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Local stability criterion for a gravitating disk of stars

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Abstract. Computer N-body experiments are desribed which test the validities of the original Toomre's (1964) criterion and of a generalized criterion for local stability of Jeans-type perturbations in a self-gravitating, infinitesimally thin, and practically collisionless disk of stars. The fact that the nonaxisymmetric perturbations in the differentially rotating system are more unstable than the axisymmetric ones is taken into account in this generalized criterion. It is shown that for differentially rotating disks, the generalized criterion works as well as Toomre's ordinary criterion does for rigidly rotating ones.

A modest discrepancy is observed between the analytical stability criteria and the numerical results. We tentatively attribute this to the shortcomings of the asymptotic density wave theory and possibly additional ones introduced by approximations in the local numerical code employed here. In addition, the linear stability theory of small oscillations of a disk of stars is reexamined by using the method of particle orbit theory. This representation gives new insight into the problem of gravitating disk stability. Certain applications of the theory and the N-body simulations to actual disk-shaped spiral galaxies are explored as well.

Key words: galaxies: kinematics and dynamics – galaxies: structure

1. Introduction

1.1. Local stability criterion

Spirals are common in rapidly (and nonuniformly) rotating galaxies. The origin and maintenance of the spiral structure of such highly flattened systems has proved to be a difficult problem in galactic dynamics. Even though no definitive answer can be given at the present time, the study of the stability of small-amplitude waves in disk-shaped galaxies of stars is the first step towards an understanding of the phenomena. This is because in the Milky Way Galaxy and many other giant galaxies the bulk of the optical mass, probably $\gtrsim 90\%$, is composed of stars, and therefore stellar dynamical phenomena play a basic role.

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More than three decades ago, Lindblad (1963, and earlier references) proposed that spiral arms of a galaxy are quasi-stationary density waves propagating through differentially rotating parts of a collisionless disk of stars with a constant phase velocity. Subsequently, Lin & Shu (1966), Lin et al. (1969), Shu (1970), and others (e.g., Nakamura et al. 1975) further developed the density wave theory by studying collective effects in self-gravitating stellar systems; see reviews by Toomre (1974, 1977) and Athanassoula (1984). It seemed reasonable to attribute galactic spiral arms to Lin-Shu type small-wavelength density waves driven by the classical Jeans instability in a rapidly rotating system of young, dynamically cold stars.

Initially, in the asymptotic Lin-Shu density wave theory of tightly-wound spirals, important effects of the azimuthal gravitational forces in nonuniformly rotating systems were not properly taken into account. As a result of this simplification, the well-known Toomre's (1964) local criterion for stability against only *axially symmetric* (radial) Jeans-type perturbations of the gravitational potential can be derived from the original Lin-Shu-Kalnajs dispersion relation (Lin & Shu 1966; Lin et al. 1969; Shu 1970; Toomre 1977).[1]

The original Toomre's criterion states that the radial residual (random)-velocity dispersion of stars c_r, which is proportional to the square root of the "temperature" of the system, will suppress the axisymmetric Jeans perturbations in the rapidly rotating, nearly homogeneous, and very thin ($h/2 \ll R$, where h is a typical thickness and R is a characteristic radius of the system) disk, if

$$c_r \geq c_T \equiv \frac{3.4G\sigma_0}{\kappa}. \tag{1}$$

In Eq. (1), G is the gravitational constant and σ_0 is the equilibrium surface mass density. The local epicyclic frequency $\kappa(r)$ is given by $\kappa = 2\Omega\,[1 + (r/2\Omega)\,(d\Omega/dr)]^{1/2}$ where the quantity $\Omega(r)$ denotes the angular velocity of rotation at the distance r from the galactic center. The epicyclic frequency decreases from 2Ω for the rigid body rotation to Ω for the Keplerian one.

[1] In plasma physics an instability of the Jeans type is known as the negative-mass instability of a relativistic charged particle ring or the diocotron instability of a nonrelativistic ring that caused azimuthal clumping of beams in synchrotrons, betatrons, and mirror machines (Landau & Neil 1966; Nocentini et al. 1968; Davidson 1992).

The disk with the velocity dispersion $c_r = c_T$ is on the verge of the gravitational Jeans-type instability with respect to the short-scale, $\ll R$, purely radial or ringlike perturbations only. The local criterion (1) gives a necessary condition for radial stability. It does not obviously address the stability of nonaxisymmetric, relatively large-scale modes, $\lesssim R$, particularly open spiral modes in the bar form of the differentially rotating ($d\Omega/dr \neq 0$) disks.

In a series of papers Morozov (1980, 1981a, 1981b) extended the works of Toomre (1964), Lin & Shu (1966), Lin et al. (1969), and Shu (1970) by including the azimuthal forces. It was demonstrated by Morozov that the presence of the differential rotation (or shear) results in quite different dynamical properties of the axisymmetric and nonaxisymmetric (spiral) perturbations. A dispersion relation for arbitrary perturbations which propagate in the plane of a differentially rotating stellar disk is derived using a kinetic approach. This generalized Lin-Shu type dispersion relation leads to the following modified local stability criterion obtained by Morozov:

$$c_r \geq c_M \equiv c_T \left\{1 + \left[(2\Omega/\kappa)^2 - 1\right] \sin^2 \psi\right\}^{1/2}, \tag{2}$$

where the condition $2\Omega/\kappa > 1$ always holds in the differentially rotating system. In flat galaxies,

$$(2\Omega/\kappa)^2 - 1 \approx -(r/2\Omega)(d\Omega/dr),$$

$|(r/2\Omega)(d\Omega/dr)| < 1$, and $d\Omega/dr < 0$. The pitch angle ψ between the direction of the wave front and the tangent to the circular orbit of a star in Eq. (2) is $\psi = \arctan(m/k_r r)$, where the nonnegative azimuthal mode number m is the number of spiral arms, while k_r and $k_\varphi \equiv m/r$ are the radial and the azimuthal wavenumbers, respectively. The parameter $\left\{1 + \left[(2\Omega/\kappa)^2 - 1\right] \sin^2 \psi\right\}^{1/2}$ is an additional stability parameter which depends on both the pitch angle and the amount of differential rotation in the galaxy (cf. the parameter $\mathcal{J}$ introduced by Lau & Bertin 1978, Lin & Lau 1979, and Bertin 1980, 1994).

It is clear from the modified criterion (2) that in a nonuniformly rotating disk, namely when $2\Omega/\kappa > 1$, for nonaxisymmetric perturbations ($\psi \neq 0$) the modified velocity dispersion c_M of a marginally Jeans-stable system is larger than c_T (although still of the order of c_T). Moreover, Morozov took into account the additional weak destabilizing effect of a density inhomogeneity, and stabilizing effects of a radial gradient of a velocity dispersion and of a finite disk's thickness. The result is that these effects practically cancel out each other, at least in the solar vicinity of our own Galaxy. In the present study, we therefore neglect these small corrections. In addition to Morozov's studies, Griv (1992) has obtained a value of critical dispersion to the next leading order in the asymptotic expansion by including higher-order terms in the epicyclic amplitude. Recently, Griv (1996) and Griv & Peter (1996) clarified the basic assumptions of the asymptotic approximation and rederived the criterion (2) by using the kinetic approach. A relationship exists between Eq. (2) and what Toomre (Toomre 1981; Binney & Tremaine 1987, p. 375) called "swing amplification" in which the material at radius r_0 is pulled forward by the azimuthal forces of the material ar $r < r_0$ that it trails.

Apparently, Toomre (1964, p. 1222) first noted the different dynamical properties of perturbations with different ψ in the nonuniformly rotating stellar disk. Later the destabilizing effect of the azimuthal forces has been studied using an analysis based both on a gas dynamical model by Lau & Bertin (1978) and Lin & Lau (1979), and a stellar dynamical model by Bertin & Mark (1978) and Bertin (1980) by using an improved potential theory. They have explained the physical origin of the difference between radial and spiral perturbations in a nonuniformly rotating system, e.g., Lau & Bertin (1978, p. 509). Briefly, in order to fit in with the gravitational field in flat systems, galactic rotation has to be differential and such shear has important kinematic and dynamic consequences. They pointed out that the generalized stability criterion in the form of Eq. (2) takes properly into account the combined influence of self-gravity, thermal motions, shear, and azimuthal forces. The reader should consult Lau & Bertin (1978), Bertin (1980, 1994), and Lin & Bertin (1984) for a detailed discussion of the problem. Recently, the problem has been nicely reviewed by Polyachenko & Polyachenko (1997). Note only that the free kinetic energy associated with the differential rotation of the system under study is only one possible source for the growth of the energy of these spiral Jeans-type perturbations, and appears to be released when angular momentum is transferred outward.

As one can see from Eq. (2), the modified critical velocity dispersion c_M grows with ψ. Consequently, in order to suppress the most "dangerous," in the sense of the loss of gravitational stability, nonaxisymmetric perturbations in a form of a bar ($\psi \to 90°$), c_r should obey the following generalized criterion:

$$c_r \geq c_G \equiv \frac{2\Omega}{\kappa} c_T. \tag{3}$$

One should keep in mind that Eq. (3) is clearly only an approximate one, since it was obtained in the framework of the moderately tightly-wound Lin-Shu perturbations approximation (Lin & Lau 1979; Griv 1996; Griv & Peter 1996). Strictly speaking, the above expression (2) for c_r cannot be used when the pitch angle is large, since in the asymptotic theory it is necessarily assumed that

$$\tan^2 \psi \ll 1. \tag{4}$$

The condition (4) limits the analysis of the actual low-m galaxies (in the standard Fourier analysis of the azimuthal coordinate) with $m < 5$–7 to a consideration of disturbances with a pitch angle smaller than about $45°$ only (Lin & Lau 1979; Griv & Peter 1996). Such a requirement naturally arises within the WKB approximation we are interesting. Polyachenko (1989) and Polyachenko & Polyachenko (1997) tried to find a stability criterion for arbitrary localized perturbations beyond the limitation of the WKB approximation by considering a hydrodynamical model. Note, however, for the disk with flat rotation curve at least, Polyachenko's marginal stability condition and (3) are practically coincidental.

Although the expression (2) only indicates the tendency of growth of the critical dispersion with increasing ψ, it is clear that the generalized criterion for the local stability of a stellar disk against arbitrary Jeans-type perturbations (including the most unstable barlike ones) should be approximately of the form of Eq. (3). In this case, in a Jeans-stable differentially rotating disk, the widely used Toomre (1964, 1977) critical stability parameter (which guarantees the suppression of arbitrary Jeans-type perturbations in a rapidly rotating disk by the thermal velocities of stars)

$$Q_{\rm crit} = \frac{c_{\rm crit}}{c_T}, \qquad (5)$$

where $c_{\rm crit}$ is the critical radial-velocity dispersion, must be greater than 1 and equal to about $2\Omega/\kappa$. (Toomre's Q-value is a measure of the ratio of thermal and rotational stabilization to self-gravitation and is defined below.) In particular, in a gravitating system with the Keplerian rotation ($2\Omega/\kappa = 2$), Toomre's critical parameter is $Q_{\rm crit} \approx 2$. In the case of the flat rotation curve $2\Omega/\kappa = \sqrt{2}$ and hence $Q_{\rm crit} \approx \sqrt{2}$ also. According to Eqs. (2), (3), and (5), the value of Toomre's critical stability parameter becomes $Q_{\rm crit} = 1$ only for arbitrary perturbations in the rigidly rotating disk ($2\Omega/\kappa = 1$) and/or for axisymmetric perturbations in the differentially rotating one (Bertin & Mark 1978; Lau & Bertin 1978).

It is obvious that in differentially rotating galaxies, disks manage to keep their local stability parameter close to the critical value, $Q_{\rm crit} \approx 2\Omega/\kappa \approx 2$ or $c_r \approx (2\Omega/\kappa)c_T \approx 2c_T$, respectively. In this case, once the entire differentially rotating disk has been heated to values $c_r \approx 2c_T$, no further spiral waves can be sustained by virtue of the Jeans instability – unless some "cooling" mechanism is available leading to Toomre's Q-value,

$$Q = \frac{c_r}{c_T},$$

under approximately 2 or to the value of c_r smaller than approximately $2c_T$, respectively (e.g., by the dissipation in the gas and accretion, and/or by the star formation in a "cold" interstellar medium). By using N-body simulations, first Hohl (1971) and then, e.g., Sellwood & Carlberg (1984) and Griv & Chiueh (1998), have already shown that the process of formation of new dynamically cold stars, which move on nearly circular orbits, plays a vital role in prolonging spiral activity in the plane of the disk by reducing the random velocity dispersion of the entire stellar component. Thus, the cold interstellar medium may play a dominant role in determining the observed spiral structure in galaxies because it is the site of the generation of new, dynamically cold stars. In Saturn's rings such a cooling mechanism is also operating: inelastic physical collisions between particles reduce the magnitude of the relative velocity of particles.[2] Salo (1992) already investigated numerically the role of the Jeans instability mechanism in long-lived sculpting of Saturn's rings by including inelastic (dissipative) interparticle impacts.

[2] Common dynamical processes act in the stellar disks of flat galaxies and in a planetary rings system of mutual-gravitating particles (Tremaine 1989).

1.2. Physics motivation

The value of Toomre's stability parameter Q is critically important for any gravitational theory of spiral structure in galaxies (and for dynamics of planetary rings). The generalized local stability criterion as well as Toomre's critical Q-value has been discussed at length by Morozov (1980, 1981a, 1981b), Polyachenko (1989), and recently by Griv (1996), Griv & Peter (1996), and Polyachenko & Polyachenko (1997). Surprisingly, their ideas on the generalized local stability criterion have not attracted a great deal of attention, and other explanations were involved to confront the observations and N-body simulations. For instance, Bertin & Romeo (1988) invoked the destabilizing effect of a sufficient amount of cold interstellar material to explain the observed large value of the parameter Q for NGC 488. Although this explanation can be accepted for the gas-rich galaxies, it certainly cannot be universal. For example, Cinzano & van der Marel (1994) showed that even in such a gas-poor spiral galaxy like NGC 2974, sometimes classified as E4, the Q value considerably exceeds unity, and probably is larger than 3. The problem seemed so complicated that Bottema (1993) even claims that it is very difficult to relate the pure observational results, that Q between 2 and $2\frac{1}{2}$ over a large range of galactic disks, to any existing theoretical concept.[3]

Recently, the dynamical behavior of weakly collisional, planetary rings system has been studied via an N-body simulation (Osterbart & Willerding 1995; Salo 1992, 1995). It was found that the stability number Q of Toomre in relaxed equilibrium disks does not fall below a critical value, which lies about $Q_{\rm crit}$ = 2–2.5. No adequate explanation of the latter fact has been presented. (Interestingly, observational data on the Saturnian rings system, obtained with the Voyager 2 spacecraft, have indicated about the same value of $Q \approx 2$ for the densest B ring; Lane et al. 1982, p. 543.)

We conclude that even though the criterion for local stability in a gravitating, rapidly rotating particulate disk is a relatively old issue in galactic and planetary rings dynamics, it is necessary to address the problem again. In the present work, we turn to studies of localized gravity perturbations by using both N-body simulations and an analytical approach. The linear disk's stability theory is reexamined and conditions which guarantee the lack of all Jeans-type unstable perturbations in a disk of stars are found. We restrict ourselves to the simplest case of practically collisionless stellar system which is spatially homogeneous and two-dimensional. The effects of inhomogeneity and three-dimensional motion will be investigated in a forthcoming paper.

The main objective of the current work is to check the generalized local stability criterion (3) numerically using the method of direct many-body simulations. Moreover, the dispersion relations (9) and (20) of Morozov (1980) and Griv & Peter (1996),

[3] Jog (1996) obtained the criterion for local stability against gravity perturbations in gravitationally coupled stars and gas in a galactic disk by treating the stars and gas as two isothermal fluids. Again, the stability of a disk only with respect to axisymmetric perturbations has been studied.

respectively, and the stability criterion (3) obtained in the framework of the linear kinetic theory do not reveal what kind of structure can emerge due to the gravitational instability. Simulations should be able to identify these structures. As is thought, a very simple model should be taken in the numerical work to compare the analytical stability criterion with the one obtained numerically (see the next section of the paper). In addition, for the sake of completeness in Appendix A.1 of the current paper the local stability criteria (2) and (3) are rederived by employing the Lagrangian formalism of magnetized plasma theory which deals with similar problems. In our theory, the simplest theoretical method of plasma physics is used. This is the so-called method of particle dynamics (or particle orbit theory) in which the motion of an "average" star-"particle" is considered (Rosenbluth & Longmire 1957; Alexandrov et al. 1984, p. 46). The essential part of the method is to regard ρ/r_0 as a small parameter and to expand the solution in terms of it, where ρ is the mean epicyclic radius of the star and r_0 is the epicyclic center (the guiding center in plasma physics) displacement from the galactic center.

Occasionally doubts have been raised about the validity of strictly two-dimensional N-body simulations of stellar disks of galaxies (White 1988; Romeo 1997). For example, White (1988) found a few computer models in the exactly planar simulations which are probably affected by noise and two-body relaxation (see, however, Hohl 1973). The physical effect of relaxation in the N-body simulations is to generate viscosity and heat conduction. One obvious effect of short-term relaxation is a heating of the disk, and therefore some of the two-dimensional N-body simulations probably cannot be trusted (White 1988). We show, however, in Appendix A.2 of the present work that in general the effect of such rare, $\nu_c \ll \Omega$, say, $\nu_c \sim 0.01\Omega$, elastic gravitational collisions (encounters) is very small, and may be important only on a timescale of the order of the mean time of many galactic rotations, typically ~ 100 rotations. Here ν_c is the effective frequency of interparticle collisions. Thus, two-body relaxation effects in such N-body models probably do not yield any interesting physics on a timescale of several first rotations when the gravity perturbation may be already large as a result of Jeans instability, i.e., in weakly collisional systems with $\nu_c \ll \Omega$ the collective effects may be apparent before the collisional timescale is reached.

In Appendix A.3 of the present paper, following Rybicki (1971) and Hohl (1973), we shall use an experimental method of testing a computational procedure by repeating calculations using a mass spectrum. The latter would clearly show whether computations are sensitive to the undesirable particle relaxation effects.

The organization of the paper is as follows. In Sect. 2 the details of the numerical simulation model are discussed. The results of computer simulations are shown in Sect. 3 and compared with the predictions of the basic theory as outlined in Appendix A.1. Sect. 4 is devoted to a discussion of the principal results of the work and their application to observational data. Through Appendix A.2 the effect of interparticle encounters on the dispersion law of Jeans perturbations is estimated. In Appendix A.3 we check if the system is being correctly modeled as a collisionless Boltzmann (Vlasov) system.

2. Numerical experiments: descriptions

Simulations of galaxies of stars can be divided into two basic categories: global and local. The former have been done to simulate the global dynamics and the development of large-scale spiral and bending structures (Hohl 1971, 1972, 1978; Sellwood & Carlberg 1984; Grivnev 1985; Peter et al. 1993; Griv & Chiueh 1998). Certainly, some aspects of dynamical behavior of stellar systems can be studied by global simulations only (nonlinear effects, etc.). An obvious shortcoming of the global simulation approach is that the numbers of stars in a simulation is orders of magnitude smaller than in a typical galaxy. This might not permit revelation of the small-scale $\sim \rho$ spiral structure (see Appendix A.1 of the present paper). Here $\rho \approx c_r/\kappa$ is the mean epicyclic (Coriolis) radius (Larmor radius in magnetized plasmas, respectively). As a rule, in spiral galaxies $\rho \sim 0.5$ kpc, and $\rho \gtrsim h$ and $\rho \ll R$.

On the other hand, to study some aspects of particulate disk dynamics when inhomogeneity is relatively weak, a different numerical approach may be taken: local N-body simulations. The latter galactic N-body experiments in a local or Hill's approximation has been pioneered by Toomre (1990) and Toomre & Kalnajs (1991). In these simulations dynamics of particles in small regions of the disk are assumed to be statistically independent of dynamics of particles in other regions. The local numerical model thus simulates only a small part of the system and more distant parts are represented as copies of the simulated region. Wisdom & Tremaine (1988) applied the same numerical technique in studying the equilibrium properties of planetary rings. In addition, Salo (1992, 1995) and Griv (1997) studied the dynamical behavior of collisional self-gravitating rings systems by using the same method. In contrast to global simulations, in local ones complicated effects of disk inhomogeneity and finite thickness may be studied separately. This is the main reason why in the present work the local N-body simulations are used. In our opinion, this simple model is useful for clarifying the physics of the phenomenon, and provides us with results which can serve as a convenient starting point for more complicated theory and numerical simulations.

In fact, Morozov (1981a) has already attempted to confirm the criterion (3) numerically. However, because of the very small number of model stars, $N = 200$, Morozov's results are subject to considerable uncertainties, and additional simulations are clearly required to settle the issue. Furthermore, for that number of particles, the two-body relaxation timescale is comparable to the crossing time, even with Morozov's modest softening parameter, raising some question about the applicability of his simulations to actual almost collisionless galaxies. Increasing the number density of model stars is definitely a more reliable procedure. This paper presents the results of such simulations.

Since Chandrasekhar's (1960) fundamental "molecular-kinetic" studies, in stellar systems such as the solar vicinity of our own Galaxy, binary star-star encounters are well recognized to have no influence on the evolution. That is, if other perturbations were absent the motion of a star in its orbit in the regular gravitational field of a galaxy would be determined at every moment of time by the initial conditions that prevailed when the star was "born."[4] Thus, in sufficiently dynamically hot and rarefied stellar systems of flat galaxies interparticle collisions can be neglected on the timescale of interest $\lesssim 100\Omega^{-1}$, where in galaxies $\Omega^{-1} \sim 10^8$ yr. Then, the equations of motion for an individual star of unit mass in the inertial frame with the origin at the disk center have the form (Chandrasekhar 1960, chap. 3):

$$\frac{d^2r}{dt^2} = r(\dot{\varphi})^2 - \frac{\partial\Phi_0}{\partial r} - \frac{\partial\Phi_1}{\partial r}, \tag{6}$$

$$\frac{d}{dt}(r^2\dot{\varphi}) = -\frac{\partial\Phi_0}{\partial\varphi} - \frac{\partial\Phi_1}{\partial\varphi}, \tag{7}$$

where the $\dot{\varphi}$ indicates time derivatives of φ with respect to time. Here and below r, φ, and z are the galactocentric cylindrical coordinates and the axis of the galactic rotation is along the z-axis. In the equations above, the gravitational potential has been divided into the smoothed part $\Phi_0(r)$ satisfying the equilibrium condition

$$\frac{\partial\Phi_0}{\partial r} = r\Omega^2, \tag{8}$$

and the fluctuating small perturbation $\Phi_1(\mathbf{r}, t)$ with $|\Phi_1/\Phi_0| \ll 1$ for all $\mathbf{r}$ and t.

As has been mentioned, the local simulation has been developed by Wisdom & Tremaine (1988), Toomre (1990), Toomre & Kalnajs (1991), and Salo (1992, 1995). Following them, let us assume that the radial extent of any region of interest is much smaller than its distance from the center of rotation and any relative motion is only a small fraction of the full rotation velocity. In such a model, the linearized Newtonean Eqs. (6) and (7) in Hill's approximation can be rewritten in the suitable form:

$$\frac{d^2x}{dt^2} - 4\Omega A_0 x - 2\Omega\frac{dy}{dt} = -F_x, \tag{9}$$

$$\frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} = -F_y. \tag{10}$$

In the equations above,

$$x = r - r_0, \;\; y = r_0(\varphi - \Omega t),$$

r_0 is the reference radius, $\Omega = \Omega(r_0)$, and $A_0 = -(r_0/2)(d\Omega/dr)_0$ is the first Oort constant of the differential rotation which is a measure of the shear strength. In actual galaxies $0 < A_0 < (3/4)\Omega$ and typically $A_0 \simeq 0.5\Omega$. In general, $-F_x$ and $-F_y$ are the forces due to interactions with other stars. The gravitational forces are

$$\mathbf{F_i} = -Gm_s^2 \sum_{j\neq i}^{N} \frac{\mathbf{r_i} - \mathbf{r_j}}{[(\mathbf{r_i} - \mathbf{r_j})^2 + r_{\rm cut}^2]^{3/2}},$$

where $\mathbf{r_i}$ is the position of the i-th particle, $\mathbf{r_j}$ is the position of the j-th particle, and m_s is the mass of a particle. The cutoff radius $r_{\rm cut}$ of the potential was introduced in order to avoid numerical difficulties caused by rare very close encounters between the model particles. This "softening" parameter reduces the interaction at short ranges and puts a lower limit on the size of the model stars, i.e., the stars in the system can no longer be considered as point-masses – they are in fact Plummer spheres with a scale size $r_{\rm cut}$. In addition, a sufficiently high value of $r_{\rm cut}$ makes the two-dimensional system a "collisionless" one (see below). Of course, the linearized equations of motion (9) and (10) are valid only if $|x| \ll r_0$. Such equations do not allow for investigation of nonlinear effects, the such as the well-known (in plasma physics) quasilinear collective-type relaxation.

The system of equations of motion (9)-(10) for N identical particles was integrated by the standard Runge-Kutta method of the fourth order. A rotating Cartesian coordinate system with origin at the reference position r_0 was chosen, the x axis pointing radially outward, and the y axis pointing in the direction of the rotation (for details see Toomre 1990 and Salo 1995). The particles were initially placed on nearly circular orbits with an anisotropic Schwarzschild distribution of small radial and azimuthal random velocities components. The last statement means that according to the set of equations (16) and (17) of Appendix A.1, the ratio of the velocity dispersions in the azimuthal and the radial directions (in the rotating frame we are using) is given by (Spitzer & Schwarzschild 1953)

$$\frac{c_\varphi}{c_r} = \frac{\kappa}{2\Omega}.$$

This is close to that observed in the solar vicinity of the Galaxy. In conformity with observations we set the Gaussian distribution of small random velocities along each coordinate in momentum space both in the theory and in the numerical experiments. Thus, equilibrium is established in a simple manner in such disks, i.e., it is governed mainly by the balance between the centrifugal and gravitational forces. It is this metaequilibrium that is to be examined for stability by local simulations.

The initial distribution of stars (x_i, y_i) was generated by means of pseudo-random number generator placing particles uniformly in the box in real space. The box should be thought of as being embedded in a galactic disk which has a constant angular velocity gradient in the x direction, that is, the velocities obey initially a linear shear profile, the stationary solution $v_{i,x} = \tilde{v}_{i,x}$, $v_{i,y} = -2A_0 x_i + \tilde{v}_{i,y}$, where $\tilde{v}_{i,x}$ and $\tilde{v}_{i,y}$ are the random velocities in the x-direction and y-direction, respectively. To maintain the system under the shearing stress in a steady state, the cyclic boundary conditions are used in the form suggested by Wisdom & Tremaine (1988) and Salo (1995).[5] A star leaving

[4] On the other side, the modern observational data convincely indicate the presence of a strong perturbative mechanism disturbing the stellar orbits; see Binney & Tremaine (1987, p. 470) as a review of the problem. The majority of the experts in the field is yield to the opinion that this dynamical relaxation may be explained naturally by collective interactions of stars with unstable density waves.

[5] The "sliding brick" technique of Wisdom & Tremaine (1988), Toomre (1990), Toomre & Kalnajs (1991), and Salo (1995) has been used in the past to simulate transport properties of simple fluids under the action of a strong shearing force (Lees & Edwards 1972; Evans & Morriss 1984).

the computational domain at one side will enter again at the opposite side with the suitable velocity components.

In all the experiments reported in this paper, we have taken the box to be rectangular, $L_x \times L_y = \lambda_J \times \lambda_J$, $4\lambda_J \times 6\lambda_J$, $5\lambda_J \times 5\lambda_J$ or $6\lambda_J \times 8\lambda_J$, where $\lambda_J = 4\pi^2 G\sigma_0/\kappa^2$ is the ordinary Jeans-Toomre radial wavelength (Toomre 1964, 1990). The direction of the disk rotation was taken to be clockwise and units are such that $Gm_s^2 = 1$. Time $t = 1$ corresponds to a single revolution of the disk, and the orbital period is $T_{\rm orb} = 2\pi/\Omega$. All the particles move with the same constant Runge-Kutta time step $\Delta T = 0.001 T_{\rm orb}$. We did not included any artificial extra damping force on the right side in Eqs. (9) and (10) suggested by Toomre (1990) to reduce the computational time.

In our simulations (within the local simulation technique), a particle at (x, y) has images at $(x \pm lL_x,\ y \pm p2A_0L_x\Omega t \pm sL_y)$, where t is the time and the values of l, s, and p were chosen to be equal 1 (Wisdom & Tremaine 1988; Toomre 1990). Gravitational forces on a given target particle are calculated from all the other particles whose nearest image lies within the distance $r_{\max} \leq \frac{1}{2}\ \min\{L_x, L_y\}$ (Salo 1995, Fig. 1 in his paper). Then, more distant images $|l|$, $|s|$, and $|p| > 1$ do not contribute to gravitational forces.

Within the simple molecular-kinetic theory by Chandrasekhar (1960, chap. 2), the classical collisional relaxation time for a three-dimensional system,

$$\tau \approx \frac{c^3}{2\pi G^2 m_f^2 n_f \ln \Lambda}, \qquad (11)$$

should be replaced by the collisional relaxation time in a two-dimensional system (Rybicki 1971; Hohl 1973; Grivnev 1985):

$$\tau \approx \frac{c^3 \delta}{\pi G^2 m_f^2 n_f}. \qquad (12)$$

Here c is the averaged velocity dispersion, δ is the minimum impact parameter, m_f is the mass of a field particle, and n_f is the two-dimensional (Eq. [12]) or the three-dimensional (Eq. [11]) number density of field particles. Also $\ln \Lambda$ is the so-called Newton's (or Coulomb's in plasmas) logarithm, by means of which the long-range nature of the gravitational force is taken into account. In galaxies $\ln \Lambda \approx \ln N$, and N is the total number of field particles (Binney & Tremaine 1987, pp. 187 and 420). Theis (1998) has presented semi-analytical calculations for the two-body relaxation in softened potentials based on a Plummer mass distribution and compared these calculations with N-body simulations. It has been shown that with respect to a Keplerian potential the increase of the relaxation time given by Eq. (11) in the modified potentials is generally less than one order of magnitude, typically only between 2 and 5, if the softening length is of the order of the mean interparticle distance. Consequently, we expect that the expressions (11) and (12) for the time of two-body relaxation in the case of the softenend potential we are using, are correct at least to the order of magnitude.

In contrast to three-dimensional models, the collisional relaxation time for exactly two-dimensional computer models being calculated from Eq. (12) is very short, of the same order as the rotation period (Rybicki 1971). However, Rybicki (1971) has already been pointed out that fortunately numerical calculations are themselves subject to further approximations, and a discretization effect minimizes the difficulty with relaxation time. Indeed, in contrast to the three-dimensional case (Eq. [11]), there is no Newton's logarithm in the expression (12) for the relaxation time via the binary encounters in a two-dimensional system. On the other hand, in a two-dimensional system, encounters with small impact parameters play the main role for collisional relaxation, $\tau \propto \delta$; consequently, in two-dimensional systems there is no problem with the maximum impact parameter (in plasma physics the upper limit is the Debye radius). It is natural therefore to set $\delta = r_{\rm cut}$ in Eq. (12) [Grivnev 1985]. Clearly, by choosing a sufficiently large value of δ, one can construct a two-dimensional model which is practically collisionless on the timescale of interest.

It is very important to realize that the numerical model of a galaxy should properly simulate almost *collisionless* systems. According to Eq. (12), a way to achieve this in a two-dimensional system is to reduce the gravitational attraction at short distances so that the relaxation time $\tau \gg T_{\rm orb}$. Otherwise, as it was shown by Griv & Peter (1996), Griv & Chiueh (1996), Griv & Yuan (1996), and Griv et al. (1997a) in a disk with frequent collisions, in which $\tau\Omega \ll 1$, another secular dissipative-type instability may develop effectively. This dissipative instability may produce structures completely unrelated to the effects we would like to model (e.g., Sterzik et al. 1995).

Eq. (12) indicates that our two-dimensional N-body system will remain practically collisionless for more than several revolutions, if $c_r \geq 0.2c_T$, $\delta \equiv r_{\rm cut} \geq 0.03\lambda_J$, and $N \geq 2000$. In this case, in the lowest approximation one does not need to include the effect of interparticle collisions in the calculation of the accelerations on the right-hand sides of Eqs. (9) and (10). Below we describe the results of simulations of different computer models containing a sufficiently large number of particles $N = 2400$–6480 (and with $c \geq 0.2c_T$).

The value of $r_{\rm cut}$ was chosen to be $0.03\lambda_J$, but the results are not sensitive to the choice of $r_{\rm cut}$ in the range $(0.005$–$0.05)\lambda_J$. (Note that such a value of $r_{\rm cut}$ does not suppress the axisymmetric Jeans instability if random motions are completely absent; see Toomre 1990 for an explanation.) We did not find any dependence of critical stability criterion on the amplitude of the smoothing parameter (in the range $r_{\rm cut} = 0.005$–0.05) as advocated recently by Romeo (1997).

Summarizing, similar to actual galaxies of stars our model is a collisionless one to a good approximation at least on a timescale of several $T_{\rm orb}$.

In all the experiments the simulation had been performed up to a time $t = 3$, but we shall present snapshots for times $t < 1$ only, since after the first rotation the system is always stabilized and no further rapid evolution is visually detectable. We performed a few runs for systems containing $N = 20\,000$ model stars and for smaller systems containing only $N = 2000$ ones. It was found that the results obtained for those systems are qualitatively indistinguishable: we did not detect in our experiments any dependence of the type $\aleph \propto N^{-1/2}$, where $\aleph$ is the amplitude of the density variations. The last is clearly inconsistent with Toomre's (1990) hypothesis that the spirals observed in local simulations can be explained by the swing-amplified particle noise ("spiral chaos in an orbiting patch" or "kaleidoscope of chaotic arm features" which are responses to the random density irregularities orbiting within the particulate disk). In the following discussion we advocate a way to describe the rapidly evolving structures, such as those reported in the simulations (Sect. 3), in terms of local true instabilities of Jeans-type perturbations.

We claim that our numerical results are insensitive to the value of N at least in the range 2000–20 000 and therefore two-body relaxation effects are not important. Also we did not find any difference between the results of simulations with or without applying the so-called quiet starts procedure to select the initial coordinates of particles. The methods of quiet starts were developed first in plasma simulations by Byers & Grewal (1970). Basically, by applying the method of quiet starts, one uses no random numbers in the initial conditions to suppress the noise level in a system. Such techniques have proven useful in obtaining realistic noise levels without the use of a large number of particles. Moreover, tests indicated that the results were insensitive to changes in other gross parameters (the area of unit cell, etc.). A test gives a good check on the numerical stability of the code as well as the accuracy of the program; the code conserves energy to within $1\% - 2\%$ during the first 3 rotations of the system.

3. Numerical experiments: results

In this section we report on the numerical study of the spontaneous appearance of the Jeans modes in a collisionless stellar disk representing the disks of highly flattened galaxies. In particular, we focus on the random motion effect both in rigidly rotating disks and in differentially rotating disks. The structures that appeared in computer models are interpreted by us in terms of the stability theory. It will be shown that the stability criterion obtained in numerical experiments is close to the theoretical generalized stability criterion (3).

3.1. Rigidly rotating disk

First the rigidly rotating disk was investigated. In Fig. 1 we show a series of eight snapshots from a run with the cool model, i.e., the rigidly rotating model, $d\Omega/dr = 0$, in which stars all move along almost circular orbits and the radial dispersion c_r of the random velocities is smaller than the critical Toomre's one, namely $c_r = 0.2c_T$, or Toomre's Q-value is equal to 0.2, respectively. As has been predicted in the theory (see the Introduction), the Jeans-type instability develops quickly in the system during the time of the first rotation, $\sim \Omega^{-1}$. As one can see from Fig. 1, the system's evolution can be qualitatively divided into three stages. At the beginning, at times $t \lesssim 0.1$ the particle distribution in the (x, y)-plane is nearly random. Then at times $t \sim 0.2$ the linelike structures develop. At times $t \sim 0.4$ most of the stars are accumulated inside the lines which form a "honeycomb" network with large voids between the filaments.

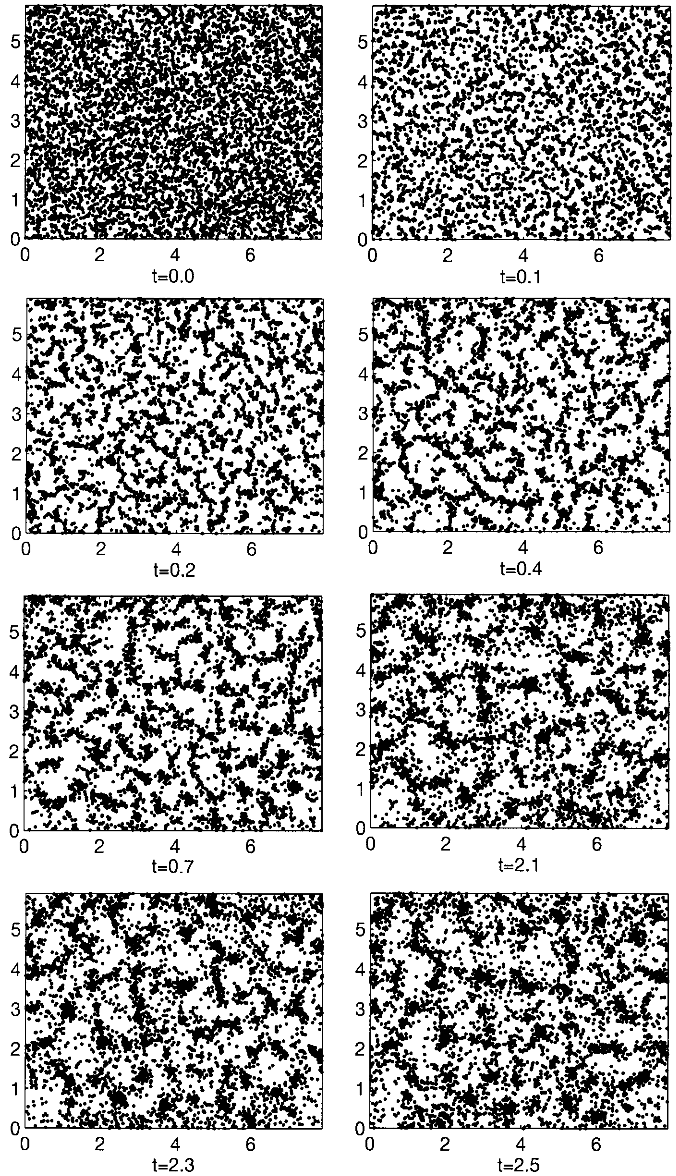

Fig. 1. N-body ($N = 6480$) gravitational simulation snapshots at normalized times t for the rigidly rotating disk model $(2\Omega/\kappa = 1)$ with the radial dispersion of random velocities of particles $c_r = 0.2c_T$, where c_T is the marginal Toomre's dispersion (Eq. [1]). The time here and everywhere is normalized so that $t = 1.0$ corresponds to a single revolution of the disk. The direction of disk rotation is taken to be clockwise. The box here as well as in calculations shown in Figs. 2–5 and 10–11 is taken to be rectangular, $L_x \times L_y = 6\lambda_J \times 8\lambda_J$, where λ_J is the ordinary Jeans-Toomre wavelength (Sect. 2). The system is violently unstable to a gravitational mode with the wavelength $\sim \lambda_J$. These results agree with previous studies, and are the manifestation of the classical Jeans instability in rapidly rotating disks.

The size of a typical void (or a typical distance between the filaments) both in the radial direction and in the azimuthal direction is $\sim \lambda_J$, indicating that perturbations with wavelength λ_J have the fastest growth rate. Such a size of a void is in agreement with the theory (Appendix A.1, Eq. [28]). To stress, the analogy to honeycomb should not be taken too far since glob-

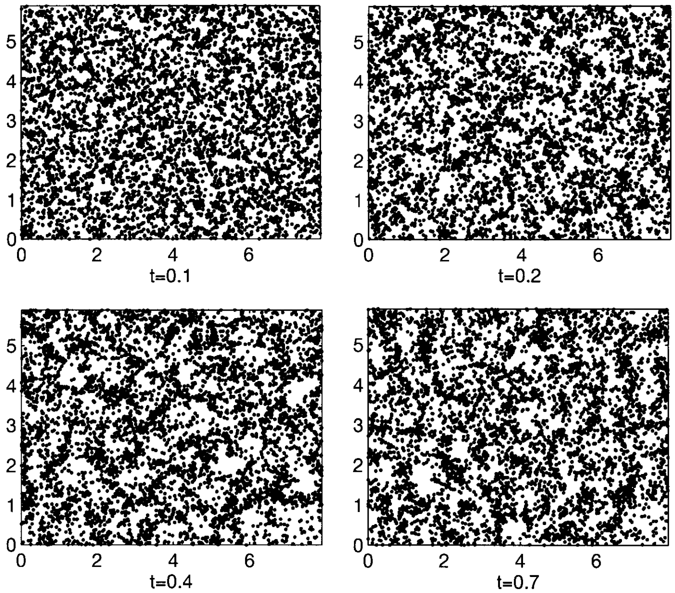

Fig. 2. The rigidly rotating disk in which the usual Toomre's condition for stability toward axisymmetric perturbations, i.e., $c_r = c_T$ holds at each point. The model is only weakly unstable with respect to Jeans-type modes.

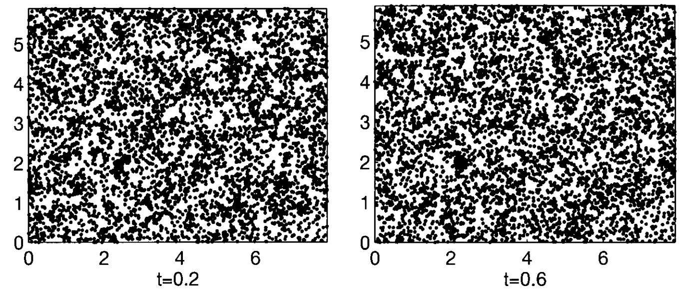

Fig. 3. Snapshots from the evolution of the rigidly rotating disk with $c_r = 1.5c_T$. In contrast to what one can see in Figs. 1 and 2, the system now is practically stable with respect to all local gravitational perturbations.

ally the filaments are quite randomly oriented and there is no preferred direction. At the third, later stage, a tendency towards what Toomre (1990) called "moon-making" is clearly seen in Fig. 1 at times $t > 2.0$. That is, the lines disintegrate into several pointlike "moons," while voids are filled with few stars. It is interesting to note that moon's number density is approximately $1/\lambda_J^2$.

In the second set of experiments with the rigidly rotating disk, we simulated a system which is stable according to Toomre (1964): $c_r = c_T$ or $Q = 1$, respectively. The evolution of the model is shown in Fig. 2. In such a system this relatively high temperature essentially reduces (but does not eliminate completely) the growth rate of the instability, i.e., the instability is sensitive to Toomre's Q-value. Now the disk is near the stability threshold. Again, in agreement with the theory, the size of the voids is about λ_J and the number density of moons is $\sim 1/\lambda_J^2$.

In the final experiment with the rigidly rotating system we set at $t = 0$ the velocity dispersion $c_r = 1.5c_T$ (or $Q = 1.5$). The results are shown in Fig. 3. The system becomes practically stable. Therefore, we conclude that the critical dispersion is near $1.5c_T$. Why the disk is still unstable (more correct, weakly unstable) when $c_r > c_T$ but $c_r \lesssim 1.5c_T$, or $1 < Q \lesssim 1.5$, respectively, remains an open question; the modifications due to the effect of azimuthal forces do not help in this case of rigid rotation (Appendix A.1).

To summarize, computer configurations with increased velocity dispersion change more slowly than their low velocity dispersion counterparts. This result is consistent with the theoretical prediction (Eq. [30]).

Thus, Toomre's local stability criterion is only reasonably accurate in the case of the rigidly rotating disk, and the critical dispersion $c_{\rm crit}$ might be a bit larger than c_T. Although this value of $c_{\rm crit}$ is in general agreement with the theory, it is also disappointing that it is not exactly equal to c_T. Strictly speaking, we do not know the reason for such a larger value of $c_{\rm crit}$, while it might be attributed either to the linearization on the theory side or the linearization on experiment side (Hill's approximation). We should note also that no better agreement can be expected from the theory described in the Introduction and our greatly simplified theory presented in Appendix A.1 (see, for example, the approximate expressions for the Bessel functions). In addition, one obvious shortcoming of the numerical procedure used here is that gravitational forces on a given target particle are calculated only from other particles whose nearest image lies within the distance $r_{\rm max} \le \frac{1}{2} \min\{L_x, L_y\}$. The accuracy of such an approximation may need further investigation because the gravitational forces are the long range ones. Clearly, in a more accurate model one also has to include more distant images in the calculation of the gravitational forces.

3.2. Differentially rotating disk

In the current subsection the development of the Jeans instability in the nonuniformly rotating disk, $d\Omega/dr \neq 0$, is studied. Figs. 4 and 5 clearly show that in a disk with the Keplerian shear profile ($2\Omega/\kappa = 2$) a fast nonaxisymmetric instability develops both in a Toomre unstable system ($c_r = 0.2c_T$; Fig. 4) and in a Toomre stable one ($c_r = c_T$; Fig. 5). A spiral pattern (more accurately, a chainlike structure or "wakes") develops rapidly in the initially featureless disk on a dynamical timescale, $\sim \Omega^{-1}$. Unlike in the case of rigid rotation (Figs. 1 and 2), the structure now consists of elongated trailing filaments. Note that in agreement with the theory as described in the Introduction, even Toomre's stable nonuniformly rotating disk (in which $c_r = c_T$ or $Q = 1$, respectively; Fig. 5) is still violently unstable to spiral perturbations growing on a dynamical timescale. This fierce instability of the system with $c_r = c_T$ indicates that in a differentially rotating system the instability of nonaxisymmetric perturbations cannot be suppressed by the ordinary Toomre's critical dispersion c_T.

The pitch angle of spiral wakes $\psi \lesssim 35°$, thus $\tan^2\psi \ll 1$ and the asymptotic Lin-Shu approximation of moderately

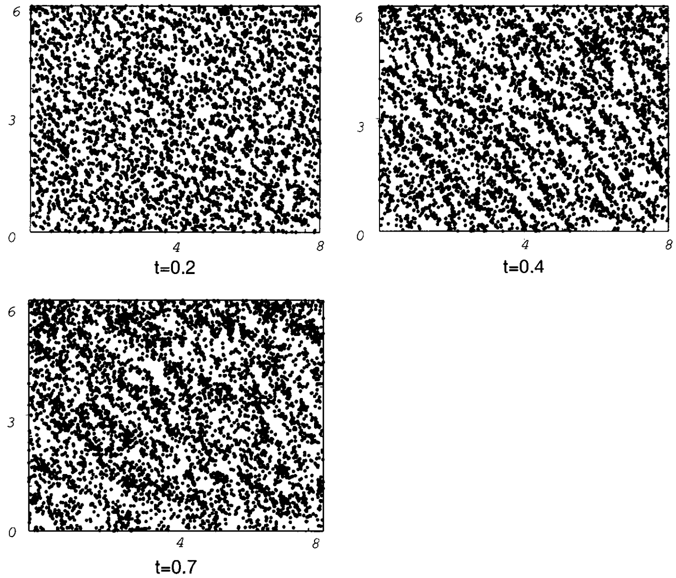

Fig. 4. The differentially rotating disk with the Keplerian rotation ($2\Omega/\kappa = 2$) and the radial-velocity dispersion $c_r = 0.2c_T$. The model is violently unstable against rapidly growing, trailing ($\psi > 0$) nonaxisymmetric (spiral) perturbations (m and $\psi \neq 0$) of the Jeans type. A typical radial distance between spiral filaments is $\sim \lambda_J$; the number of filaments $\simeq \beta L_x/\lambda_J \approx 12$ (Eq. [29]). Notice the change in the structure of the instability in compare with that seen in Figs. 1 and 2.

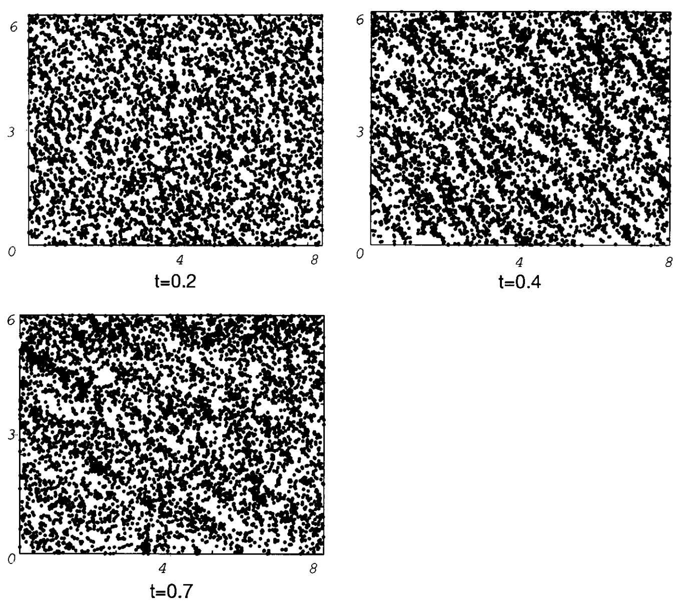

Fig. 5. The disk with the Keplerian rotation curve and dispersion $c_r = c_T$. Even though the velocity dispersion is equal to Toomre's critical one c_T, this differentially rotating model is still violently unstable against Jeans-type modes.

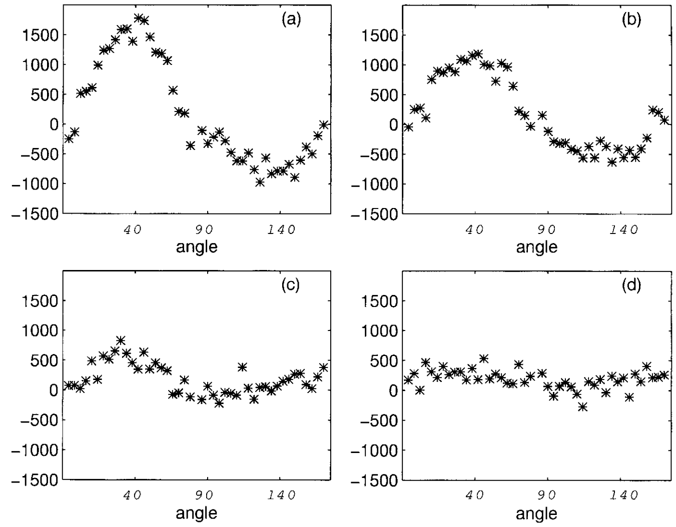

Fig. 6a–d. The orientational correlator of particles $n(\psi)$ (Eq. [13]) vs. the perturbation angle (in degrees) at the calculation time $t = 0.5$ for differentially rotating models with the velocity dispersion: **a** $c_r = 0.2c_T$, **b** $c_r = c_T$, **c** $c_r = (2\Omega/\kappa)c_T$, and **d** $c_r = 1.5 \times (2\Omega/\kappa)c_T$.

tightly-wound perturbations used throughout the theory described in the Introduction and Appendix A.1, does not fail.

In agreement with the theory (Eq. [30]), the disks become progressively more stable as the initial velocity dispersion is increased. This is clearly seen in Figs. 4 and 5: the spiral structure in the cool model (Fig. 4) develops at times $t \sim 0.2 - 0.3$ but in the warm model (Fig. 5) it develops at somewhat later times $t \sim 0.4$–0.5, i.e., the structure in the simulation illustrated in Fig. 5 is weaker and takes longer to form than in the equivalent cooler model shown in Fig. 4. Hence, the growth rate of unstable perturbations in the warm model is smaller than the growth rate in the cool model. This is a natural consequence of greater random motions as is shown in the theory (Appendix A.1).

Also, as one can see visually in Figs. 4 and 5 these elongated trailing filaments, which are similar to those found in spiral galaxies, are very different from the "honeycomb" structures which appear in rigidly rotating disks (Figs. 1 and 2). The former definitely have a prefered direction ψ_s. To quantitatively study the breaking of the rotation symmetry we plot in Figs. 6a and 6b the orientational correlator of stars at the calculation time $t = 0.5$ for models with $c_r = 0.2c_T$ and $c_r = c_T$, respectively, defined by

$$n(\psi) = \langle n_i(\psi) \rangle - \frac{1}{2\pi} \int_0^{2\pi} n_i(\psi) d\psi, \qquad (13)$$

where $\langle \cdots \rangle$ denotes averaging over all the stars, $n_i(\psi)$ is number of stars inside a segment around ψ with distances not exceeding $r_m = \lambda_J$ (cf. the usual "friends-to-friends" method to find groups in particle distribution). This value of r_m is chosen to exhibit short range order: it is conveniently larger than the width of the filaments but smaller than the scale on which filaments develop the curvature. It seems, therefore, that the signal is optimal. The peak at the values $\psi_s = 30° - 40°$ is clearly seen both for the model with $c_r = 0.2c_T$ and for $c_r = c_T$. Moreover, in the direction perpendicular to ψ_s at the values $\psi = 120°$–$130°$ one sees negative correlation.

Of course, we do not see a network of parallel equidistant lines. So we cannot rely on the standard Fourier treatment to analyze the structures seen in Figs. 4 and 5. One can

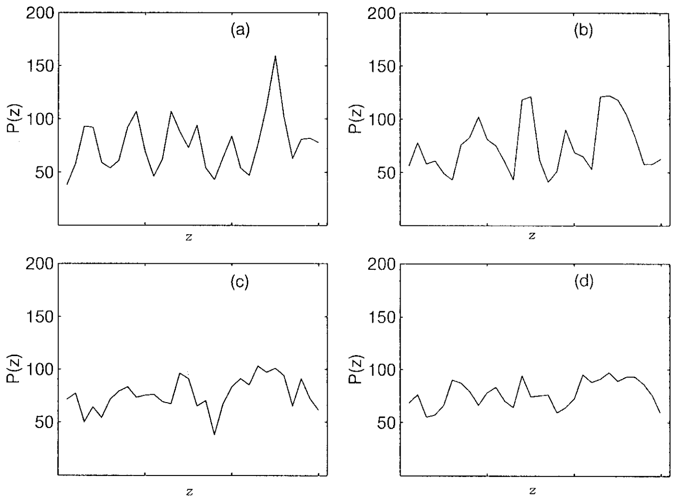

Fig. 7a–d. The correlator $P(z)$ at the calculation time $t = 0.5$ for differentially rotating models with the velocity dispersion: **a** $c_r = 0.2c_T$, **b** $c_r = c_T$, **c** $c_r = (2\Omega/\kappa)c_T$, and **d** $c_r = 1.5 \times (2\Omega/\kappa)c_T$.

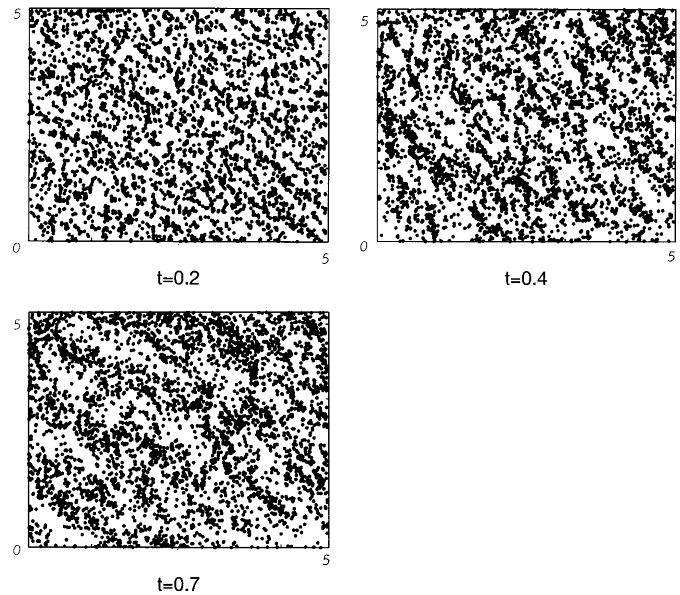

Fig. 8. Time-development of the nonuniformly rotating model of stars ($2\Omega/\kappa = 2$) distributed over the $L_x \times L_y = 5\lambda_J \times 5\lambda_J$ unit cell, $N = 5000$, and $c_r = c_T$.

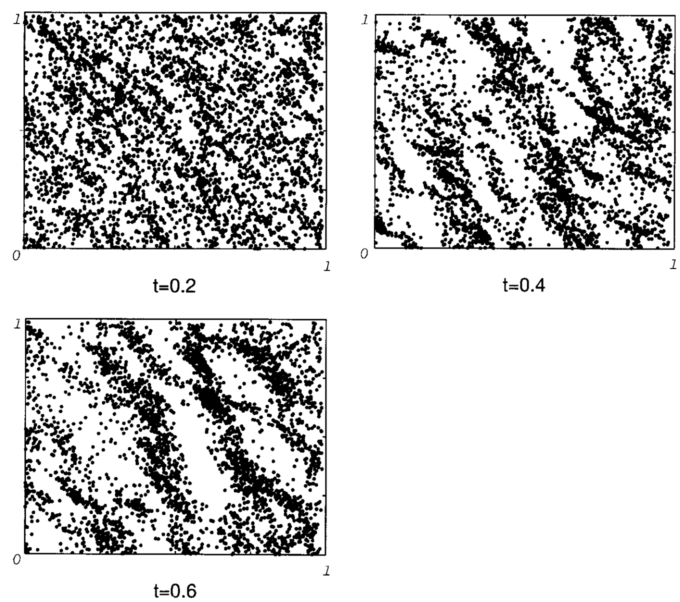

Fig. 9. Time-development of the nonuniformly rotating model of stars ($2\Omega/\kappa = 2$) distributed over the $L_x \times L_y = \lambda_J \times \lambda_J$ unit cell, $N = 5000$, and $c_r = 0.2c_T$. Notice the change in the number of spiral wakes in Figs. 8 and 9 in compare with that seen in Fig. 4.

clearly see, however, that a characteristic distance between the nearby (nearly parallel) lines is of order of $\beta\lambda_J$ (Eq. [29]). As for the present study, the latter fact convincingly indicates that we have dealt with a gravitational instability rather than with a random process. The translation symmetry in the direction ψ_s is clearly unbroken, while in the perpendicular direction a series of alternating peaks and dips is seen. To describe the latter phenomenon quantitatively we calculated the correlator $P(z)$, where $z = x \cos\psi_s + y \sin\psi_s$ and x, y are the coordinates of a particle, in the direction $\psi_s + \pi/2$. We counted the average number of stars P at the time $t = 0.5$ in strips of width $\Delta = L_x/30$ around z; the results are presented in Fig. 7a–d. From Figs. 7a and 7b one measures the average distance between nearby peaks $\sim \beta\lambda_J$ in agreement with the theory (Eq. [29]). In the latter figures we see 5–6 dips and peaks. Peaks are clearly higher than one standard deviation. This is above any possible noise and shows a developing instability process. It is natural to attribute the observed instability to the Jeans instability so far discussed in the present paper.

To show another direct indication that it is so, we repeated calculations with the nonuniformly rotating models of smaller sizes $L_x \times L_y = 5\lambda_J \times 5\lambda_J$, $N = 5000$, $c_r = c_T$ and $L_x \times L_y = \lambda_J \times \lambda_J$, $N = 5000$, $c_r = 0.2c_T$. The results of these simulations are shown in Figs. 8 and 9, respectively. As one can see, now in full agreement with the theory, the number of spiral wakes n_w is considerably smaller than with that in simulations of the $L_x \times L_y = 6\lambda_J \times 8\lambda_J$ model (Figs. 4 and 5) but still corresponds to the theoretical prediction $n_w \simeq \beta L_x/\lambda_J \approx 10$ (Fig. 8) and $n_w \simeq \beta L_x/\lambda_J \approx 2$ (Fig. 9).

We then simulated a differentially rotating disk which is stable in accordance with the generalized local stability criterion (3). Fig. 10 shows the observed evolution of the model with $c_r = (2\Omega/\kappa)c_T$. As can be seen, in contrast to the previous simulations (Figs. 4, 5, 8, and 9) the model is indeed more stable gravitationally. The orientational correlator of stars $n(\psi)$ shows that the disk is near the stability threshold (Fig. 6c). This is also seen quantitatively in Fig. 7c on which the correlator $P(z)$ clearly has less structure on the standard deviation scale.

Finally, similar to the case of the rigidly rotating disk, the value of the generalized stability criterion (3) should probably be increased by the factor ~ 1.5. To prove this suggestion, in Fig. 11 we show the evolution of the model with $c_r = 1.5 \times (2\Omega/\kappa)c_T$. As one can see in Figs. 6d, 7d and 11, now the model

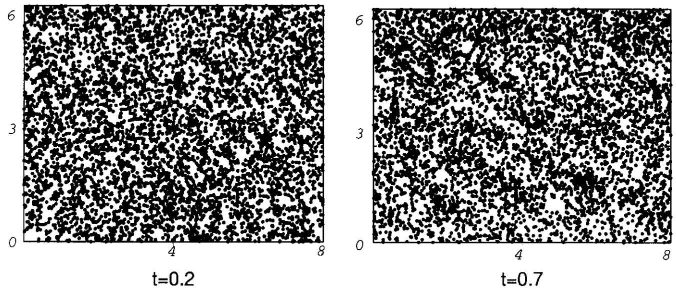

Fig. 10. Evolution of the mildly unstable disk with the Keplerian rotation curve and dispersion $c_r = (2\Omega/\kappa)c_T$.

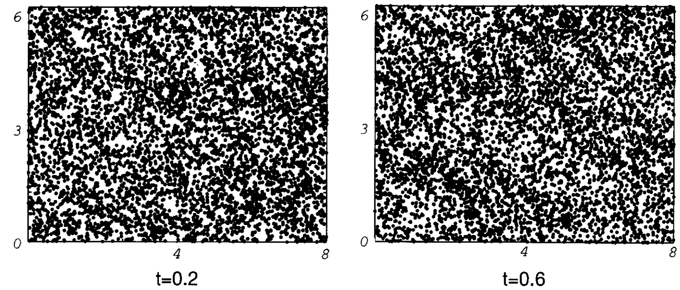

Fig. 11. The disk with the Keplerian rotation and velocity dispersion $c_r = 1.5 \times (2\Omega/\kappa)c_T$. All gravitational perturbations are practically suppressed, including the most unstable nonaxisymmetric ones. The result agrees with the theoretical explanation described in the Introduction and Appendix A.1.

is practically stable.[6] The contrast between Figs. 4 and 10, and Figs. 4 and 11 establishes experimental evidence to support the theory outlined in the Introduction and Appendix A.1.

We conclude that the basic theory explains the results of local N-body simulations. That is, in order to suppress the instability of arbitrary but not only axisymmetric Jeans-type gravity perturbations in a differentially rotating stellar disk, including the most unstable nonaxisymmetric perturbations in the bar form, the value of the radial velocity dispersion must exceed $c_{\rm crit} = \alpha(2\Omega/\kappa)c_T$, where $\alpha \approx 1.5$. The latter will guarantee lack of any exponentially increasing perturbations of the Jeans type.

In closing of the subsection we would like make the following points. The Jeans-unstable perturbations in a disk grow aperiodically: they are aperiodically shrinked (Eq. [27]). On the other hand, Jeans-stable perturbations are not damped, so that in the plane of the stellar disk the undamped waves can propagate similar to magnetoacoustic waves in a plasma. It was the initial idea of Lin & Shu (1966), Lin et al. (1969), and Shu (1970) to explain the phenomenon of the spiral structure of galaxies by these neutral waves which propagatae in the plane of a system.[7]

Griv (1996), Griv & Peter (1996), Griv & Yuan (1996), Griv et al. (1997b), and Griv (1998) recently investigated the influence on the disk stability of the so-called drift motion of particles in planetary rings and stars in galaxies. This addition to the basic circular minor systematic motion proportional to the square of c_r (proportional to the temperature of the system), whose value can be defined in the high-order approximation of Lindblad's epicyclic theory, is analogous to the magnetic (or grad B) drift of an electrically charged particle of a plasma, and is due to the nature of the differential rotation of the system (Grivnev 1988; Griv 1996; Griv & Peter 1996). It was shown that even in a Jeans-stable differentially rotating, nearly homogeneous disk, that is, when the generalized local stability condition (3) is satisfied, other spiral perturbations of the kinetic type will grow. The cause of this kinetic instability of small-amplitude perturbations is the resonant interaction of drifting stars with the field of the spiral Jeans-stable waves at the corotation in a spatially inhomogeneous particulate system. In other words, the cause of the instability is the resonant wave-particle interaction in a hydrodynamically (Jeans-) stable stellar disk. It is similar to the instabilities caused by a Cherenkov effect (an inverse Landau damping effect) in a magnetized plasma. In plasma physics the analogous instability is already known as the transverse magneto-drift instability of an inhomogeneous plasma (Krall & Rosenbluth 1963; Chamberlain 1963). Since such a wave-particle interaction, being intrinsically a kinetic interaction, involves resonant stars, it cannot be derived from the ordinary epicyclic equations of motion of a mean particle considered in Appendix A.1 or from fluid-like equations used by Lau & Bertin (1978), Lin & Lau (1979), Drury (1980), Lin & Bertin (1984), and others. The next, postepicyclic approximation must be used for this purpose both in the analytical approach of the Boltzmann-Vlasov kinetic equation and of particle dynamics (Griv 1996; Griv & Peter 1996; Griv & Yuan 1996; Griv et al. 1999).

We expect that resonant wave-particle scattering of stars as outlined above will lead to further heating of the disk up to values of $Q > 2\Omega/\kappa$. Similar to the case of Jeans instability, a phase of kinetic instability may also increase the central condensation of the disk and assist in the formation of a condensed nucleus of the galaxy (and a diffused outer envelope).

It may be suggested that this type of Landau instability may be related to that discovered in global N-body simulations by Sellwood & Lin (1989) and Donner & Thomasson (1994) [see Griv 1998 for a discussion]. What one can see in Figs. 4 and 5 are not the recurrent spiral waves found by Sellwood & Lin and Donner & Thomasson. We suggest that in order to find the recurrent Landau-type instability in local simulations, one has to include in the linearized equations of motion (9)–(10) and (14)–(15) terms proportional to the square of the epicyclic radius ρ along with other terms of order ρ/r_0 (one has to include

[6] In conjunction with the last result, it should be emphasized again that in view of the Lin-Shu type asymptotic theory the analysis presented here provides only an approximate estimation of the local stability criterion.

[7] In turn, Toomre (1969) has shown that density waves of the kind originally proposed by Lin and Shu (Lin & Shu 1966; Lin et al. 1969; Shu 1970) cannot be stationary, and a wave theory can explain the phenomena of spiral patterns only if some instability exists which could cause small perturbations to grow to observable amplitudes.

the effects of spatial inhomogeneity as well).[8] The reason for this is that the kinetic instability of Jeans-stable perturbations is expected to be associated with the resonant condition $\omega_* = kv_D$, where v_D is the velocity of the star drift proportional to $\rho^2(d\Omega/dr)$ [Griv 1996; Griv & Peter 1996; Griv et al. 1999].

4. Conclusions and discussion

We described some many-particle experiments concerning numerical computations on the dynamics of the stellar layer of a differentially rotating, almost centrifugally-supported galaxy. Our usage of the oversimplified model of the layer (two-dimensional disk) is justified because global N-body simulations have been shown that the inclusion of motions normal to the plane makes little difference to the evolution of the rapidly rotating thin disk (Hohl 1978). We argued that in general computer experiments presented here confirm the predictions of the linearized stability theory of small-amplitude gravity perturbations developed by Bertin, Lau, Lin, Mark, Morozov, Polyachenko, and others: the differentially rotating, marginally Jeans-stable disk of stars (and a planetary disk with rare collisions between mutual-gravitating particles) is dynamically hotter than the original Toomre's local stability criterion predicts. That is, in a nonuniformly rotating disk of stars the critical Toomre's stability parameter $Q_{\rm crit} \approx 2\Omega/\kappa$ is appreciably greater than (although still of the order) unity. In actual galaxies and planetary rings $Q_{\rm crit} \sim 2$.

A dynamically cold rigidly rotating disk with the initial radial dispersion of random velocities of stars $c_r < c_T$ is found to be gravitationally unstable as predicted first by Toomre's (1964) stability analysis. Namely, small-scale almost radial perturbations grow exponentially during the time of the first rotation of the system under consideration. In agreement with the theory, in the numerical model of the warm ($c_r = c_T$) rigidly rotating disk the relatively high temperature leads to significant reduction of the growth rate of the Jeans instability; such a disk is near the stability threshold. In the hot numerical model ($c_r \gtrsim 1.5c_T$) all Jeans-type gravity perturbations are stabilized.

By way of contrast, even the Jeans-stable (by the original Toomre's criterion) differentially rotating disk is still violently unstable to the relatively large-scale nonaxisymmetric modes when $1 < Q < Q_{\rm crit}$. In such a system the spiral structure develops rapidly during the first rotation of the system only. Finally, differentially rotating, spatially homogeneous models with the initial value of Toomre's stability parameter $Q \gtrsim Q_{\rm crit}$ (or $c_r \gtrsim (2\Omega/\kappa)c_T$, respectively) show little structure that can be associated with the Jeans instability. This basically agrees with the theory discussed in the Introduction and Appendix A.1.

In both cases, rigidly and differentially rotating systems, some residual instability is observed for Q up to a factor ~ 1.5 times the critical value $Q_{\rm crit}$. The reason for such a minimally larger value of the critical velocity dispersion might be partly due to the shortcomings of the asymptotic Lin-Shu density wave theory which is used here. Accordingly, we restricted our analysis to the approximation of moderately tightly-wound spirals (Appendix A.1). Indeed, as is known, since all the above results are given for moderately tightly-wound spirals, they are subject to an uncertainty of a factor of $1 + O[\tan^2\psi]$, where $\tan^2\psi \lesssim 1$. Straightforward estimates show that in the case of spirals shown in Figs. 4, 5, 8, and 9 $\tan^2\psi$ is about 0.2–0.3; thus, we can have reasonable confidence in theoretical and experimental results perhaps to within $20\% - 30\%$ only. In this regard, it is interesting to note that at least for a disk with a constant rotation velocity Polyachenko (1989), who did not use the approximations of the Lin-Shu theory, has found a slightly greater value of the critical velocity dispersion than the criterion (3) gives.[9] Interestingly, such a slightly greater value of the critical velocity dispersion is also consistent with the results of Toomre's (1981) numerical experiments with stellar disks, in which the disks with a flat rotation curve became completely stable specifically when $Q \gtrsim 3$.

Also, following Griv (1992), to obtain a more accurate value of critical velocity dispersion one has to consider the next leading order in the asymptotic expansion by including higher-order terms in the epicyclic amplitude.

In addition, the shortcomings of local experiments in Hill's equations context are quite obvious. For instance, almost certainly in contrast to our calculations, one has to include gravitational forces on a given target particle from other particles whose nearest image lies out of the distance $r_{\rm max} = \frac{1}{2}\,\min\{L_x, L_y\}$ (see Sect. 2). This is because of the long range of gravitational forces. Further theoretical and experimental N-body studies to clarify the problem are desirable. At the present, however, the causes of these relatively small discrepancies between the results of our theory and local N-body simulations are not clear, but may be due to both theoretical and computational factors just mentioned above.

According to Eqs. (27) and (30), the Jeans-unstable perturbations in a spatially homogeneous disk grow aperiodically with the growth rate $\Im\omega_* \sim \Omega$. This means that as a rule the Jeans instability develops rapidly on a dynamical timescale $\sim \Omega^{-1}$; in galaxies $\Omega^{-1} \sim 10^8$ yr $\ll T$, where $T \sim 10^{10}$ yr is the Hubble timescale. Inevitably, the velocity dispersion of particles would be expected to increase in the field of unstable waves with an amplitude increasing with time as a result of "hydrodynamic" (nonresonant) collective interactions between Jeans-unstable perturbations and stars: the Jeans instability grows on a dynamical timescale and presumably heats the disk until $Q \sim Q_{\rm crit}$. In

[8] One has to recognize, however, that correct N-body simulation of resonant effects is a very difficult problem in stellar dynamics because of lack of fine resolution in the phase space. Perhaps, the better way to study the resonant wave-star interaction involved in the support or damping of the modes is to solve numerically the collisionless Boltzmann equation similar to that by Nishida et al. (1984).

[9] According to Polyachenko (1989) the marginal stability condition for Jeans perturbations of an arbitrary degree of axial asymmetry has been available since 1965 (Goldreich & Lynden-Bell 1965), though in a slightly masked form. See Polyachenko & Polyachenko (1997) for a detailed discussion of the problem.

addition, the Jeans instability, which can effectively heat the medium without raising the entropy, leads to the mass redistribution of the system by increasing the central condensation of the disk (and a diffused outer envelope). The diffusion of stars in the velocity space and the coordinate space takes place because stars gain additional oscillatory energy of the gravitational field in the unstable density waves (see Griv et al. 1994 for a discussion).

It is interesting to note that about the same value of $Q \gtrsim 2$ brings both the observations of actual rapidly (and nonuniformly) rotating galaxies of stars we are investigating including our own Galaxy (Toomre 1974, 1977; van der Kruit & Freeman 1986; Bottema 1993) and the global N-body simulations (Hohl 1971, 1972, 1978; Sellwood & Carlberg 1984; Griv et al. 1994; Griv & Chiueh 1998). Also observations and local simulations of the Saturnian ring system show about the same value of Q (Lane et al. 1982; Salo 1992, 1995; Griv 1996, 1997; Griv & Yuan 1996). Therefore, we conclude that in general both the theory and our N-body simulations are in agreement with observational data.

In closing, the differences between actual inhomogeneous gravitating systems and computer models used in our simulations may result in ambiguity in the applications of the N-body calculations and the theory to galaxies and planetary rings. In order to resolve the ambiguity, it will be possible in the future to make more realistic simulations of this type and to extend the theory so as to allow for spatial inhomegeneity and a finite thickness of the disk.

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Appendix A

A.1 The local stability criterion

A highly flattened disk of stars in almost circular orbit about the galactic center will be subject to self-gravity, which will tend to cause clumping of the matter. This tendency will be counteracted by centrifugal force due to the rotational motion of the mass, and stellar "pressure" due to the thermal motion. If the "binding energy"

$$E = E_{\rm gravity} + E_{\rm rotation} + E_{\rm thermal}$$

of a clump of matter of radius $r_{\rm clump}$ in orbit about the galactic center at radius r_0 is negative, collapse will occur. If $E > 0$, any perturbation in density will be damped out. Below, through the studying of dispersion relations, we reexamine the theory of small-amplitude gravity oscillations and their stability in a practically collisionless, two-dimensional, and spatially homogeneous galactic disk of stars.

In order to find the dispersion relation describing the collective oscillations of a medium near its metaequilibrium state within the method of particle orbit theory, one must determine first the perturbed particle trajectories.[10] Therefore, we start by deriving formulae for nearly circular stellar trajectories in the rotating galactic disk with nonaxisymmetric perturbations due to spiral density waves. The perturbation of the main smoothed galactic potential will be assumed small, and the star's motion will be represented, as usual, by epicyclic free oscillations plus additional forced ones under the action of the gravitational field of the waves. Then, the perturbed (or forced) velocities will be used in the continuity equation to determine perturbation of the surface density. Equating the result with the surface density given by the asymptotic solution of the Poisson equation, the dispersion relation will be obtained. Finally, from the dispersion relation the local generalized stability criterion will be derived. The criterion guarantees the lack of arbitrary but not only axisymmetric Jeans-type unstable perturbations in a disk of mutual-gravitating particles.

In the absence of any perturbing gravity, a nearly circular orbit of a star (and a particle in planetary rings) may be represented as an epicyclic motion along the Coriolis ellipse (epicycle) with the simultaneous rotation of the ellipse (the guiding center) about the galactic center (Lindblad 1963; Chandrasekhar 1960; Binney & Tremaine 1987). In the epicyclic approximation the dispersion of random velocities of stars is taken to be small compared to the circular velocity of regular rotation $V = r\Omega$ determined by the smooth potential (Eq. [8]). This condition of nearly circular stellar orbits is normally satisfied in disks of

[10] Note that the equivalence of the particle orbit theory and the more rigorous Boltzmann kinetic equation approach in the absence of collisions has been demonstrated, and in the astronomical literature is referred to as Jeans' theorem; see Longmire (1963) and Chandrasekhar (1965) for explanations. Similarly, the particle orbit theory reflects both single-particle dynamics and the overall continuity of the system of mutual-gravitating particles. It can be applied only to strongly rarefied particulate systems with practically uncorrelated unperturbed particle motion. The great advantage of using this purely Lagrangian formulation lies in the fact that the equation of continuity and the Poisson equation can be simplified in the asymptotic limit of moderately tightly-wound spiral waves. The analytic calculations presented here are carried out in a new and insightful way, and it is hoped that many workers in the field will find it useful.

flat galaxies that are seen in the sky. With the exception of resonances, the small perturbing gravitational field of a wave causes small forced oscillations in addition to the usual free epicyclic motion. Of course, it is doubtful that the approximation of nearly circular orbits adopted above is valid for the very central regions of flat galaxies. Assuming the nearly axially symmetric model, the vertical, normal to the plane motion in the rapidly rotating self-gravitating disk can be neglected (Shu 1970; Griv & Peter 1996). This assumption is partially supported by the global N-body simulations showing that the inclusion of the vertical motion makes little difference to the evolution of the thin, rapidly rotating disk (Hohl 1978).

The disk is subject to the equation of continuity and the equations of motion along the radial and azimuthal directions. The linearized equations of two-dimensional motion (9)–(10) in the frame of reference rotating with angular velocity Ω can be rewritten in Hill's approximation as (Spitzer & Schwarzschild 1953; Toomre 1990):

$$\frac{dv_r}{dt} - 2\Omega v_\varphi + 2r_0 r_1 \Omega \frac{d\Omega}{dr} = -\frac{\partial \Phi_1}{\partial r}, \qquad (14)$$

$$\frac{dv_\varphi}{dt} + 2\Omega v_r = -\frac{1}{r_0}\frac{\partial \Phi_1}{\partial \varphi}, \qquad (15)$$

where r_0 is the radius of the chosen circular orbit in the (r, φ) plane, $\Omega \equiv \Omega(r_0)$, and r_1 and φ_1 are small perturbations of the coordinates. Eqs. (14) and (15) must be solved simultaneously with the continuity equation and the Poisson equation.

In the model described by Eqs. (14) and (15), the case of rare gravitational collisions between particles is considered when

$$\kappa \simeq \Omega \gg \nu_c,$$

where ν_c is the effective collision frequency. That is, collisions are so infrequent that their effects on both unperturbed and perturbed particle motions can be neglected. Evidence in favour of such an almost collisionless galactic model is provided by observations (Chandrasekhar 1960). The evolution of the system described by Eqs. (14) and (15) is determined by pure stellar encounters with collective modes.

Neglecting all the terms containing the small perturbation Φ_1, the homogeneous differential equations (14)–(15) yield the ordinary Lindblad's expressions for unperturbed coordinates and velocities of a star along the elliptic-epicyclic orbit:

$$r_0^{(0)} = r_0 - \frac{v_\perp}{\kappa}[\sin(\phi_0 - \kappa t) - \sin\phi_0]; v_r^{(0)} = v_\perp \times \cos(\phi_0 - \kappa t), \qquad (16)$$

$$\varphi = \Omega t + \frac{2\Omega}{\kappa}\frac{v_\perp}{r_0 \kappa}[\cos(\phi_0 - \kappa t) - \cos\phi_0]; v_\varphi^{(0)} = \frac{\kappa}{2\Omega} \times v_\perp \sin(\phi_0 - \kappa t), \qquad (17)$$

where $\rho/r_0 = v_\perp/\kappa r_0 \ll 1$ and $v_\perp, \phi_0$ are constants of integration (Spitzer & Schwarzschild 1953). The set of Eqs. (16)–(17) describes the rotation of a star along the epicycle with frequency κ and the mean epicyclic radius $\rho \approx v_\perp/\kappa$. Griv (Grivnev) (1988) and Griv & Peter (1996) have obtained expressions for the galactic orbits of stars to the second order of the epicyclic theory, when terms proportional to $(\rho/r_0)^2$ are also retained in the linearized equations of motion.

Now in order to find an inhomogeneous solution of Eqs. (14) and (15) we have to choose a particular form of the gravitational perturbation Φ_1. Bearing in mind that the equilibrium distribution does not depend on the φ (and the z) coordinate, in a rotating frame, the perturbation Φ_1 may be expanded in a Fourier series

$$\Phi_1(r, \varphi, t) = \sum_{m=-\infty}^{\infty} \tilde{\Phi}_m(r)\, e^{im\varphi - i\omega_* t},$$

where $\omega_* = \omega - m\Omega$ is the Doppler-shifted complex frequency of excited waves as seen by the moving star and the term $m\Omega$ takes into account the possibility of different harmonics in the rotating system (many-armed waves), and $\Re\omega_*$ and $i\Im\omega_*$ are the real and imaginary parts of the wavefrequency, respectively. Evidently Φ_1 is a periodic function of φ, and hence the azimuthal number m must be an integer. The criteria for stability differ for each m, and must be determined by a detailed analysis. In the framework of the linear theory, we can select one of the harmonics: $\tilde{\Phi}(r)\exp(im\varphi - i\omega_* t)$, which rotates at a uniform rate $\Omega_p = \omega_*/m$ and m is the number of spiral arms.

For such a form of Φ_1 the particular solution of the system (14)–(15) is (e.g., Lin & Lau 1979, Sellwood & Kahn 1991, and Griv et al. 1999):

$$v_r^{(1)} = \frac{-i}{\omega_*^2 - \kappa^2}\left[\omega_* \frac{\partial \tilde{\Phi}}{\partial r} - 2\Omega k_\varphi \tilde{\Phi}\right] \times e^{im\varphi - i\omega_* t}, \qquad (18)$$

$$v_\varphi^{(1)} = \frac{1}{\omega_*(\omega_*^2 - \kappa^2)}\left[\left(4\Omega^2 - \kappa^2 + \omega_*^2\right) k_\varphi \tilde{\Phi} + 2\Omega\omega_* \frac{\partial \tilde{\Phi}}{\partial r}\right] e^{im\varphi - i\omega_* t}. \qquad (19)$$

The solutions (18) and (19) describe the forced velocities of a star in the radial and azimuthal directions under the action of the small gravity perturbation, $|v_r^{(1)}|$ and $|v_\varphi^{(1)}| \ll r_0\Omega$. Thus, the present theory suggests some systematic radial and azimuthal motions of the stars distributed in the form of a spiral-like flow field which is a small correction to the basic almost circular galactic motion.

To stress, the solutions (18) and (19) define the forced velocities $\mathbf{v^{(1)}}\left(\mathbf{v^{(0)}}\right)$ of an individual star. In order to obtain the perturbed density, by using the continuity equation, we shall wish to average Eqs. (18) and (19) over the distribution of initial velocities. Such a distribution (the so-called modified Schwarzschild distribution) has been derived by Shu (1970) as follows:

$$f_0(\mathcal{E}, J_z) = \frac{2\Omega(r_0)}{\kappa(r_0)}\frac{\sigma_0(r_0)}{2\pi c_r^2(r_0)} \exp\left\{-\frac{v_\perp^2}{c_r^2(r_0)}\right\}.$$

Here $\mathcal{E} = v_\perp^2/2$ and J_z are well defined integrals of motion, that is, the epicyclic energy integral and the angular momentum integral, and a distance r_0 is defined by the relation

$$J_z = r(r\Omega + v_\varphi) = r_0^2\Omega(r_0).$$

Then, a star in circular motion at a distance r_0 has precisely the given value of J_z. Such a distribution function for the unperturbed system is particularly important because it provides a fit to observations (Shu 1970).

The continuity equation for a small density perturbation $\sigma_1(\mathbf{r},t)$ in a spatially homogeneous, two-dimensional disk is

$$\sigma_1 = -\int_{-\infty}^{t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\sigma_0 r v_r^{(1)}\right)+\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\sigma_0 v_\varphi^{(1)}\right)\right]dt' \approx -\sigma_0\int_{-\infty}^{t}\left(\frac{\partial v_r^{(1)}}{\partial r}+\frac{1}{r_0}\frac{\partial v_\varphi^{(1)}}{\partial\varphi}\right)dt', \qquad (20)$$

where $|\sigma_1/\sigma_0| \ll 1$ and we omitted the term $\sigma_0 v_r/r$, i.e., we neglected the curvature effect. This is a valid approximation if r is large (Lin & Lau 1979; Sellwood & Kahn 1991). To find a solution of Eq. (20) one has to choose an amplitude of the perturbation $\tilde{\Phi}(r)$ in the set (18)–(19). If a medium is only weakly inhomogeneous on the scale of the radial oscillation wavelength λ, i.e.,

$L \gg \lambda$ and $L \gg \rho$,

where $L = |\partial \ln\sigma_0/\partial r|^{-1}$ is the radial scale of the spatial inhomogeneity, the wave behaves approximately as a plane one (Alexandrov et al. 1984). In this case, the analysis can be greatly simplified by using the convenient WKB approximation. We seek thus the radial variation of the wave amplitude in a form:

$$\tilde{\Phi}(r) = \delta\Phi(r)\, e^{i\int^r k_r dr'}, \qquad (21)$$

where $k_r(r)$ is the radial wavenumber (Shu 1970; Griv & Peter 1996). In Eq. (21), $\delta\Phi(r)$ is a slowly varying amplitude, while the rapidly varying part of $\tilde{\Phi}(r)$ resides in the phase, i.e., $\left|\int k_r dr'\right| \gg 1$. Since the amplitude and the wave vector depends weakly on the coordinates, we can construct the solutions of dynamic problems for weakly inhomogeneous disks in the form of an expansion in the parameter λ/L; when calculating the terms of higher order one can simultaneously solve the field equations with any desired degree of accuracy (Alexandrov et al. 1984, p. 243). Further, by applying the zero-order or the so-called local approximation of the WKB method we shall assume that $\delta\Phi$ and k_r are homogeneous, $\delta\Phi = \text{const}$ and $k_r = \text{const}$. In other words, in the local WKB approximation the wave is considered plane: all terms of the order λ/L and of higher order are fully neglected (or all derivatives of $\delta\Phi(r)$ and $k_r(r)$ are neglected).

Thus, from here on we consider localized dispersion relations only. The reason for doing so is that localized solutions seem to describe the physical situation in what follows in a natural way. The meaning of localized dispersion relation has been discussed in plasma physics (Krall & Rosenbluth 1963; Alexandrov et al. 1984, p. 243; Krall & Trivelpiece 1986, p. 418).

Utilizing the above expansion of $\tilde{\Phi}$, we can approximate $\mathbf{k}\cdot\mathbf{r}$ by substituting the unperturbed orbits from Eqs. (16) and (17).[11] Such a substitution is permissible in the framework of the linear theory. Then by averaging over initial random velocities with the equilibrium Schwarzschild distribution $f_0(v_\perp^2/2, r_0)$, the integral in Eq. (20) can be approximated as:

$$\sigma_1 = \frac{\Phi_1\sigma_0}{\omega_*^2-\kappa^2}\left[k_r^2+\frac{4\Omega^2-\kappa^2+\omega_*^2}{\omega_*^2}k_\varphi^2\right]\omega_* \times \sum_{l=-\infty}^{\infty}\frac{e^{-x}I_l(x)}{\omega_*-l\kappa} + i\Phi_1\frac{4\Omega k_r k_\varphi\sigma_0}{\omega_*^2-\kappa^2}\sum_{p=-\infty}^{\infty}\frac{e^{-x}I_p(x)}{\omega_*-p\kappa}, \qquad (22)$$

where $I_l(x)$ is the Bessel function of imaginary argument of the order l. Its argument is $x = k_*^2 c_r^2/\kappa^2 \approx k_*^2\rho^2$ with the effective wavenumber k_* defined by $k_*^2 = k^2\{1+[(2\Omega/\kappa)^2-1]\sin^2\psi\}$. To obtain Eq. (22) we introduced the polar coordinates in wavenumber space $k_r = k\cos\psi$ and $k_\varphi = k\sin\psi$. The integral in Eq. (20) was estimated using the forced coordinates of stars $r_1 = \int v_r^{(1)}dt$ and $\varphi_1 = (1/r_0)\int v_\varphi^{(1)}dt$ (Eqs. [18] and [19]), the identity

$$e^{\pm i(k_* v_\perp/\kappa)\sin\phi} = \sum_{l=-\infty}^{\infty} J_l(k_* v_\perp/\kappa)e^{\pm il\phi}$$

and the formula:

$$\int_0^\infty e^{-r^2x^2}J_l(\alpha x)J_l(\beta x)x dx = \frac{1}{2r^2}\exp\left(-\frac{\alpha^2+\beta^2}{4r^2}\right)I_l\left(\frac{\alpha\beta}{2r^2}\right), \qquad (23)$$

where $J_l(k_* v_\perp/\kappa)$ is the Bessel function of the first kind of the order l. Note that analogous integrals appear in the theory of magnetic plasma oscillations when one integrates the perturbed phase-space distribution function along the unperturbed particle trajectories (Krall & Rosenbluth 1963; Alexandrov et al. 1984, p. 110; Krall & Trivelpiece 1986, p. 402).

In Eq. (22) the denominators vanish when $\omega_* - l\kappa = 0$. At these values one gets hydrodynamic-type "wave-fluid" resonances, and thereby this solution obtained in the framework of linear approximation cannot be used. The most important resonances are the corotation one, for which $l = 0$ and correspondingly $\omega_* = 0$, and the inner and outer Lindblad's resonances, for which $l = \pm 1$ and $\omega_* = \pm\kappa$. Resonances of a higher order, $l = \pm 2, \pm 3, \cdots$, are dynamically less important (Griv & Peter 1996). It is obvious that all the terms except $l = 0$ in the sum over the Bessel functions in Eq. (22) can be ignored for the most important long-wavelength oscillations, for which $x \lesssim 1$. (But, of course, in order to be appropriate for a WKB wave approximation we consider the perturbations with $|k_r|r \gg 1$; typically, in galaxies $r/\rho \sim 20$.) For example, comparing the contributions of $|l| = 1$ to that of $l = 0$, in the long-wavelength limit one obtains (see below):

$$\frac{I_1(x)}{I_0(x)}\frac{\omega_*^2}{\kappa^2} \approx \frac{x}{2}\frac{\omega_*^2}{\kappa^2}.$$

[11] Since we work within the two-dimensional disk model, the wave vector $\mathbf{k}(\mathbf{r})$ is perpendicular to the rotation axis, that is, k is given by $k = \left(k_r^2+k_\varphi^2\right)^{1/2}$.

As we shall see later, one has to consider the case of perturbations $\omega_*^2/\kappa^2 \sim x < 1$ only. Therefore the above ratio is of order $x^2 \ll 1$ and in accordance with the earlier assumption terms with $|l| \geq 1$ can be neglected.

In Eq. (22) we should consider the low-frequency perturbations $|\omega_*| < \kappa$ only. Indeed, in the opposite case of the high perturbation frequencies, $\omega_*^2 \gg \kappa^2$, the effect of the disk rotation (or of magnetic field in plasmas) is negligible and therefore not relevant to us. This is because in this case the star motion is approximately rectilinear on the time and length scales of interest which are the wave growth/damping periods and wavelength, respectively. In this rotationless case instead of Eq. (22) another expression for the perturbed surface density can be found. In plasma physics the analogous problem has been described, e.g., by Alexandrov et al. (1984, p. 110).

To summarize, starting from equations of motion and the continuity equation we obtained the perturbed surface density (Eq. [22]). Self-consistency requires that it should be equal to the solution of the Poisson equation. Such an improved solution of the Poisson equation in the two-dimensional case in which we are interested has been obtained to the second order of the Lin-Shu asymptotic approximation of moderately tightly-wound spirals ($k_r \gtrsim k_\varphi$ or $\tan^2\psi \ll 1$, respectively):

$$\sigma_1 = -\frac{|k|\Phi_1}{2\pi G}\left\{1 - \frac{i}{k_r r}\frac{d\ln}{d\ln r}\left[r^{1/2}\delta\Phi\right]\right\} \tag{24}$$

(e.g., Lin & Lau 1979 and Bertin 1980).

Equating the "in-phase" parts of Eq. (22) and Eq. (24), we get the generalized Lin-Shu local dispersion relation for low-frequency oscillations with $|\omega_*| < \kappa$ near a certain arbitrary radius r in the following form:

$$\omega_*^2 \approx \kappa^2 - \frac{2\pi G\sigma_0}{|k|}\left(k_r^2 + \frac{4\Omega^2 - \kappa^2 + \omega_*^2}{\omega_*^2}k_\varphi^2\right) \times \sum_{l=-1}^{1}\omega_*\frac{e^{-x}I_l(x)}{\omega_* - l\kappa}. \tag{25}$$

It is valid even for relatively open spirals and barlike structures throughout a disk excluding the resonance zones. Only the principal part of the disk between the inner $l = -1$ (where $\omega_* = -\kappa$) and outer $l = 1$ (where $\omega_* = \kappa$) wave-fluid Lindblad's resonances considered. Note that Morozov (1980) by using a kinetic approach numerically calculated the contributions of the $|l| > 1$ terms and found them to be smaller than $(0.05\text{–}0.07)(\omega_*^2/\kappa^2) \ll 1$ (see also Griv et al. 1999, Fig. 1 in their paper).

The basic dispersion relation above is highly nonlinear in the frequency ω_*. Following the plasma physics method (Lifshitz & Pitaevskii 1981, p. 128), let us consider various limiting cases of perturbations described by some simplified variations of Eq. (25), that have a special interest for us. For instance, we solve this equation by successive approximations. In the first approximation, one can omit all terms which depend on k_r and k_φ. Under this condition, the zeroth-order approximation solution is

$$\omega_*^2 = \kappa^2. \tag{26}$$

Such a form for the trivial solution seems fairly straightforward. Indeed, when $G = 0$, that is, when the self-gravitation of the disk is neglected, from the generalized dispersion relation (25) we have ordinary epicyclic oscillations:

$$\frac{d^2 r_1}{dr^2} + \kappa^2 r_1 = 0,$$

where $r_1 \propto \exp(-i\omega_* t)$ is a small perturbation of the radius of the initially circular orbit, $r(t) = r_0 + r_1(t)$, at the motion in the central field with the effective potential energy $P = \Phi_0 + J_z^2/2r^2$ (Griv & Peter 1996).

Using the elementary solution (26), in the next approximation the squared wavefrequency is

$$\omega_*^2 = \omega_J^2 \equiv \kappa^2 - 2\pi G\sigma_0|k|e^{-x}I_0(x) \times\left\{1 + \left[(2\Omega/\kappa)^2 - 1\right]\sin^2\psi\right\}, \tag{27}$$

where ω_J^2 is the square of the so-called Jeans frequency. This is the required simplified dispersion relation, which describes the physics and the condition of the gravitational (Jeans) modes in the two-dimensional disk. The hydrodynamic-type Jeans instability occurs when $\omega_J^2 < 0$.

Generally, there are two branches to our solution (27): the case of long waves, $x \lesssim 1$ or $\lambda \gtrsim 2\pi\rho$, in which we are especially interested, and the opposite case of short waves, $x \gg 1$. The short-wavelength instabilities (those with $x \gg 1$) are not dangerous in the problem of the galactic disk stability, since they lead to the very small-scale $\lambda^2 \ll \rho^2$ perturbations of the density only. Therefore from now on, we consider just the long-wavelength (or the hydrodynamical) limit $x^2 \ll 1$, for which the following expansions can be used

$$e^{-x}I_0(x) \simeq 1 - x + \frac{3}{4}x^2 \text{ and } e^{-x}I_1(x) \simeq \frac{x}{2}.$$

In the short-wavelength limit,

$$e^{-x}I_0(x) \approx e^{-x}I_1(x) \approx \frac{1}{\sqrt{2\pi x}}\left[1 + O\left(\frac{1}{x}\right)\right],$$

while in a more rigorous approximation $I_l(x)$ is a monotonically decreasing function of l for a fixed $x \gg 1$.

The local dispersion relation in the simple form (27) generalizes that of the Lin-Shu one (Lin et al. 1969; Shu 1970). This type of the dispersion relation for spiral waves, derived in a similar form, e.g., by Morozov (1980, 1981b) who used a kinetic approach, takes into account effects of azimuthal forces (m and $\psi \neq 0$). It goes beyond the original Lin-Shu relation in that it is now applicable to the critically important case of the nonaxisymmetric perturbations concerning spiral structures. This relation is qualitatively similar to the standard dispersion relation of Lin-Shu in that $\omega_*^2 \to \kappa^2$ both in the long-wavelength, or fluid limit $x \to 0$, and in the short-wavelength limit $x \to \infty$. Similar dispersion relation can also be derived from the Lynden-Bell & Kalnajs (1972, Eq. [A11] in their paper) dispersion relation for open spirals. Unlike Lynden-Bell & Kalnajs, Morozov, and Griv & Peter, we used here a simplified method of particle orbit theory.

In Eq. (27), $e^{-x}I_0(x)$ is the so-called reduction factor, which is approximately equal to unity in dynamically cold systems ($c_r = 0$) and is always smaller than unity in dynamically hot disks ($c_r > 0$). Lin & Shu (1966) first introduced such a reduction factor; they have already pointed out that the high-dispersion stars would not participate in the spiral pattern in full, and this effect can be described with the help of the reduction factor. Different forms of the reduction factor are given by Athanassoula (1984). The existence of solutions of the dispersion relation with $\omega_J^2 < 0$ implies the aperiodic Jeans instability. In this case of gravitational instability the wavefrequency is purely imaginary, so that the wave propagation cannot occur. The solutions with $\omega_J^2 > 0$ describe long-lived natural (harmonic) oscillations. The marginal condition between these cases is given by $\omega_J^2 = 0$. To emphasize, this instability is hydrodynamical in nature and has nothing to do with any resonant effects. In a general sense, the instability represents the ability of a gravitating disk to relax from a nonthermal (or an almost nonthermal) state by collective collisionless processes in much less time than the binary collision time.

Apart from the obvious replacement of k_r by k, which originates from the consideration of the nonaxisymmetrical modes, the relation (27) differs from the corresponding standard Lin-Shu expression by the appearance of the factor $\left\{1 + \left[(2\Omega/\kappa)^2 - 1\right]\sin^2\psi\right\}$. This factor indicates an extra clumping associated with the azimuthal forces in the differentially rotating media: spiral perturbations, in contrast with radial ones, are subject to the influence of the nonuniform character of the rotational motion. Lau & Bertin (1978) first obtained a somewhat similar expression for the extra clumping in a gas dynamical model (see also Bertin & Mark 1978, Lin & Lau 1979, Bertin 1980, and Lin & Bertin 1984).

Let us further analyze the consequences of this simple dispersion relation on the dynamical behavior of disks of stars. First, by using the condition $\omega_J^2 \geq 0$ for all possible k to second order in asymptotic theory, a generalized stability criterion can be immediately obtained. Indeed, if the nonaxisymmetric Jeans-type perturbations are to be stable, the value of the stellar radial-velocity dispersion $c_r(r)$ should be greater or at least equal to that given by Eq. (2).[12] To repeat, it is clear from the criterion (2) that stability of the nonaxisymmetric perturbations (m and $\psi \neq 0$) in a nonuniformly rotating disk ($2\Omega/\kappa > 1$) requires a larger velocity dispersion than the ordinary Toomre's critical value c_T (cf. Fridman & Polyachenko 1984, Vol. 1, p. 323). It is crucial to realize that the various dynamical properties of the perturbations with different ψ are peculiarities of the differentially rotating disks only. In a way of contrast, in the rigidly rotating disk $2\Omega/\kappa = 1$ and the critical velocity dispersion (2) is in fact equal to c_T.

[12] To obtain Eq. (2) by using the dispersion relation (27), one first finds the critical wavenumber $k_{\rm crit}$ from the relation $\partial\omega_J^2/\partial k = 0$. Then this $k_{\rm crit}$ is substituted into the dispersion relation and from the condition $\omega_J^2 \geq 0$ the critical velocity dispersion is found.

Second, according to the dispersion relation (27), the growth rate of the axisymmetric gravitational modes has a maximum at the radial wavenumber $k_J \approx \kappa/c_r$ or at the radial wavelength

$$\lambda_J \approx \frac{2\pi c_r}{\kappa} \approx 2\pi\rho. \tag{28}$$

The above equation reflects the well-known fact that the velocity dispersion shifts the threshold of gravitational stability toward a longer wavelength. At the limit of stability with respect to axisymmetric gravity perturbations the critical radial velocity dispersion $c_r \approx 3.4G\sigma_0/\kappa$ and the critical wavelength becomes approximately equal to $4\pi^2 G\sigma_0/\kappa^2$. This reproduces the usual Toomre's stability criterion to have a stable disk against axisymmetric collapse and the usual Jeans-Toomre critical radial wavelength (Toomre 1964, 1977).

On the other hand, in the case of nonaxisymmetric perturbations of a differentially rotating disk, the critical wavelength is a slightly longer:

$$\lambda_{\rm crit} \approx \left\{1 + \left[(2\Omega/\kappa)^2 - 1\right]\sin^2\psi\right\}^{1/2}\lambda_J \equiv \beta\lambda_J, \tag{29}$$

where in galaxies as a rule $\beta = 1.2$–1.5.

Third, the growth rate of the Jeans instability is

$$\Im\omega_* \approx \sqrt{2\pi G\sigma_0|k|\{1 + [(2\Omega/\kappa)^2 - 1]\sin^2\psi\}e^{-x}I_0(x)}. \tag{30}$$

Generally, $\Im\omega_* \sim \Omega$. That is, the instability growth rate is high and the instability develops rapidly on the dynamical timescale (which is the time of one galactic rotation $\sim \Omega^{-1}$). Eq. (30) indicates that open "barlike" modes are seem to be the most unstable, $\Im\omega_* \propto |\sin\psi|$. It is important to point out that the growth rate decreases as the radial velocity dispersion grows approximately as $\Im\omega_* \propto \exp(-c_r^2)$. It is also interesting that in the case of differentially rotating disks the growth rate is dependent on the mode number m; it is only in a rigidly rotating disk that the growth rate is independent of the mode number m. In addition, for the Jeans-unstable perturbations ($\omega_J^2 < 0$) the wavefrequency is purely imaginary, $\Re\omega_* = 0$ and $\Im\omega_* > 0$, and therefore the instability develops aperiodically.

Finally, in Sect. 4 of the present paper, we confirmed the generalized local stability criterion (2) for the case of the most unstable spiral perturbations – barlike ones with $\psi \to 90°$ – by local N-body computer simulations.

A.2 The effect of interparticle collisions

Thus far, we have studied the dynamics of the collisionless disk. Let us here estimate the influence of interparticle collisions on the dispersion law of Jeans perturbations using the simple method of particle orbit theory. Of course, the Boltzmann kinetic equation provides a more rigorous but much more complicated treatment of the problem of a collisional disk oscillations (Griv & Chiueh 1996; Griv & Yuan 1996; Griv et al. 1997a).

Including non-physical elastic (gravitational) interparticle collisions, the equations of motion (6)–(7) for an individual star in inertial frame with the origin at the disk center take the form:

$$\frac{d^2r}{dt^2} \ = \ r(\dot{\varphi})^2 - \frac{\partial \Phi}{\partial r} - \nu_c v_r, \tag{31}$$

$$\frac{d}{dt}(r^2\dot{\varphi}) \ = \ -\frac{\partial \Phi}{\partial \varphi} - \nu_c r v_\varphi, \tag{32}$$

where the friction term $\mathbf{F} = -\nu_c \mathbf{v}$ approximates the force produced by collisions, $\nu_c = n\langle sv\rangle$ is the effective collision frequency, n is the number density of particles, s is the effective "radius" of a particle, $\langle\cdots\rangle$ denotes the average over particles of all random velocities v in a Maxwellian distribution, and the terms with ν_c are small corrections (in the case of rare, $\nu_c \ll \Omega$, and weak collisions, $\nu_c \ll |\omega_*|$, in which we are especially interested). This is just the opposite of the procedure in ordinary gas dynamics, where collisions are the dominant effect. This approach is valid for high temperatures and low densities, when the mean potential between neighboring particles is small compared with the thermal energy. The collision model of the form (31)–(32) does not take into account the detailed mechanism of the gravitational long-range interaction such as the spatial distribution of particles, non-rectilinear orbits of particles in a rapidly rotating system, etc. (Griv et al. 1997a). It seems that this model can give qualitatively correct results in considered rarefied disks where the detailed effects of gravitational collisions may be ignored.

The linearized Eqs. (31) and (32), $\Phi(\mathbf{r}, t) = \Phi_0(r) + \Phi_1(\mathbf{r}, t)$ and $\mathbf{r}(t) = \mathbf{r_0} + \mathbf{r_1}(t)$, take the form:

$$\frac{d^2r_1}{dt^2} \approx \frac{h_0^2 - 2h_0 \int_{-\infty}^t (\partial\Phi_1/\partial\varphi)\, dt'}{(r_0+r_1)^3} - \frac{\partial\Phi_0}{\partial r} - \frac{\partial\Phi_1}{\partial r} - \nu_c v_r,$$

$$(r_0+r_1)^2(\dot{\varphi}_0 + \dot{\varphi}_1) \approx h_0 - \int_{-\infty}^t \frac{\partial\Phi_1}{\partial\varphi} dt' - \nu_c r_0^2 \int_{-\infty}^t \frac{\partial\varphi_1}{dt} dt',$$

where $h_0 = r_0^2\dot{\varphi}_0$ is the area constant and $r_0\dot{\varphi}_0^2 = (\partial\Phi_0/\partial r)_0$. Equations above describe the small departure $\mathbf{r_1}(t)$ of the actual radius $\mathbf{r}(t)$ from $\mathbf{r_0}$, which is chosen so that the constant of areas for the circular orbit h_0 is equal to the angular momentum integral $J_z = r^2\dot{\varphi}$. From these equations we get

$$\frac{d^2r_1}{dt^2} + \kappa^2 r_1 = -\frac{2\Omega}{r_0}\int_{-\infty}^t \frac{\partial\Phi_1}{\partial\varphi} dt' - \frac{\partial\Phi_1}{\partial r} - \nu_c \frac{dr_1}{dt}, \tag{33}$$

$$(r_0+r_1)^2(\dot{\varphi}_0 + \dot{\varphi}_1) - \Omega r_0^2 = -\int_{-\infty}^t \frac{\partial\Phi_1}{\partial\varphi} dt' - \nu_c r_0^2 \int_{-\infty}^t \frac{d\varphi_1}{dt} dt', \tag{34}$$

where $|\mathbf{r_1}/\mathbf{r_0}| \ll 1$ and $|\Phi_1/\Phi_0| \ll 1$ for all $\mathbf{r}$ and t.

The homogeneous differential Eqs. (33) and (34) yield the ordinary Lindblad's elliptic-epicyclic orbits:

$$r \ = \ r_0 - \frac{v_\perp}{\kappa}[\sin(\phi_0 - \kappa t) - \sin\phi_0],$$

$$\varphi \ = \ \Omega t + \frac{2\Omega}{\kappa}\frac{v_\perp}{r_0\kappa}[\cos(\phi_0 - \kappa t) - \cos\phi_0],$$

where $v_\perp/r_0\kappa \sim \rho/r_0 \ll 1$ (Spitzer & Schwarzschild 1953).

The particular solutions yield the expressions for perturbed velocities (cf. Eqs. [18] and [19])

$$v_r^{(1)} \ = \ \frac{-i\omega_*}{\omega_*^2 - \kappa^2 + i\omega_*\nu_c}\frac{\partial\tilde{\Phi}}{\partial r} e^{im\varphi - i\omega_* t}, \tag{35}$$

$$v_\varphi^{(1)} \ \approx \ k_\varphi \tilde{\Phi} \frac{4\Omega^2 - \kappa^2 + \omega_*^2}{\omega_*(\omega_*^2 - \kappa^2 + i\omega_*\nu_c)} e^{im\varphi - i\omega_* t}, \tag{36}$$

where $|\omega_*| \sim \Omega \gg \nu_c$ and only the "in-phase" terms are included. As we can see from the equations above, in comparison with the collisionless disk in the collisional system one needs to replace the wavefrequency ω_* by $\omega_* + i\nu_c$; thus if $\nu_c/|\omega_*|$ is small enough we can ignore these collisions.

Paralleling the analysis leading to Eq. (27) and making use of Eqs. (35) and (36), it is straightforward to show that the simplified dispersion relation can now be expressed as

$$\omega_*^2 + i\omega_*\nu_c - \omega_J^2 = 0, \tag{37}$$

where as usual ω_J^2 is the squared Jeans frequency. The solution of Eq. (37) is

$$\omega_* \simeq \pm p|\omega_J| - i\frac{\nu_c}{2}, \tag{38}$$

where $p = i$ for Jeans-unstable perturbations ($\omega_J^2 < 0$) and $p = 1$ for Jeans-stable ones ($\omega_J^2 > 0$), $|\omega_J| \sim \Omega$, and $\nu_c \ll |\omega_J|$.

Eq. (38) describes the weak damping of Jeans-stable perturbations, $\Im\omega_* < 0$. Such a stabilizing influence is quite obvious, because in general the effect of collisions is to disrupt the organized wave motion (Alexandrov et al. 1984). Accordingly, as a result of collisions, a Jeans-stable wave tends to be damped on a timescale of the order of the mean time between collisions $\sim 1/\nu_c$. Clearly, however, these rare, $\nu_c \ll \Omega$, and weak, $\nu_c \ll |\omega_*|$, gravitational collisions between particles do not affect the local stability criterions (2)–(3).

It follows from Eq. (38) that the collisional effects do not depend on the wavenumber k. The latter contradicts our recent results obtained with the exact Landau integral of collisions (Griv et al. 1997a). Therefore, gravitational collisions are poorly represented by an approximate method presented here. The results obtained in this Appendix indicate only a tendency of Jeans-stable perturbations to be damped in a colisional system, and the damping rate given by Eq. (38) is correct only to the order of magnitude.

Thus, it is found that rare and weak collisions between particles lead to the weak stabilization of Jeans-stable modes in a stellar disk. The effect is small: the time necessary for the wave amplitude to fall to $1/e$ of its initial value τ_f is about the collision time, ν_c^{-1}. We have assumed $|\omega_J| \sim \Omega$ and $k^2\rho^2 \lesssim 1$. This is much longer than the characteristic time of a single revolution of a disk $\sim \Omega^{-1} \ll \nu_c^{-1}$.

According to observations, in the disk of the Galaxy the frequency of gravitational collisions between stars and giant molecular clouds $\nu_c \sim 10^{-9}\,\mathrm{yr}^{-1}$ (Grivnev & Fridman 1990). Therefore, even though the time τ_f is longer than the characteristic time of a single revolution of the Galaxy in the solar vicinity, it is quite sufficient to damp the standard Lin-Shu quasi-stationary density waves on the Hubble time $T \sim 10\nu_c^{-1} \sim 10^{10}$ yr. By this way, the effects of even rare (and weak) encounters may become essential.

A.3 Relaxation time in strictly two-dimensional simulations

Consider a system of mutual-gravitating particles. The local distribution functions $f(\mathbf{r}, \mathbf{v}, t)$ must satisfy the Boltzmann kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \frac{\partial \Phi}{\partial \mathbf{r}}\cdot\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial v}\right)_{\mathrm{coll}}, \qquad (39)$$

where $\Phi(\mathbf{r}, t)$ is the total gravitational potential determined self-consistently from the Poisson equation, $(\partial f/\partial v)_{\mathrm{coll}} \propto \nu_c(f_0 - f)$ is the so-called collisional integral which defines the change of f arising from ordinary interparticle collisions, ν_c is the collision frequency, and f_0 is the quasi-steady state distribution function.

In plasma physics, Lifshitz & Pitaevskii (1981, p. 115) have discussed phenomena in which interparticle collisions are unimportant, and such a plasma is said to be collisionless (and in the lowest-order approximation of the theory one can neglect the collision integral in the kinetic equation). It was shown that a necessary condition is that $\nu_c \ll |\omega_*|$: then the collision operator in the kinetic equation (39) is small in comparison with $\partial f/\partial t$. In Appendix A.1, we have shown that generally speaking the frequency of collective Jeans-type oscillations in a stellar disk $|\omega_*| \sim \Omega$. Therefore, in the gravitation case in the lowest-order approximation of the theory we can neglect the effects of collisions between particles on a timescale of many rotations if $\nu_c \ll \Omega$. Lifshitz & Pitaevskii (1981) have pointed out that collisions may be neglected also if the particle mean free path is large compared with the wavelength of collective oscillations. Then the collision integral in Eq. (39) is small in comparison with the term $\mathbf{v} \cdot (\partial f/\partial \mathbf{r})$.

In this Appendix we test numerically if the models used in our N-body simulations are being correctly modelled as collisionless Boltzmann (Vlasov) systems. The direct method of checking if the system is being modelled as a collisionless system is to repeat a calculation using a mass spectrum (Rybicki 1971). It is obvious that as a result of gravitational collisions there is a tendency towards energy equipartition between the various masses. Hohl (1973) has determined the experimental relaxation time and compared it with a theoretical prediction for the collisional relaxation time of a two-dimensional disk by using the method of global simulations; here we do such a comparison by using the method of local simulations.

Let us consider the strictly two-dimensional computer model consisting of 2% stars of mass $m_3 = 10m_s$, 18% stars of mass $m_2 = 2m_s$, and 80% stars of mass $m_1 = 0.55m_s$. The total number of stars, which are distributed in the rectangular box with $L_x \times L_y = 4\lambda_J \times 6\lambda_J$, is small, $N = 2400$, in comparison with the number of stars in simulations presented in Sect. 3. Initially, the different mass groups of stars are distributed with the same velocity dispersion (with different temperatures).

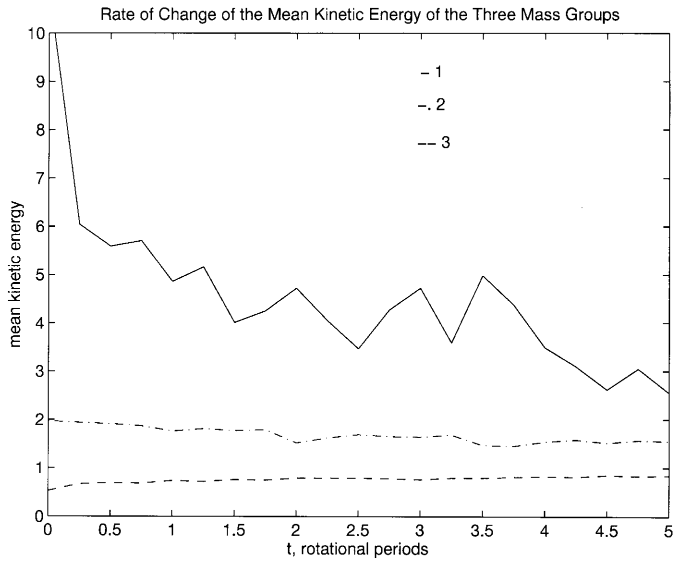

Fig. A1. Rate of change of the mean kinetic energy for stars of the three mass groups of the rigidly rotating model with $N = 2400$, $L_x \times L_y = 4\lambda_J \times 6\lambda_J$, and $c_r = 1.5c_T$; 1 – kinetic energy of stars with the mass of a star $10m_s$, 2 – with the mass of a star $2m_s$, and 3 – with the mass of a star $0.55m_s$. The two groups of heavy stars lose kinetic energy while the group of lightest stars gains an approximately corresponding amount of kinetic energy. The mean slope of the curves will result in energy equipartition after about 20 rotation periods. This result suggests that interparticle collisions do not play a significant role for instabilities studied in the paper.

In Sect. 3.1 of the present paper it has been found that the rigidly rotating disk becomes almost stable gravitationally for $c_r \gtrsim 1.5c_T$. In such a Jeans-stable system collective effects associated with the classical gravitational instability will not affect the random velocity dispersion of particles (Griv et al. 1994; Griv & Peter 1996): the change of velocity dispersion can be explained only by usual two-body encounters.[13] For this reason the initial condition was chosen to be a quasi-stable uniformly rotating disk with $c_r = 1.5c_T$. Following Hohl (1973), let us define the relaxation time τ_E as the time required for the mean change of the kinetic energy per unit mass of the test star to equal the initial kinetic energy.

In Fig. A1 we show the change of the ratio of the mean particle (kinetic) energy, $< m_1 V_1^2 >$, $< m_2 V_2^2 >$, and $< m_3 V_3^2 >$ (in units of the total kinetic energy of the system), for the different mass groups, where V_i is the total velocity of a given mass group. As is expected, the two groups of heavy stars lose energy while the group of lightest stars gains an approximately corresponding amount of kinetic energy. Also as is expected, one can see the decrease in the change of the kinetic energy with time. This is because the collisional frequency $\nu_c \approx 1/\tau_E$ is inversely proportional to the velocity dispersion (Eq. [12]),

[13] In a plasma, it has already been known that the rate of relaxation toward equilibrium can be greatly enhanced by collective processes (Kulsrud 1972; Alexandrov et al. 1984, p. 408; Krall & Trivelpiece 1986, p. 512).

and thus the encounters only weakly affect the stars with high random velocities.

As one can see, the mean slope of the curves shown in Fig. A1 will result in energy equipartition after about 20 rotation periods. It is crucial to realize that these relaxation times even for this relatively small number of model stars are much longer that the time of a single disk revolution. We conclude that the two-dimensional computer models used in the present study may indeed be considered as collisionless ones to a good approximation at least during the first 8–10 rotations which are of especial interest in spiral-galaxy simulation. Therefore, we argue that the collective effects studied in this paper were apparent before the collisional timescale was reached.

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