

# The relation of the Allan- and $\Delta$ -variance to the continuous wavelet transform

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**Abstract.** This paper is understood as a supplement to the paper by Stutzki et al. 1998, where we have shown the usefulness of the Allan-variance and its higher dimensional generalization, the  $\Delta$ -variance, for the characterization of molecular cloud structures. In this study we present the connection between the Allan- and  $\Delta$ -variance and a more popular structure analysis tool: the wavelet transform. We show that the Allan- and  $\Delta$ -variances are the variances of wavelet transform coefficients.

**Key words:** ISM: clouds – ISM: structure

## 1. Introduction

In Stutzki et al. 1998 (hereafter paper I) we defined the  $\Delta$ -variance as a generalization of the Allan-variance, a method traditionally used in the stability analysis of electronic devices. The Allan- or  $\Delta$ -variance method allows a clear separation between the global drift characteristics of the signal and other contributions, like white noise. It is specially suited to estimate the spectral index of power law spectral distributions. We demonstrated the use of the two-dimensional  $\Delta$ -variance for the analysis of molecular cloud images, e.g. maps of CO line emission. Application of the  $\Delta$ -variance analysis to the integrated CO intensity images of various molecular clouds shows a power law behaviour of power spectrum of the intensity distribution with spectral index  $\beta$  close to 2.8. This intensity distribution has a structure well described by a *fractional Brownian motion (fBm)* model and in paper I it was shown that the power law index  $\beta$  derived by the  $\Delta$ -variance is related to the drift exponent  $H$  of the corresponding *fBm* model by  $\beta = E + 2H$  (where  $E$  is the Euclidean dimension of the model) and to other parameters characterizing the fractal, such as the area-perimeter index.

Another tool well established over the last decade in the areas of structure analysis and data processing is the wavelet transform. While the discrete wavelet transform is mainly used in the field of signal compression, the continuous wavelet transform has applications in texture analysis and feature extraction. The connection between the local regularity of a function and the scaling behaviour of its wavelet transform coefficients, as the scale considered becomes smaller and smaller,

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makes the wavelet technique a natural tool for the analysis of self-similar distributions with (multi-)fractal properties (eg. Holschneider 1995).

In the context of molecular cloud structure analysis the wavelet transform was first used by Gill & Henriksen 1990, who also pointed out the advantages of this method over traditional structure indicators like the autocorrelation function. In their study they found a power law behaviour for the centroid velocities of  $^{13}\text{CO}$  spectral data of L1551, as well as for the peak intensities. Langer et al. 1993, presented a clump decomposition of CO maps of the cloud Barnard 5 using the Laplacian pyramid transform, which is also a wavelet transform. The number distribution of clump masses derived by the Laplacian pyramid transform follows a power law, in agreement with the results presented by Kramer et al. 1998, obtained by using gaussian clump decomposition. These results give additional evidence for the hierarchical structure of molecular clouds.

The purpose of this paper is to show that both these concepts are closely related, namely that the Allan- and  $\Delta$ -variance can be expressed using the variance of wavelet transform coefficients for suitably chosen wavelets.

After a short introduction of the Allan-variance and the wavelet transform in one dimension we show the relation between these concepts (Sect. 2). The use of the  $\Delta$ -variance and the continuous wavelet transform generalizes our result to two dimensions (Sect. 3). We summarize our results in Sect. 4.

## 2. Allan-variance and wavelets

### 2.1. Allan-variance

We first repeat some facts about the Allan-variance; for a broader introduction the reader may consult appendix A of paper I.

To calculate the Allan-variance of a random signal  $s(t)$  we consider the differences  $d(t, T)$  of succeeding averages over a (time) interval  $T$  and calculate their variance. Defining the *down-up rectangle function* by

$$\sqcap_T(t) := \begin{cases} \frac{-1}{2T} & : -T \leq t < 0 \\ \frac{1}{2T} & : 0 \leq t \leq T \\ 0 & : \text{elsewhere,} \end{cases}$$

this difference can be written as a convolution

$$\frac{1}{T} \left( \int_{t-T}^t s(t') dt' - \int_t^{t+T} s(t') dt' \right) = - \left\{ s * 2 \sqcup_T \right\} (t) := d(t, T). \quad (1)$$

Without loss of generality we here assume the average  $\bar{s}$  to be zero. Then the Allan-variance  $\sigma_A^2(T)$  is defined as

$$\sigma_A^2(T) := \frac{1}{2} \langle d^2(t, T) \rangle_t, \quad (2)$$

where  $\langle \dots \rangle_t$  denotes the average over all times (or positions)  $t$ .

## 2.2. Wavelets

The Fourier transform of a function  $\psi$  is defined by  $\tilde{\psi}(\omega) = \int_{\mathbb{R}} e^{-i2\pi\omega t} \psi(t) dt$ . A square integrable function  $\psi$ , which satisfies

$$0 < c_\psi := \int_{\mathbb{R}} \frac{|\tilde{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \quad (3)$$

is called a *wavelet*. For  $a \in \mathbb{R} \setminus \{0\}$ ,  $b \in \mathbb{R}$  the wavelet transform of a signal  $s$  with respect to  $\psi$  is defined by

$$L_\psi s(a, b) := \frac{1}{\sqrt{c_\psi}} |a|^{-1/2} \int_{\mathbb{R}} s(t) \psi \left( \frac{t-b}{a} \right) dt. \quad (4)$$

Each wavelet transform coefficient  $L_\psi s(a, b)$  is thus the scalar product of the signal with a dilated and translated version of the “mother function”  $\psi$ .

## 2.3. The Allan-variance as variance of wavelet transform coefficients

As a special case of a mother function consider a translated and dilated version of the well known Haar wavelet  $\psi_{\tilde{H}}$ , which we define here as follows:

$$\psi_{\tilde{H}}(t) := \begin{cases} 1 & : -1 \leq t < 0 \\ -1 & : 0 \leq t \leq 1 \\ 0 & : \text{elsewhere} \end{cases}.$$

For this the constant defined by (3) is  $c_{\psi_{\tilde{H}}} = 8 \ln 2$  (see Appendix). In the case of this modified Haar wavelet the transform coefficient  $L_{\psi_{\tilde{H}}} s(a, b)$  given by (4) corresponds to the difference of succeeding integrals over  $s(t)$ , which is the same as the averaging process leading to the definition of the Allan-variance. It follows that

$$\int_{\mathbb{R}} \psi_{\tilde{H}} \left( \frac{t-b}{a} \right) s(t) dt = \int_{b-a}^b s(t) dt - \int_b^{b+a} s(t) dt,$$

which equals the term in brackets in Eq. (1) with  $b \equiv t$  and  $a \equiv T$ . Further we have

$$L_{\psi_{\tilde{H}}} s(a, b) = \frac{1}{\sqrt{c_{\psi_{\tilde{H}}}}} |a|^{-1/2} \int_{\mathbb{R}} s(t) \psi \left( \frac{t-b}{a} \right) dt$$

$$= \frac{1}{\sqrt{c_{\psi_{\tilde{H}}}}} |a|^{-1/2} \left( \int_{b-a}^b s(t) dt - \int_b^{b+a} s(t) dt \right)$$

$$\stackrel{(1)}{=} \sqrt{\frac{|a|}{c_{\psi_{\tilde{H}}}}} \cdot d(b, a)$$

$$\text{i.e. } d(b, a) = \sqrt{\frac{c_{\psi_{\tilde{H}}}}{|a|}} L_{\psi_{\tilde{H}}} s(a, b). \quad (5)$$

Using Eq. (2) it follows that the Allan-variance can be expressed as

$$\sigma_A^2(a) = \frac{1}{2} \frac{c_{\psi_{\tilde{H}}}}{|a|} \langle L_{\psi_{\tilde{H}}}^2 s(a, b) \rangle_b, \quad (6)$$

that is the Allan-variance at lag  $a$  equals the average of the wavelet transform coefficients over the translation index  $b$  for constant dilation index  $a$ .

## 3. $\Delta$ -variance and $\Delta$ -wavelet

The main idea of the preceding section was, that the difference of adjacent averages corresponds with the convolution product of the signal  $s$  with an oscillating function  $\psi$  of zero mean, that is a wavelet transform coefficient  $L_\psi s$ . This concept will now be extended to the symmetric version of the Allan-variance, the  $\Delta$ -variance as introduced in paper I. We recall from paper I (Appendix A, p. 713 ff.) the definition of the one-dimensional *down-up-down rectangle function*

$$\sqcup \sqcup (x) := \begin{cases} 1 & : |x| \leq 1/2 \\ -1/2 & : 1/2 < |x| \leq 3/2 \\ 0 & : \text{elsewhere} \end{cases}.$$

The *double difference*  $\Delta(t, T)$ , that is the average over succeeding intervals is defined as follows:

$$\begin{aligned} \Delta(t, T) &:= \left\{ s(\dots) * \frac{1}{T} \sqcup \sqcup \left( \frac{\dots}{T} \right) \right\} (t) \\ &= \int_{\mathbb{R}} dx \frac{1}{T} \sqcup \sqcup \left( \frac{t-x}{T} \right) s(x). \end{aligned} \quad (7)$$

The  $\Delta$ -variance is the variance of the *double difference*  $\Delta(t, T)$ :

$$\sigma_\Delta^2(T) := \frac{1}{2} \langle \Delta^2(t, T) \rangle_t. \quad (8)$$

To find the relation to a wavelet transform we choose the *down-up-down rectangle function*  $\sqcup \sqcup (x)$  as mother function, which defines the family of one-dimensional “ $\Delta$ -wavelets”<sup>1</sup>:

$$\psi_{\Delta a, b}(x) := \sqcup \sqcup \left( \frac{x-b}{a} \right),$$

where again  $a \in \mathbb{R} \setminus \{0\}$  and  $b \in \mathbb{R}$ . The value of the admissibility constant is  $c_{\psi_\Delta} \approx 3.37$ . The corresponding wavelet transform coefficients are

$$L_{\psi_\Delta} s(a, b) = \frac{1}{\sqrt{c_{\psi_\Delta}}} |a|^{-1/2} \int_{\mathbb{R}} dx \sqcup \sqcup \left( \frac{x-b}{a} \right) \cdot s(x),$$

<sup>1</sup> A scaled version of this  $\Delta$ -wavelet is used as “French Hat” wavelet in the literature, cf. Gill & Henriksen 1990.

and with  $a \equiv T, b \equiv t$  and  $\square\square(x) = \square\square(|x|)$  one gets

$$\Delta(b, a) \stackrel{(7)}{=} \sqrt{\frac{c_{\psi\Delta}}{|a|}} L_{\psi\Delta} s(a, b).$$

With Eq. (8) this leads to the result analogous to Eq. (6):

$$\sigma_{\Delta}^2(a) = \frac{1}{2} \frac{c_{\psi\Delta}}{|a|} \langle L_{\psi\Delta}^2 s(a, b) \rangle_b. \quad (9)$$

### 3.1. Two-dimensional continuous wavelet transform

To generalize our results to higher dimensions we make use of the definition of two-dimensional (directional) wavelets given by Antoine & Murenzi 1996. As in the one-dimensional case a two-dimensional function  $\psi$  is called a wavelet, if the following admissibility condition holds:

$$0 < c_{\psi} := \int_{\mathbb{R}^2} \frac{|\tilde{\psi}(\mathbf{f})|^2}{\|\mathbf{f}\|^2} d^2\mathbf{f} < \infty.$$

From Antoine & Murenzi 1996 we adopt the notation for translation, dilation and rotation operators:

$$\begin{aligned} \text{translation:} \quad & (T^{\mathbf{b}}s)(\mathbf{x}) = s(\mathbf{x} - \mathbf{b}), \quad \mathbf{b} \in \mathbb{R}^2 \\ \text{dilation:} \quad & (D^a s)(\mathbf{x}) = \frac{1}{a} s\left(\frac{\mathbf{x}}{a}\right), \quad a > 0 \\ \text{rotation:} \quad & (R^{\theta} s)(\mathbf{x}) = s(r_{-\theta}(\mathbf{x})), \quad \theta \in [0, 2\pi), \end{aligned}$$

where  $r \in SO(2)$ . A combination  $\Omega(a, \theta, \mathbf{b}) = T^{\mathbf{b}} D^a R^{\theta}$  of these operators acts on a function  $s$  in the following way:

$$(\Omega(a, \theta, \mathbf{b})s)(\mathbf{x}) = s_{a, \theta, \mathbf{b}}(\mathbf{x}) \equiv \frac{1}{a} s\left(\frac{1}{a} r_{-\theta}(\mathbf{x} - \mathbf{b})\right).$$

With this notation the two-dimensional continuous wavelet transform (CWT) of a function (or “image”)  $s$  with respect to  $\psi$  is given by

$$L_{\psi} s(a, \theta, \mathbf{b}) := \frac{|a|^{-1}}{\sqrt{c_{\psi}}} \int_{\mathbb{R}^2} d^2\mathbf{x} \psi\left(r_{-\theta}\left(\frac{\mathbf{x} - \mathbf{b}}{a}\right)\right) s(\mathbf{x}) \quad (10)$$

with  $a \in \mathbb{R} \setminus \{0\}, \mathbf{b} \in \mathbb{R}^2$ . For a complex valued mother function  $\psi$  in Eq. (10) is replaced by its complex conjugate  $\psi^*$ , but we will only make use of real valued functions here. At present only isotropic wavelets  $\psi(\mathbf{r}) = \psi(r), r = |\mathbf{r}|$  will be considered, so that we omit the index  $\theta$ :

$$L_{\psi} s(a, \mathbf{b}) := \frac{|a|^{-1}}{\sqrt{c_{\psi}}} \int_{\mathbb{R}^2} d^2\mathbf{x} \psi\left(\frac{\mathbf{x} - \mathbf{b}}{a}\right) s(\mathbf{x}).$$

### 3.2. $\Delta$ -variance in two dimensions

This section is based on paper I, Appendix B (p. 717), where we now consider the case  $E = 2$ , that is two-dimensional signals or images.

The two-dimensional *down-up-down rectangle function* is given by:

$$\square\square(\mathbf{r}) := \frac{3^2}{(3^2 - 1)} \left[ \square(\mathbf{r}) - \frac{1}{3^2} \square\left(\frac{\mathbf{r}}{3}\right) \right]$$

$$= \begin{cases} 1 & : & |\mathbf{r}| \leq 1/2 \\ -1/8 & : & 1/2 < |\mathbf{r}| \leq 3/2 \\ 0 & : & \text{elsewhere} \end{cases}.$$

For the two-dimensional unit sphere the “volume” is  $\mathcal{V}_2 = \pi$  and the “surface” is  $\mathcal{S}_2 = 2\pi$ . Using the nomenclature of paper I we have

$$\odot_{D,2}(\mathbf{r}) = \frac{1}{\pi\left(\frac{D}{2}\right)^2} \square\square\left(\frac{\mathbf{r}}{D}\right). \quad (11)$$

Convolving  $s(\mathbf{r})$  with  $\odot_{D,2}(\mathbf{r})$  leads to the difference of averages at average distance  $D$ :

$$\begin{aligned} \Delta_s(\mathbf{r}, D, 2) &:= \left\{ s(\dots) * \odot_{D,2}(\dots) \right\}(\mathbf{r}) \\ &= \int_{\mathbb{R}^2} d^2\mathbf{x} \odot_{D,2}(\mathbf{r} - \mathbf{x}) \cdot s(\mathbf{x}) \\ &= \frac{1}{\pi\left(\frac{D}{2}\right)^2} \int_{\mathbb{R}^2} d^2\mathbf{x} \square\square\left(\frac{\mathbf{r} - \mathbf{x}}{D}\right) \cdot s(\mathbf{x}) \quad (12) \end{aligned}$$

As in the one-dimensional case the  $\Delta$ -variance is given as the variance, i.e the autocorrelation function at zero lag of the filtered signal:

$$\sigma_{\Delta_s}^2(D) := \frac{1}{2\pi} A_{\Delta_s(r, D, 2)}(0). \quad (13)$$

### 3.3. $\Delta$ -variance

and two-dimensional continuous wavelet transform

To relate the  $\Delta$ -variance to the two-dimensional CWT one naturally considers the *down-up-down-rectangle function* given by Eq. (11) as mother wavelet and defines

$$\psi_{\Delta_{a,b}}(\mathbf{r}) := \square\square\left(\frac{\mathbf{r} - \mathbf{b}}{a}\right);$$

$\psi_{\Delta_{a,b}}$  will also be called (two-dimensional)  $\Delta$ -wavelet; the admissibility constant for the two-dimensional function is  $c_{\psi\Delta} \approx 4.11$ . Like in the one-dimensional case the wavelet transform coefficients  $L_{\psi\Delta} s(a, \mathbf{b})$  are directly related to the differences of average values at average distance  $D$   $\Delta(\mathbf{r}, D, 2)$  (cf. Eq. (12)):

$$\begin{aligned} L_{\psi\Delta} s(a, \mathbf{b}) &= \frac{1}{\sqrt{c_{\psi\Delta}}} \frac{1}{a} \int_{\mathbb{R}^2} d^2\mathbf{x} \psi_{\Delta_{a,b}}(\mathbf{x}) \cdot s(\mathbf{x}) \\ &= \frac{1}{\sqrt{c_{\psi\Delta}}} \frac{1}{a} \int_{\mathbb{R}^2} d^2\mathbf{x} \square\square\left(\frac{\mathbf{x} - \mathbf{b}}{a}\right) \cdot s(\mathbf{x}) \quad (14) \end{aligned}$$

We identify  $D \equiv a$  and  $\mathbf{r} \equiv \mathbf{b}$  and because of  $\square\square(\mathbf{r}) = \square\square(|\mathbf{r}|)$  we find using Eq. (12) and Eq. (14):

$$\begin{aligned} L_{\psi\Delta} s(a, \mathbf{b}) &= \frac{1}{\sqrt{c_{\psi\Delta}}} \frac{1}{a} \cdot \frac{\pi a^2}{4} \cdot \Delta_s(\mathbf{b}, a, 2) \\ &= \frac{\pi a}{4 \cdot \sqrt{c_{\psi\Delta}}} \cdot \Delta_s(\mathbf{b}, a, 2). \end{aligned}$$

Looking at the autocorrelation function as the two-dimensional generalization of the variance as we have done in Eq. (13), we find as the main result of this section:

$$\sigma_{\Delta_s}^2(a) = \frac{1}{2\pi} \frac{16 c_{\psi_\Delta}}{\pi^2 a^2} A_{L_{\psi_\Delta s}(a,b)}(0), \quad (15)$$

i.e. the  $\Delta$ -variance for a given lag  $a$  can be expressed in terms of the autocorrelation function at zero lag of the wavelet transform coefficients with dilation parameter  $a$ .

#### 4. Summary and outlook

The main results of our study are Eqs. (6), (9) and (15) which show, that the Allan- and  $\Delta$ -variance can be expressed using the coefficients of special wavelet transforms and that both these concepts are very closely related. Pando & Fang 1998 have shown that the *discrete* wavelet transform is a good power spectrum estimator especially in the case of finite sized samples. The same result was found by Bensch et al. (priv. comm.) for the  $\Delta$ -variance analysis of CO integrated intensity maps as small as  $32 \times 32$  pixel. Detailed investigations by Bensch et al. and Ossenkopf et al. 1998 show that the  $\Delta$ -variance remains a robust measure of astronomical images if the effects of noise, smearing due to the finite size antenna beam pattern and radiative transfer are taken into account.

Pando et al. pointed out that evaluating the power spectrum with an over-complete basis function system of the CWT can lead to correlations “that are not in the sample, but due to correlations among the wavelet coefficients”. This will also be true for the Allan- and the  $\Delta$ -variance, because the definitions of both employ such an over-complete system. It has to be checked in a further study, how the results change, if only coefficients on a discrete, wavelet adapted grid (a “wavelet frame”) are considered, i.e. if the continuous wavelet transform is replaced by a discrete one.

While the choice of isotropic wavelets was natural and sufficient in our work on the isotropic scaling behaviour of molecular cloud structure, the directional sensitivity of non-axial symmetric wavelets will allow the determination of preferred orientations, eg. of filamentary substructures, in astronomical images.

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#### Appendix A: evaluation of the factors $c_\psi$

For all wavelets discussed in this paper the factors

$$c_\psi := \int_{\mathbb{R}} \frac{|\tilde{\psi}(\omega)|^2}{|\omega|} d\omega \quad \text{and} \quad c_\psi := \int_{\mathbb{R}^2} \frac{\|\tilde{\psi}(\mathbf{f})\|^2}{\|\mathbf{f}\|^2} d^2 \mathbf{f}$$

for the one- and two-dimensional case, respectively, can be calculated analytically. The integrals were solved using *Mathematica 3.0 for students*.

#### A.1. Modified Haar-wavelet

Looking at the mother function  $\psi(x) = -2\lfloor \square(x) \rfloor$  of the modified Haar wavelet system we find (paper I, p.714)

$$|\tilde{\psi}(\omega)|^2 = 4 \frac{\sin^4(\pi\omega)}{(\pi\omega)^2} \quad \text{and therefore}$$

$$c_{\psi_{\tilde{H}}} = \int_{-\infty}^{\infty} 4 \frac{\sin^4(\pi\omega)}{(\pi\omega)^2 \cdot |\omega|} = \int_{-\infty}^{\infty} 4\pi \frac{\sin^4(\pi\omega)}{|\pi\omega|^3} = 8 \ln 2$$

#### A.2. $\Delta$ -wavelet

##### A.2.1. $\Delta$ -wavelet in one dimension

Looking at the mother function  $\psi(x) = \square \square(x)$  we find (paper I, p.715)

$$|\tilde{\psi}(\omega)|^2 = 4 \frac{\sin^6(\pi\omega)}{(\pi\omega)^2} \quad \text{and therefore}$$

$$\begin{aligned} c_{\psi_\Delta} &= \int_{-\infty}^{\infty} 4 \frac{\sin^6(\pi\omega)}{(\pi\omega)^2 \cdot |\omega|} = \int_{-\infty}^{\infty} 4\pi \frac{\sin^6(\pi\omega)}{|\pi\omega|^3} \\ &= 12 \ln 2 - \frac{9 \ln 3}{2} \approx 3.37 \end{aligned}$$

##### A.2.2. $\Delta$ -wavelet in two dimensions

The power spectrum of the two-dimensional *down-up-down rectangle function* is (paper I, p. 717)

$$\|\widetilde{\square \square}(\mathbf{f})\|^2 = \left[ \frac{J_1(\pi f)}{(\pi f)} - \frac{J_1(3\pi f)}{(3\pi f)} \right]^2,$$

where  $J_1$  denotes the first order *Bessel*-function of first kind. With this we get

$$\begin{aligned} c_{\psi_\Delta} &= \int_{\mathbb{R}^2} \frac{\|\widetilde{\square \square}(\mathbf{f})\|^2}{\|\mathbf{f}\|^2} d^2 \mathbf{f} = 2\pi \int_0^\infty \frac{\left[ \frac{J_1(\pi f)}{(\pi f)} - \frac{J_1(3\pi f)}{(3\pi f)} \right]^2}{f} df \\ &= 2\pi \left( \ln(3) - \frac{4}{9} \right) \approx 4.11 \end{aligned}$$

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