omega)$ is the moment of inertia of a shell at the surface of a rotating star of angular velocity Ω while $J_{\rm anis}(\Omega)$ is the moment of inertia of the mass which is ejected by stellar winds. $\mathcal{L}_{\rm excess}$ will be positive for a polar ejection an negative for an equatorial one.

$$J(\Omega) = 2\pi \int_0^\pi \int_r^{r+\Delta r} r^4(\vartheta)\rho \sin^3\vartheta d\vartheta dr \quad (5.43)$$

Using $x(\vartheta) = r(\vartheta)/r_{pb}$ and considering that the shell is very thin with a mass ΔM, we get

$$J(\Omega) = r_{pb}^2 \frac{\int_0^\pi x^4(\vartheta)\sin^3\vartheta d\vartheta}{\int_0^\pi x^2(\vartheta)\sin\vartheta d\vartheta} \quad (5.44)$$

where $x(\vartheta)$ are the solutions of the surface equation (A7). In a spherical case, $x = 1$ and we get, as expected, $J = \frac{2}{3}r^2\Delta M$. Turning to $J_{\rm anis}$, we have

$$\frac{J_{\rm anis}(\Omega)}{J(\Omega)} = \frac{\int_0^\pi x^4(\vartheta)\dot{M}(\vartheta)\sin^3\vartheta d\vartheta}{\int_0^\pi \dot{M}(\vartheta)d\vartheta \int_0^\pi x^4(\vartheta)\sin^3\vartheta d\vartheta} \quad (5.45)$$

where $\dot{M}(\vartheta)$ is given by Eq. (4.30). If κ, α and $\Gamma(\vartheta)$ are constant over the stellar surface (e.g. for O-stars), we can write $g_{\rm eff}(\vartheta)$ instead of $\dot{M}(\vartheta)$ in the above equation. With the expressions (5.42), (5.44) and (5.45), we can express the gain or loss of angular momentum at the stellar surface in the form (5.40) or (5.41).

The application of this new boundary condition to stellar evolutionary models of rotating stars will modify the angular momentum during the evolution of mass losing stars. Polar ejections in massive hot stars evolving directly to Ofpe/WN and WN stars could increase the specific angular momentum in the remaining star by orders of magnitude. In addition, conservation of angular momentum during the contraction of the stellar core (the $\dot{r}$–term in 5.40) as well as the inwards transport of angular momentum by meridional circulation (the U–term in 5.40) will also contribute to produce fast–spinning cores in the advanced stages of massive star evolution. Thus, it is likely that some WR stars, which after the O–phase have always stayed on the blue side of the HR diagram are fast rotators, since polar ejection is likely to have been dominant. At the opposite, WR stars resulting from an evolution through the red-supergiant phase, where equatorial mass loss dominates, may show lower rotational velocities on the average. Of course, if magnetic coupling between the star and the matter ejected occurs, sinks of angular momentum will counteract the above effects, but at present there is no evidence for significant magnetic fields in hot stars (cf. Mathys, 1998).

Finally, we point out that anisotropic mass loss may also strongly influence meridional circulation. As a matter of fact, for a typical meridional circulation rising along the polar axis and sinking at the equator, an equatorial mass loss will constructively push the "conveyor belt" and enhance the meridional circulation. The reason is that matter will likely come from polar regions to compensate for the equatorial losses rather than from the interior, which would require more work against gravitation. At the opposite, polar mass loss will drain matter from the equator and thus will oppose the circulation. Such an effect could in some cases be large enough to damp, stop or even reverse the circulation.

In this context, it is clear that the usual treatment of meridional circulation is not compatible with a strong anisotropic mass loss. New developments of the theory of meridional circulation should account for the effects of anisotropic mass loss in future.

6. Future perspectives

The anisotropic shape of many nebulae around hot stars, WR stars, red supergiants, AGB stars and nuclei of planetary nebulae may owe their features to the anisotropies in the mass ejection and wind density from rotating stars. The exact relevance and contribution of the "$g_{\rm eff}$" and of the "κ–effect" in shaping these nebulae, in addition to other effects such as binarity and nonradial oscillations, must be better ascertain in future, a goal which seems feasible in view of the high resolution images now available. It was interesting to notice the possibility that opacity peaks may lead to the formation of non equatorial symmetrical rings.

Regarding stellar evolution, an interesting objective for the numerical models, is to determine the relative importance of all the various effects intervening: hydrostatic effects, shears, meridional circulation, internal transport of angular momentum and of chemicals, "$g_{\rm eff}$", "κ–effect" and anisotropic losses of mass and angular momentum, etc... We emphasize that for massive stars which could lose up to 90% of their mass (cf. Maeder 1992), the question of the anisotropies in the mass loss is not an academic problem. Some WR stars may be fast rotators, while other ones may be slow rotators, depending whether they have passed only through hot evolutionary phases, with mainly polar ejection, or through cooler phases with a lot of equatorial mass ejection.

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Appendix A: general equation for the stellar surface

In a cylindrically or uniformly rotating star the equipotentials are given in first approximation by

$$\Psi = \frac{GM}{r(\vartheta)} + \frac{1}{2}\Omega^2 r^2(\vartheta)\sin^2\vartheta = \mathrm{const} \tag{A1}$$

where $r(\vartheta)$ is the stellar radius at colatitude ϑ. In such a case the equipotentials and isobars coincide and the model is barotropic. In (A1) the distorsion of the gravitational potential is not taken into account (Roche model).

In case of shellular rotation with $\Omega = \Omega(r)$ the problem is no longer conservative and the star is baroclinic. We recall that expressing shellular rotation with a law of the form $\Omega = \Omega(r)$ implies that we consider rather low rotational velocities. As noted by Meynet & Maeder (1997) the isobars for shellular rotation are identical with the equipotentials (A1) of the conservative case. We now search for the equation of the stellar surface in the case of shellular rotation. This equation will be obtained by expressing the condition that a displacement on the surface neither requires nor produces energy, i.e.

$$\boldsymbol{g_{\mathrm{eff}}} \cdot \boldsymbol{ds} = 0 \tag{A2}$$

In spherical coordinates the components of the effective gravity are

$$\begin{aligned} g_{\mathrm{eff}} &= \frac{\partial\Phi}{\partial r} + \Omega^2 r \sin\vartheta \\ g_{\mathrm{eff},\vartheta} &= \frac{1}{r}\frac{\partial\Phi}{\partial\vartheta} + \Omega^2 r\sin\vartheta\cos\vartheta \end{aligned} \tag{A3}$$

where $\Phi = GM/r$. One can also write (cf. Meynet & Maeder 1997)

$$g_{\mathrm{eff}} = \boldsymbol{\nabla}\Psi - r^2\sin^2\vartheta\,\Omega\boldsymbol{\nabla}\Omega \tag{A4}$$

and the product (A2) becomes

$$\begin{aligned} &\frac{\partial\Psi}{\partial r}dr + \frac{1}{r}\frac{\partial\Psi}{\partial\vartheta}r d\vartheta - r^2\sin^2\vartheta\,\Omega\frac{\partial\Omega}{\partial r}dr \\ &\quad -r^2\sin^2\vartheta\frac{\Omega}{r}\,\frac{\partial\Omega}{\partial\vartheta}r d\vartheta = 0 \end{aligned} \tag{A5}$$

For shellular rotation this equation simplifies as follows:

$$\frac{d\Psi}{dr} = r^2\sin^2\vartheta\,\Omega\frac{d\Omega}{dr} \tag{A6}$$

This is the generalized form of Eq. (A1) for the stellar surface of a model with $\Omega = \Omega(r)$. If Ω = const, one is brought back to Ψ = const., i.e. the usual Eq. (A1). In integral form the general equation of the surface can also be written

$$\begin{aligned} &\frac{GM}{r(\vartheta)} + \frac{1}{2}\Omega^2 r^2(\vartheta)\sin^2\vartheta \\ &\quad - \sin^2\vartheta\int_{r_p}^{r(\vartheta)} r^2(\vartheta)\Omega\frac{d\Omega}{dr}dr = \frac{GM}{r_p} \end{aligned} \tag{A7}$$

where r_p is the polar radius. This equation gives the shape $r(\vartheta)$ of the star as a function of $\Omega(r)$. The discussion of Eq. (A7) shows, since generally $d\Omega/dr < 0$, that the oblatness of a differentially rotating star is larger than that of a uniformly rotating star. The exact shape critically depends on the rotation law in the external layers.

A.1. the particular case of no mass loss

If we make the simplifying approximation that there is no transport of angular momentum through the surface, i.e. $d\Omega/dr = 0$ at the stellar surface, one is brought back to the usual Roche model, with the following surface equation

$$\frac{1}{x} + \frac{4}{27}\omega^2 x^2 \sin^2\vartheta = 1 \tag{A8}$$

with

$$x = \frac{r}{r_p} \qquad \text{and} \qquad \omega^2 = \frac{\Omega^2 r_{eb}^3}{GM}$$

If the polar radius r_p varies with ω, the second member of (A8) is $r_{pb}/r_p(\omega)$, where the subscript "b" indicates values at the break–up velocity. On such a surface the gravity is given by

$$\begin{aligned} g \;=\; \frac{GM}{r_{pb}^2}&\left[\left(\frac{-1}{x^2} + \frac{8}{27}\omega^2 x\sin^2\vartheta\right)^2\right. \\ &\left.+\left(\frac{8}{27}\omega^2 x\sin\vartheta\cos\vartheta\right)^2\right]^{1/2} \end{aligned} \tag{A9}$$

Since the radial direction does not coincide with the normal to the surface (direction of the gravity), the angle ϵ between the two directions is

$$\cos\epsilon = -\frac{\boldsymbol{g_{\mathrm{eff}}}\cdot\boldsymbol{r}}{\mid \boldsymbol{g_{\mathrm{eff}}}\cdot r\mid} = \frac{\frac{1}{x^2} - \frac{8}{27}\omega^2 x\sin^2\vartheta}{\left[\left(-\frac{1}{x^2} + \frac{8}{27}\omega^2 x\sin^2\vartheta\right)^2 + \left(\frac{8}{27}\omega^2 x\sin\vartheta\cos\vartheta\right)^2\right]^{\frac{1}{2}}} \tag{A10}$$

where x is a solution of Eq. (A8) for given values of ω and colatitude ϑ.

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