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Turbulence in differentially rotating flows

What can be learned from the Couette-Taylor experiment

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Abstract. The turbulent transport of angular momentum plays an important role in many astrophysical objects, but its modelization is still far from satisfactory. We discuss here what can be learned from laboratory experiments. We analyze the results obtained by Wendt (1933) and Taylor (1936) on the classical Couette-Taylor flow, in the case where angular momentum increases with distance from the rotation axis, which is the most interesting for astrophysical applications. We show that when the gap between the coaxial cylinders is wide enough, the criterion for the onset of the finite amplitude instability can be expressed in terms of a gradient Reynolds number. Based on Wendt's results, we argue that turbulence may be sustained by differential rotation when the angular velocity decreases outward, as in keplerian flows. From the rotation profiles and the torque measurements we deduce a prescription for the turbulent viscosity which is independent of gap width; with some caution it may be applied to stellar interiors and to accretion disks.

Key words: hydrodynamics – instabilities – turbulence

1. Introduction

Differential rotation is observed in various astrophysical objects, from planets to galaxies, and one suspects that it gives rise to turbulence, since shear flows are liable to hydrodynamical and MHD instabilities. When these instabilities are of the linear type, they are relatively easy to study by perturbing slightly the equilibrium state. But some of them occur only at finite amplitude, in which case the answer must be sought in computer simulations or laboratory experiments, with their inherent limitations.

There has been some debate recently on whether a keplerian disk, which is linearly stable (Rayleigh 1916), may be unstable to finite amplitude perturbations. It may look as if this question presents little interest, since it has been proved that a very weak magnetic field suffices to render such a disk linearly unstable (Chandrasekhar 1960; Balbus and Hawley 1991). However the properties of angular momentum transport depend sensitively on which instability dominates in the considered regime, and a finite amplitude instability can overpower the linear instability which is the first to occur, as the relevant control parameter increases. One example is the Couette-Taylor flow, with the outer cylinder at rest and the inner cylinder rotating with angular velocity Ω. When Ω is increased, the transport of angular momentum first scales as $\Omega^{3/2}$, but thereafter it varies as Ω^2, once the flow has become fully turbulent (Taylor 1936), as if the initial linear instability were superseded by a stronger shear instability (see also Lathrop et al. 1992). By extrapolating the $\Omega^{3/2}$ law to high Ω one would clearly underestimate the transport.

In the present article, we take as working assumption that any differentially rotating flow experiences, at high Reynolds number, the turbulent regime observed in the Couette-Taylor (CT) experiment, and that this turbulence will then transport angular momentum in the same way as in that experiment. The CT flow has been chosen as reference because it is the simplest flow to realize in the laboratory, with both shear and rotation that can be varied independently. We examine whether the experimental data suggest a prescription for the angular momentum transport, which may be used to model astrophysical objects. A similar approach has been taken by Zeldovich (1981), but our conclusions will differ from his (see Appendix).

2. The Couette-Taylor experiment

The CT experimental apparatus consists of two coaxial rotating cylinders of radius R_1 and R_2 separated by a gap $\Delta R = R_2 - R_1$, which is filled with a fluid of viscosity ν. The cylinders can rotate with different angular velocities Ω_1 and Ω_2; their height is in general much larger than their radius, to minimize the effect of the boundaries. The Reynolds number is usually defined in terms of the differential rotation $\Delta\Omega = |\Omega_2 - \Omega_1|$ and by taking the gap width as characteristic length:

$$Re = \frac{\Delta\Omega \, R \, \Delta R}{\nu}, \tag{1}$$

where R is the mean radius $R = (R_1 + R_2)/2$.

2.1. Critical parameters and transition to turbulence

When the inner cylinder is rotating and the outer one is at rest, angular momentum decreases outward and the flow is linearly unstable for Reynolds numbers higher than $Re_c = 41.2\,\sqrt{R/\Delta R}$

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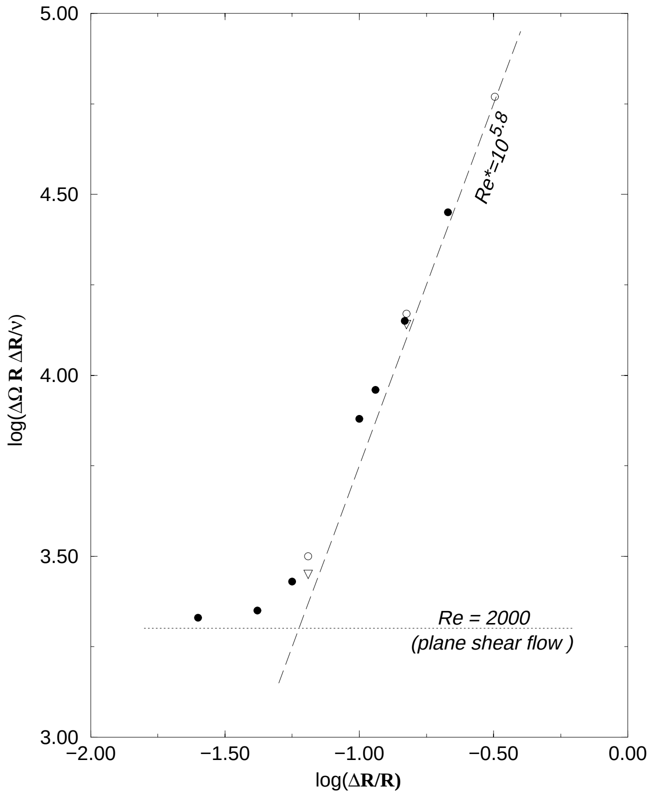

Fig. 1. Critical Reynolds number vs. aspect ratio; filled circles from Taylor (1936), open circles from Wendt (1933). Dotted line: critical Reynolds number for plane shear flow instability; dashed line: critical gradient Reynolds Re^* for circular shear flow in the limit of large gap (see text).

(for narrow gap, cf. Taylor 1923). This well known instability takes first the shape of steady toroidal, axisymmetric cells (the Taylor vortices); it is very efficient in transporting angular momentum, whose gradient is strongly reduced. At higher Re a wavy pattern appears and after a series of bifurcations the flow becomes fully turbulent. This case has been studied by many experimental teams, and it is extremely well documented (see Andereck et al. 1986). It has also been modeled successfully in three-dimensional numerical simulations (Marcus 1984; Coughlin & Marcus 1996).

In the opposite case, when the outer cylinder is rotating and the inner one is at rest, the angular momentum increases outward and the flow is linearly stable. The only theoretical prediction concerning the non-linear behavior is that by Serrin (1959), later refined by Joseph and Munson (1970), who established that the flow is stable against finite amplitude perturbations below $Re = 2\pi^2$ (in the narrow gap limit).

To this date, no numerical simulation has been able to demonstrate the finite amplitude instability. But this instability does occur in the laboratory, and it has been described already by Couette (1890). It was studied in detail by Wendt (1933) and Taylor (1936), who showed that for Reynolds numbers exceeding $Re_c \simeq 2\,10^3$, the flow becomes unstable and immediately displays turbulent motions. The critical Reynolds number depends on whether the angular speed is increased or decreased in the experiment, a typical property of finite amplitude instabilites. Moreover, it is sensitive to gap width, as demonstrated by Taylor. Fig. 1 displays results from Wendt and Taylor: Re_c is roughly constant below $\Delta R/R = 1/20$, but above it increases as $\approx (\Delta R/R)^2$, as was already noticed by Zeldovich (1981), a behavior for which an explanation was proposed by Dubrulle (1993). In the latter regime one can define another critical Reynolds number Re_c^* involving, instead of gap width, the gradient of angular velocity; since

$$Re_c = \frac{R^3}{\nu}\frac{\Delta\Omega}{\Delta R}\left(\frac{\Delta R}{R}\right)^2 = Re_c^*\left(\frac{\Delta R}{R}\right)^2, \quad (2)$$

the instability condition becomes

$$Re^* = \frac{R^3}{\nu}\frac{\Delta\Omega}{\Delta R} \geq Re_c^* \simeq 6\,10^5. \quad (3)$$

We see that two conditions must be satisfied for the finite amplitude instability to occur: the first $Re \geq Re_c$ is the classical criterion of shear instability, valid also for plane parallel flows, whereas the second $Re^* \geq Re_c^*$, involving what we shall call the *gradient* Reynolds number, is genuine to differential rotation. In addition, to trigger the instability the strength of the perturbation must exceed a certain threshold, which presumably also depends on Re or Re^*.

2.2. Transport of angular momentum

In the turbulent regime, the torque measured by Taylor scales approximately as $G \propto (\Omega_2)^n$ for a given gap width, where the exponent n tends to 2 for large Ω_2. The measurements made by Wendt confirm that scaling with $n \approx 2$. It suggests that the transport of angular momentum may be considered as a diffusive process, and that the mean turbulent viscosity $\overline{\nu}_t$ increases linearly with Ω_2, or $\Delta\Omega$. It is then natural to examine whether this viscosity may be expressed as

$$\overline{\nu}_t = \alpha R\,\Delta\Omega\,\Delta R \quad (4)$$

with α being a constant of order unity, since the largest turbulent eddies would have a size $\approx \Delta R$ and a peripheral velocity $\approx R\,\Delta\Omega/2$. The parameter α is easily derived from the torque measurements, and the surprising result is that it decreases with gap width (Fig. 2). For the smallest gaps, α scales as the inverse of $\Delta R/R$, but the slope steepens farther as if the scaling would tend asymptotically to

$$\alpha \propto \left(\frac{\Delta R}{R}\right)^{-2} \qquad \text{for } \left(\frac{\Delta R}{R}\right) \to 1. \quad (5)$$

(Comparing Wendt's experimental data of his two largest gaps, one deduces an exponent $\delta \ln\alpha/\delta\ln(\Delta R/R_2) = -1.83$.)

We may therefore conclude that, in the limit of large gap, the mean turbulent viscosity actually scales as

$$\overline{\nu}_t \simeq \left(\frac{\alpha\,\Delta R}{R}\right)^2 R^3\frac{\Delta\Omega}{\Delta R} = \beta^* R^3\frac{\Delta\Omega}{\Delta R}, \quad (6)$$

with $\beta^* \approx 4\,10^{-6}$.

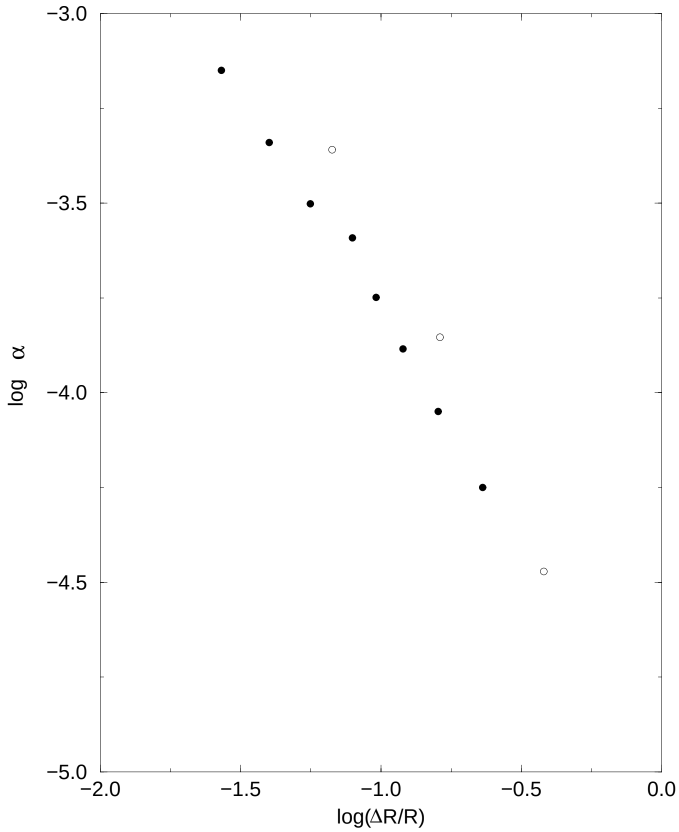

Fig. 2. Value of parameter α in the classical viscosity prescription $\nu_t = \alpha R\,\Delta\Omega\,\Delta R$, derived from Taylor (1936) in filled circles and from Wendt (1933) in open circles.

This strongly suggests that the *local* value of the turbulent viscosity is then independent of gap width, and that it is determined only by the local shear:

$$\nu_t = \beta r^3 \left| \frac{d\Omega}{dr} \right|, \tag{7}$$

r being the radial coordinate.

In principle, one should be able to verify this prescription for ν_t by examining the rotation profiles measured by Taylor and Wendt. According to (7), the conservation of angular momentum requires that its flux, given by

$$\mathcal{F} = \left[\nu + \left| \beta r^3 \frac{d\Omega}{dr} \right| \right] \; r^2 \frac{d\Omega}{dr}, \tag{8}$$

varies as $1/r$ between the cylinders. Therefore $r^3(d\Omega/dr)$ should be constant in the turbulent part of the profile (as it is in the laminar flow).

But this constancy can be expected only if the transport of angular momentum is achieved by the viscous and turbulent stresses alone. That is not the case in Taylor's experiment: as acknowledged by him, an Ekman circulation is induced by the ends of his apparatus, although he tries to minimize the boundary effects by chosing a large aspect ratio (height/radius). Moreover his rotation profiles are deduced from pressure measurements made with a Pitot tube located at half height of the cylinders, where the radial return flow has its maximum intensity. Consequently, the torque inferred from these profiles is about half of that measured directly at the inner cylinder.

The aspect ratio is less favorable in Wendt's experiment, but there the top boundary is a free surface, and most of his measurements have been made with the bottom boundary split in two annuli, attached respectively to the inner and the outer cylinders, which reduces drastically the circulation and renders his results more reliable. We examined his rotation profiles obtained for the largest gap width ($\Delta R/R = 0.38$) and with 4 different speeds of the inner cylinder ($0 \leq \Omega_1 \leq \Omega_2/2$); we found that in the bulk of the fluid, these profiles are compatible with the constancy of the gradient Reynolds number, as predicted by (8). However we cannot rule out a mild variation of β within the profile. Unfortunately, Wendt gives the torque only for the case where the inner cylinder is at rest; we draw from it the following value of the coefficient β:

$$\beta = 1.5 \pm 0.5\, 10^{-5}. \tag{9}$$

The estimated uncertainty reflects that of the velocity measurements: β results from second derivation of the shape of the fluid surface.

3. Are keplerian flows unstable?

More precisely, the question we address is whether the Couette flow is unstable when the angular velocity decreases outwards and the angular momentum increases outwards, as in keplerian flow.

There is no definite answer yet, because this regime has not been explored at high enough Reynolds number. But some information can be gleaned from Wendt's study. He reports the results of experiments where the two cylinders rotate such that $\Omega_2 R_2^2 = \Omega_1 R_1^2$. At low Reynolds number, this setup enforces a laminar flow of constant angular momentum (neutral flow), but at high Reynolds number this flow becomes turbulent for two of the three gaps used by Wendt.

The angular velocity and angular momentum profiles for one of these turbulent flows are reproduced in Fig. 3. (According to Wendt's data, $\Omega_2 R_2^2$ actually exceeds $\Omega_1 R_1^2$ by a half a percent.) The profiles clearly demonstrate the flow instability, with angular momentum being transported down the angular velocity gradient, which becomes somewhat flater far enough from the boundaries, whereas the angular momentum profile steepens substantially.

In the turbulent bulk of the flow $q = -d\ln\Omega/d\ln r \approx 1.4$, compared to the initial $q = 2$. We recall that $q = 1.5$ in keplerian flow, and that in the numerical simulations performed by Balbus et al. (1996, 1998), the instability is lost already at about $q = 1.95$. A crude estimate of the parameter β in the viscosity formulation (7) indicates that the size of the turbulent eddies is much smaller than the gap width: $\ell = \sqrt{\beta} R \approx \Delta R/100$.

The corresponding values of Re^* for the three experiments are reported in Fig. 4 together with the critical line of Fig. 1. They are located respectively above and below this line for the unstable and stable flows. The data are too scarce to locate pre-

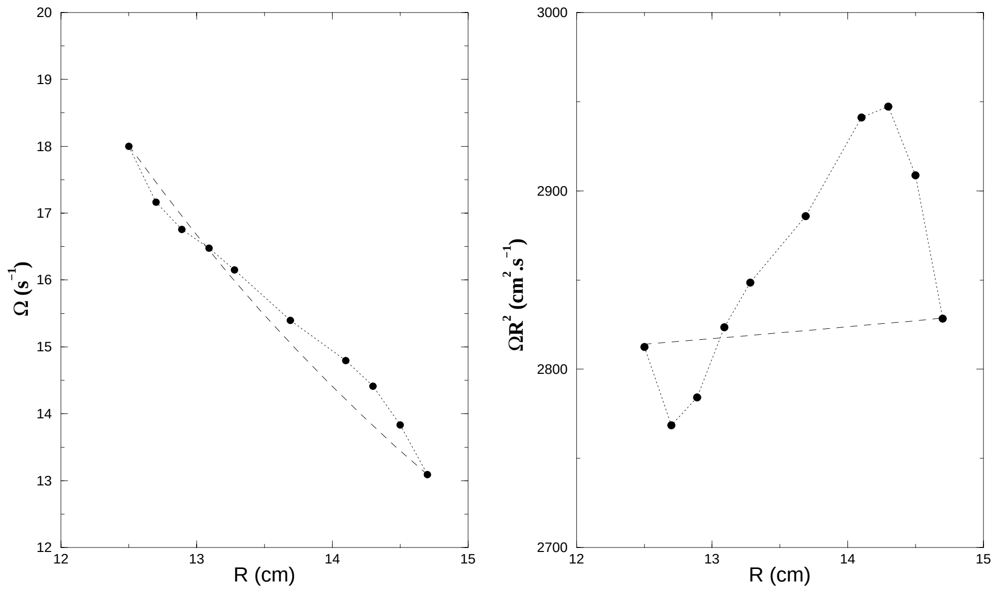

Fig. 3. Angular velocity and momentum profiles in the case of decreasing angular velocity and initially constant angular momentum – dotted line (experimental data from Wendt 1933).

cisely the critical line of these "neutral" flows, but we can conclude that the critical gradient Reynolds number Re^* then lies between $2\,10^5$ and $6\,10^5$.

4. Discussion

The firmest result of our analysis of the Couette-Taylor experiment is that the criterion for finite amplitude instability may be expressed in terms of the gradient Reynolds number $Re^* = R^3(\Delta\Omega/\Delta R)/\nu$, where the critical Reynolds number $Re^*_c \lesssim 6\,10^5$ is independent of the width of the gap between cylinders, for wide enough gap. The turbulent transport of angular momentum then seems also to be independent of gap width; it proceeds always down the angular velocity gradient, as confirmed by the behavior of the initially "neutral" flows examined by Wendt.

Though the experimental evidence is somewhat less compelling, we have established empirically an expression which links the turbulent viscosity to the local shear. The value of Re^*_c, and that of β in Eq. (7), have been derived from Wendt's experiment with the inner cylinder at rest, and it is not obvious that these parameters would be the same for different ratios of cylinder speeds. Also, the linear scaling $\nu_t/\nu = \beta Re^*$ may be valid only for those moderate gradient Reynolds numbers which could be reached in the laboratory

Nevertheless, it is tempting to apply this expression (7) to accretion disks, as an alternate for the commonly used prescription $\nu_t = \alpha c_s H$, where H is the scale height of the disk and c_s the local sound speed (Shakura & Sunyaev 1973). Some caution is required because this viscosity has been derived from experiments performed with an incompressible fluid. Moreover (7) implies that the eddies which dominate in the transport of angular momentum have a size of order $\ell \approx \sqrt{\beta}\,r$, independent of the strength of the local shear, and that their velocity is of order $\ell r\, d\Omega/dr$. When applying this prescription to a compressible flow, one has to make sure that this velocity is smaller

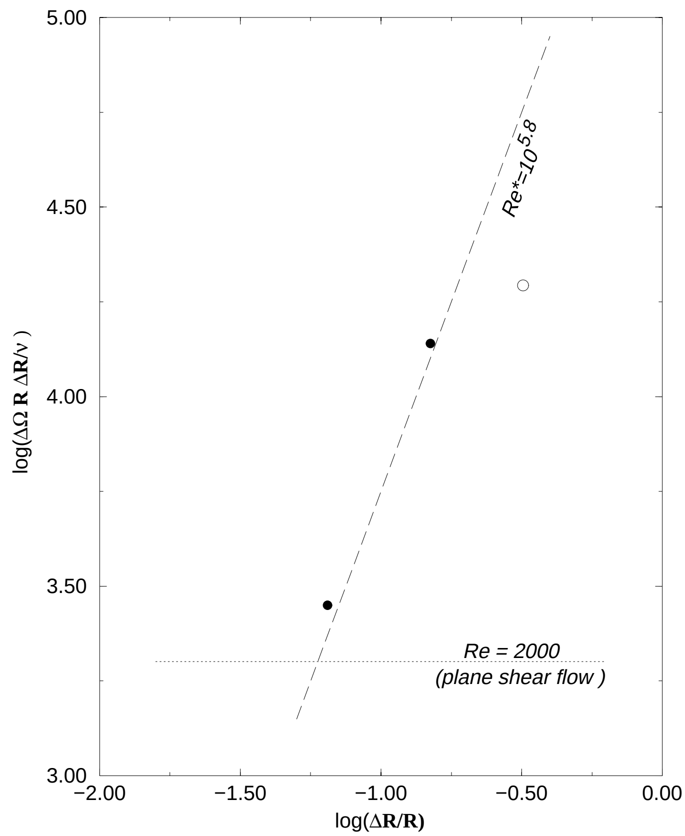

Fig. 4. Reynolds number vs. aspect ratio for the three "neutral" experiments from Wendt: the two filled circles correspond to the unstable flows, the open circle to the stable flow. The stability curve obtained for the inner cylinder at rest (Fig. 1) is displayed for comparison.

than the sound speed and, in the case of an accretion disk, that the size ℓ of the eddies, which are three-dimensional, does not largely exceed the scale height H. The behavior of "neutral" flows demonstrates that the shear instability always transports angular momentum down the angular velocity gradient, which means outward for accretion disks.

Note that in a keplerian disk our expression is equivalent to

$$\nu_t = \beta' r^2 \Omega \qquad \text{with} \quad \beta' = \frac{3}{2}\beta. \tag{10}$$

Such a prescription has been suggested originally by Lynden-Bell and Pringle (1974), and recently it was used again by Duschl et al. (1998). As a test, it is being applied to the modelling of accretion discs in active galactic nuclei (Huré & Richard 1999).

The reader may wonder why we have only used experimental results dating from the thirties, namely those of Wendt (1933) and Taylor (1936). The reason is that no one, since them, has studied in such extent the regime of outward increasing angular momentum.[1] We suspect that it is because the flow becomes then turbulent at once, without undergoing a series of bifurcations associated with enticing patterns. But we hope that experimentalists will turn again to this classical problem, which is of such great interest for geophysical and astrophysical fluid dynamics, and that they will explore the rotation regimes for which the data are so incomplete. In the meanwhile, the quest will continue to detect the finite amplitude instability in computer simulations.

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Appendix A: on Zeldovich's analysis of Taylor's results

Zeldovich (1981) had a similar goal in mind when he analyzed the results of Taylor's experiment. But he started from the idea that the turbulent flow was governed by the epicyclic frequency N_Ω and the turnover frequency ω, where

$$N_\Omega^2 = \frac{1}{r^3}\frac{d}{dr}(r^2\Omega)^2 \quad \text{and} \quad \omega^2 = \left(r\frac{d\Omega}{dr}\right)^2,$$

since they measure respectively the stability of the flow and the strength of the shear. He defines a non-dimensional parameter $Ty = N_\Omega^2/\omega^2$, akin to the Richardson number used in stratified shear flow, and he seeks the confirmation of his intuition that Ty be constant in Taylor's turbulent rotation profiles, in which case the angular velocity would obey a power law

$$\Omega \propto r^q.$$

His best fit yields $q = 5.5$. However these profiles are contaminated by the Ekman circulation mentioned above, and they differ marquedly from those obtained by Wendt.

A consequence of this constant Ty would be that the parameter α in (4) would vary as

$$\alpha \propto \left(1 - \frac{\Delta R}{R}\right)^{2q+4} = \left(1 - \frac{\Delta R}{R}\right)^{15}$$

a property which is not substantiated by the combined results of Taylor and Wendt, as can be seen in Fig. 2.

[1] For instance, the Reynolds numbers explored by Coles (1965) are one order of magnitude lower than those reached by Wendt, which explains why he did not encounter the finite amplitude instability of neutral flows.

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