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On the removal of the sign ambiguity in the photospheric transverse magnetic field

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Abstract. We present a method for removing the ambiguity in the transverse component of the photospheric magnetic field measured by vector magnetographs. The method is based upon the use of the divergence-free condition satisfied by the magnetic field in the sense of least squares. The method requires the measurement of the longitudinal component of the magnetic field at two depth levels in the photosphere. The method is is shown to be efficient when compared to other existing methods on some particular analytical force-free magnetic fields.

Key words: Magnetohydrodynamics (MHD) – Sun: corona – Sun: magnetic fields

1. Introduction

The removal of the 180^o ambiguity in the measurement of the photospheric transverse magnetic field is important for determining the magnetic field that is at the origin of most of the phenomena that occur in the solar atmosphere (Amari & Demoulin 1992, McClymont et al. 1997 and references therein). Several methods have been proposed to solve this problem, namely, the reference-field method (Gary et al. 1987, Aly 1989, Cuperman et al. 1992) the B_z-transverse-gradient method (Aly 1989, Gary et al. 1987), the decreasing energy method (Cuperman et al. 1993) and the divergence-free method (Wu & Ai 1990, Li et al. 1993). Unfortunately, the last method, based on the equation

$$\mathrm{div}\mathbf{B} = 0 \tag{1}$$

does not use all the information contained in this equation, but only a necessary condition, leaving a certain degree of arbitrariness. In this Paper, we propose a finite element method, that minimizes in the sense of least-squares, the divergence of $\mathbf{B}$, if one assumes that the longitudinal component of the magnetic field is measured at two close heights. A heuristic argument that justifies this procedure can be obtained by noticing that flipping the sign of the transverse field at any point would create a discontinuity and therefore increase the rms values of the divergence.

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The Paper is organized as follows. In Sect. 2 we present the method. In Sect. 3, we present some comparisons with few other methods on some particular known examples. Sect. 4 gathers concluding remarks.

2. The method

On the boundary $\{z = 0\}$ of the domain $\{z > 0\}$ we consider a two-dimensional rectangular grid $(x_i, y_j),\ 1 \leq i \leq n,\ 1 \leq j \leq m$, not necessarily uniform. A measured magnetic field $(B_x = \epsilon B_{x,1}, B_y = \epsilon B_{y,1}, B_z)$ is then supposed to be given on this grid. The ambiguity in the sign of the transverse magnetic component is contained in the function $\epsilon(x, y)$ that can take the two values $(+1 \mathrm{or} -1)$ depending on the position (x, y). We also assume that B_z is given at two levels separated by Δz. One can then deduce the derivative $\rho = -\partial_z B_z$ at all points of $\{z = 0\}$ (where $\partial_z f$ stands for the partial derivative when it exists of the function f with respect z:

$$\begin{aligned} \rho(x,y) &= -\partial_z B_z(x,y,0) \\ &\approx -(B_z(x,y,\Delta z) - B_z(x,y,0))/\Delta z, \end{aligned} \tag{2}$$

Let div_h stands for the horizontal divergence operator with respect to the coordinates x and y; $\mathrm{div}_h\,\mathbf{B} = \partial_x B_x + \partial_y B_y$.. Eq. (1) can then be written in the two-dimensional form

$$\mathrm{div}_h\,\mathbf{B} = \rho\ . \tag{3}$$

Let us use a finite element approximation of this equation to find ϵ. One first considers the reference rectangle (i, j) and a local coordinate system.

Let M_1, M_2, M_3 and M_4 be the vertices of this rectangle and $\lambda_m(x, y),\ 1 \leq m \leq 4$, defined by $\lambda_m(M_k) = \delta_{m,k}$ (where $\delta_{m,k} = 1$ if $m = k$, and $\delta_{m,k} = 0$, otherwise). One gets

$$\lambda_1(x,y) = \frac{(\Delta x_i - x)(\Delta y_j - y)}{\Delta x_i \Delta y_j}, \tag{4}$$

$$\lambda_2(x,y) = \frac{x(\Delta y_j - y)}{\Delta x_i \Delta y_j}, \tag{5}$$

$$\lambda_3(x,y) = \frac{xy}{\Delta x_i \Delta y_j}, \tag{6}$$

$$\lambda_4(x,y) = \frac{(\Delta x_i - x)y}{\Delta x_i \Delta y_j}. \tag{7}$$

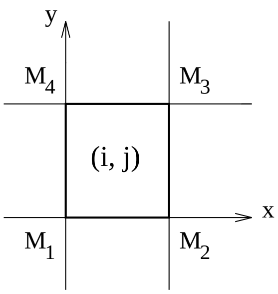

Fig. 1. The local coordinate system on the rectangle(i, j).

One can then interpolate ρ, B_x and B_y on this rectangle and writes this quantities as functions of the λ_m and the integers $\epsilon_k (= 1 \text{or} -1$for the arbitrariness in the sign of $B_t = (B_x, B_y)$ at M_k):

$$\rho = \sum_{k=1}^{4} \rho_k \lambda_k, \;\; B_x = \sum_{k=1}^{4} \epsilon_k a_k \lambda_k, \;\;\; B_y = \sum_{k=1}^{4} \epsilon_k b_k \lambda_k, \tag{8}$$

where ρ_k, a_k and b_k are the values of ρ, $B_{x,1}$ and $B_{y,1}$ at the vertices $(M_k)_{1 \le k \le 4}$. In the same way one gets:

$$\text{div}_h \mathbf{B} = \sum_{k=1}^{4} \epsilon_k (a_k \frac{\partial \lambda_k}{\partial x} + b_k \frac{\partial \lambda_k}{\partial y}) \;, \tag{9}$$

or on the basis $(\lambda_k)_{1 \le k \le 4}$ as

$$\text{div}_h \mathbf{B} = \sum_{k=1}^{4} \varphi_k \lambda_k, \tag{10}$$

where $\varphi_k = \text{div}_h \mathbf{B}(M_k)$. Using Eq. (4)-(7) and Eq. (9) one gets

$$\varphi_1 \;=\; -(\frac{a_1}{\Delta x_i} + \frac{b_1}{\Delta y_j})\epsilon_1 + \frac{a_2}{\Delta x_i}\epsilon_2 + \frac{b_4}{\Delta y_j}\epsilon_4, \tag{11}$$

$$\varphi_2 \;=\; -\frac{a_1}{\Delta x_i}\epsilon_1 + (\frac{a_2}{\Delta x_i} - \frac{b_2}{\Delta y_j})\epsilon_2 + \frac{b_3}{\Delta y_j}\epsilon_3, \tag{12}$$

$$\varphi_3 \;=\; -\frac{b_2}{\Delta y_j}\epsilon_2 + (\frac{a_3}{\Delta x_i} + \frac{b_3}{\Delta y_j})\epsilon_3 - \frac{a_4}{\Delta x_i}\epsilon_4, \tag{13}$$

$$\varphi_4 \;=\; -\frac{b_1}{\Delta y_j}\epsilon_1 + \frac{a_3}{\Delta x_i}\epsilon_3 - (\frac{a_4}{\Delta x_i} - \frac{b_4}{\Delta y_j})\epsilon_4. \tag{14}$$

The method then consists in minimizing

$$\|\text{div}_h \mathbf{B} - \rho\| \;=\; \int_0^{\Delta x_i}\int_0^{\Delta y_j} (\text{div}_h \mathbf{B} - \rho)^2 dxdy \tag{15}$$

with respect to the ϵ_k. This quantity can be expanded as:

$$J \;=\; \sum_{k,m=1}^{4} \omega_k \omega_m \beta_{k,m}, \tag{16}$$

with $\omega_k = \varphi_k - \rho_k$ and

$$\beta_{k,m} = \int_0^{\Delta x_i} \int_0^{\Delta y_j} \lambda_k \lambda_m dxdy \tag{17}$$

for $(1 \le k, m \le 4)$. Clearly $\beta_{k,m} = \beta_{m,k}$. Using the definition of ρ_k,λ_k, φ_k given above, one can deduce:

$$\beta_{k,k} \;=\; \frac{\Delta x_i \Delta y_j}{9}, \;\; 1 \le k \le 4,$$

$$\beta_{1,2} \;=\; \beta_{2,3} = \beta_{3,4} = \beta_{1,4} = \frac{\Delta x_i \Delta y_j}{18},$$

$$\beta_{1,3} \;=\; \beta_{2,4} = \frac{\Delta x_i \Delta y_j}{36},$$

The problem is thus equivalent to find ϵ_1, ϵ_2, ϵ_3, $\epsilon_4 \in \{1, -1\}$ which minimize

$$J(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \;=\; \sum_{k=1}^{4} \omega_k^2 + \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_4 + \omega_1\omega_4 + \frac{1}{2}(\omega_1\omega_3 + \omega_2\omega_4) \tag{18}$$

This minimization problem is practically straightforward since there is only $2 \times 2 \times 2 \times 2 = 16$ possible values for $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. Furthermore, by sweeping the grid, rectangle by rectangle, the number of 4-uplets to check on a each rectangle can be reduced, since the solutions on the adjacent neighbors can be used.

3. Application to a class of analytical force-free magnetic fields

The method is now tested on the set of force-free configurations ($\nabla \times \mathbf{B} = \alpha(r)\mathbf{B}$) generated by a model developed by Low (1982) and given by:

$$B_x \;=\; -\frac{B_0}{r} \cos\varphi(r), \tag{19}$$

$$B_y \;=\; \frac{B_0 x_1 y_1}{r\rho^2} \cos\varphi(r) - \frac{B_0 z_1}{\rho^2} \sin\varphi(r), \tag{20}$$

$$B_z \;=\; \frac{B_0 x_1 z_1}{r\rho^2} \cos\varphi(r) + \frac{B_0 y_1}{\rho^2} \sin\varphi(r), \tag{21}$$

where $x_1 = \dfrac{x}{a}$, $y_1 = \dfrac{y}{b}$, $z_1 = (1 + \dfrac{z}{a})$ and $r = (x_1^2 + y_1^2 + z_1^2)^{\frac{1}{2}}$. B_0 stands for the magnitude of the field at the origin and $\varphi(r)$ is the usual free generating function. $\alpha(r)$ is then given by $\alpha(r) = -\varphi'(r)$. We make the same choice as in Cuperman et al. (1992), Cuperman et al. (1993), Li et al. (1993) for the functions $\varphi_1(r), \varphi_2(r)$ and $\varphi_3(r)$; the generating function being defined as in in Low (1982):

$$\varphi_1(r) \;=\; \pi\varphi_0 \ln(r), \tag{22}$$

$$\varphi_2(r) \;=\; \begin{cases} \pi\varphi_0(r-1) & \text{if} \quad r \le 3 \\ \pi\varphi_0 & \text{else} \end{cases} \;, \tag{23}$$

$$\varphi_3(r) \;=\; \pi\varphi_0(\cos(2r) + \sin(2r)), \tag{24}$$

where φ_0 is a constant which characterize the complexity of the configuration.

We compare the results with those obtained by the three other following methods:

– **i)** The magnetic shear-based removal method of Cuperman et al. (1992) (denoted by R2 in the table below) which uses the criteria

$$\mathbf{B}_T.\mathbf{B}_{p,T} \ge 0, \tag{25}$$

Table 1. Comparison of several methods for removing the ambiguity on an exact analytical example (Low 1982)

$\varphi(r) =$		$\varphi_1(r)$			$\varphi_2(r)$			$\varphi_3(r)$		
$\varphi_0 =$		0.5	1	2	0.5	1	2	0.5	1	2
$N =$ 64	R1	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
	R2	97%	80%	65%	85%	63%	60%	76%	69%	63%
	R3	99%	98%	95%	98%	95%	90%	92%	85%	66%
	R4	99%	96%	93%	97%	93%	87%	90%	84%	65%
$N =$ 128	R1	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**	**100%**
	R3	99%	99%	97%	99%	97%	94%	95%	91%	80%
	R4	99%	98%	95%	98%	96%	93%	94%	91%	78%

where $\mathbf{B}_p$ is the unique potential field associated to the same normal component.

– **ii)** The so-called decreasing energy method of Cuperman et al. (1993) referred to hereafter as R3. In this method one uses the constraint:

$$\frac{\partial B^2}{\partial z} \leq 0 \tag{26}$$

– **iii)** The divergence free method of Li et al. (1993) whose criteria is:

$$\frac{\partial B_z}{\partial z}\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) \leq 0 \tag{27}$$

and referred to hereafter as R4

R1 will stands for the least-square method presented in this Paper.

The results presented in the Table 1 are obtained with $\Delta z = 0.05a$ on two grids, 64×64 grid ($N = 64$) and 128×128 grid ($N = 128$) for several values of φ_0.

The *percentage of success* -defined as the ratio of the number of points for which the removal of ambiguity succeeds, to the total number of grid points.

At this level it is worth noticing that the method R1 remains efficient as long as $\mathbf{B}$ varies on a length scale that is greater than Δx and Δy. To show this property let us take the following analytic class of constant-α force-free fields:

$$B_x = \frac{1}{2\omega}(\gamma \cos(\omega x)\sin(\omega y) - \alpha \sin(\omega x)\cos(\omega y))e^{-\gamma z}, \tag{28}$$

$$B_y = \frac{1}{2\omega}(\alpha \cos(\omega x)\sin(\omega y) + \gamma \sin(\omega x)\cos(\omega y))e^{-\gamma z}, \tag{29}$$

$$B_z = -\sin(\omega x)\sin(\omega y)e^{-\gamma z}, \tag{30}$$

where ω and α are constant parameters and $\gamma = (2\omega^2 - \alpha^2)^{\frac{1}{2}}$. $\mathbf{B}$ is periodic with respect to x and y (its period being equal to $\frac{2\pi}{\omega}$). We set $\alpha = \omega$, $N = 75$ (hence $\Delta x = \Delta y = 0.08a$) and $\Delta z = \Delta x$. The variation of the percentage of success in the removal of the 180^o ambiguity by the above method, as a function of the ratio $\frac{\lambda}{\Delta x}$ when λ varies, is shown in Fig. 2. The method R1 starts failing when $\frac{\lambda}{\Delta x}$ is small enough. In fact, in this case the finite element approximation is not accurate since the length scale of the magnetic field variations is of the same order as the step size δx. Therefore when the magnetograph resolution is sufficiently smaller than the length scale for magnetic field variations, the method is efficient.

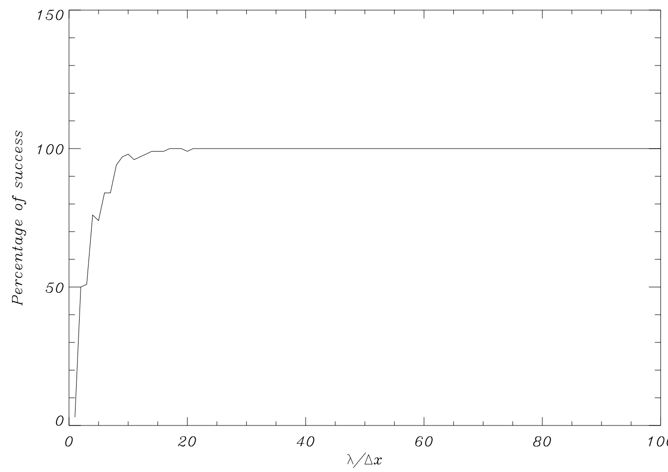

Fig. 2. The variation of the percentage of success as a function of $\frac{\lambda}{\Delta x}$.

4. Conclusions

In this paper, we have presented a numerical method for removing the ambiguity that remains in the transverse component of the photospheric magnetic field. Let us here summarize the main points we have discussed and our main results:

(a) The method is a finite-element method on a square mesh, and consists in imposing, in the sense of least-squares, the divergence-free constraint for the magnetic field.

(b) The method is shown to be efficient when applied to the particular class of Low (1982) force-free magnetic fields and compared to three other methods, (among which the other divergence-free method of Li et al. (1993). The main reason for this difference of efficiency with the method of Li et al. (1993) being that this method does not use all the information contained in the divergence-free constraint but only one inequality resulting from it.

(c) The range of efficiency of the method is fixed by the magnetograph resolution, which needs to be higher than the length scale on which varies the magnetic field. Note that this constraint is quite reasonable and should be also valid for any other method.

These results are consistent with the idea that that flipping the sign of the transverse field at any point would create a discontinuity and therefore increase the rms values of the divergence.

This method cannot yet be applied to sets of data provided by current vector magnetographs. This is because the method presented in this Paper (as well as those of Li et al. 1993) rests on the assumption that the value of the normal component of the magnetic field is known at two levels in the photosphere. Although these measurements are not yet available, current progress seem to show that they could be soon optimistically available, in particular with the telescope THEMIS (Semel 1998, private communication).

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