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Investigation on parameters of inner edge orbit of black-hole accretion disk

Ding-Xiong Wang

Department of Physics, Huazhong University of Science and Technology, Wuhan, 430074, P.R. China

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Abstract. The characteristics of evolution of the parameters of the inner edge of a black-hole accretion disk are investigated for pro- and retrograde accretion. Some useful relations among the parameters of the inner edge and those of the central black hole (CBH) are obtained. In addition, we discuss the evolution characteristics of the dimensionless angular momentum a_* of a CBH surrounded by an accretion disk with inner edge radius r_{in} ($r_{mb} < r_{in} < r_{ms}$) by considering the Blandford-Znajek process.

Key words: accretion, accretion disks – black hole physics

1. Introduction

It is well known that the black-hole accretion disk is an effective model for explaining the high energy radiation of x-ray binaries, quasars and active galactic nuclei (Rees 1984; Frank et al. 1992). In this model, the central black hole (CBH) is closely related to the surrounding disk in two respects: on the one hand, accreting matter falls into CBH after leaving the inner edge of disk, resulting in the evolution of the concerning parameters of CBH, such as mass M and angular momentum J; on the other hand, the evolution of CBH will affect the inner edge radius and the corresponding orbit parameters, such as specific energy and specific angular momentum.

In this paper, the evolution characteristics of the parameters of the inner edge orbit of black-hole accretion disk are investigated for pro- and retrograde accretion. In Sect. 2, we discuss the rate of change of r_{ms} and that of r_{mb}, which are the radius of the last stable circular orbit and the innermost bound circular orbit respectively. It is shown that both r_{ms} and r_{mb} decrease monotonously in the evolving process, being inversely proportional to the mass of the CBH. In Sect. 3, we discuss the evolution of specific energy and specific angular momentum corresponding to r_{mb} and r_{ms}, respectively. Some useful relations among the parameters of the inner edge and those of CBH are obtained. In Sect. 4, the evolution characteristics of a_* of a CBH surrounded by an accretion disk with inner edge radius r_{in} are investigated. The stable value $\left(a_*^{stable}\right)_{in}$ is proved to lie between $\left(a_*^{stable}\right)_{ms}$ and $\left(a_*^{stable}\right)_{mb}$, if the Blandford-Znajek (BZ) process (Blandford & Znajek 1977) is taken into account. In addition, we discuss the recent important revision on the BZ power (Ghosh & Abramowicz 1997, henceforth GA) and the corresponding influence on our previous results (Wang et al. 1998 henceforth W98a; Wang 1998 henceforth W98b).

Send offprint requests to: D.-X. Wang (dxwang@mail.hust.edu.cn)

2. Evolution of inner edge radius of accretion disk

It is well known that the rates of change of M and J of the CBH of an accretion disk can be expressed as:

$$(dM/dt)_{in} = E_{in} dM_0/dt \tag{1}$$

$$(dJ/dt)_{in} = L_{in} dM_0/dt, \tag{2}$$

where, in Eqs. (1) and (2), dM_0/dt is the accretion rate of rest mass, E_{in} and L_{in} are the specific energy and angular momentum corresponding to the inner edge radius r_{in}, respectively. E_{in} and L_{in} are expressed as (Page & Thorne 1974, Novikov & Thorne 1973):

$$E_{in} = \left(1 - 2\chi^{-2} + a_*\chi^{-3}\right) \Big/ \left(1 - 3\chi^{-2} + 2a_*\chi^{-3}\right)^{1/2} \tag{3}$$

$$L_{in} = M\chi\left(1 - 2a_*\chi^{-3} + a_*^2\chi^{-4}\right) \big/ \left(1 - 3\chi^{-2} + 2a_*\chi^{-3}\right)^{1/2} \tag{4}$$

where in Eqs. (3) and (4), $a_* = J/M^2$ is the dimensionless angular momentum of the CBH, $\chi = (r/M)^{1/2}$ is the dimensionless radial coordinate. The inner edge radius r_{ms} and a_* are related by the following equation:

$$\chi_{ms}^4 - 6\chi_{ms}^2 + 8a_*\chi_{ms} - 3a_*^2 = 0 \tag{5}$$

where χ_{ms} is defined as $\chi_{ms} \equiv (r_{ms}/M)^{1/2}$. r_{ms} can be expressed as an explicit function of a_*:

$$r_{ms} = M\left\{3 + A_2 \pm \left[(3 - A_1)(3 + A_1 + 2A_2)\right]^{1/2}\right\} \tag{6}$$

where $A_1 = 1 + \left(1 - a_*^2\right)^{1/3}\left[(1 + a_*)^{1/3} + (1 - a_*)^{1/3}\right]$, and $A_2 = \left(3a_*^2 + A_1^2\right)^{1/2}$. "$-$" and "$+$" of "$\pm$" in Eq. (6) are applicable to pro- and retrograde accretion, respectively. The inner edge radius, r_{in}, of a thick disk is located between r_{mb} and r_{ms} (Abramowicz et al. 1978, Abramowicz & Lasota 1980):

$$r_{mb} \le r_{in} < r_{ms} \tag{7}$$

r_{mb} is determined by M and a_* as follows:

$$r_{mb} = M\left(1+\sqrt{1-a_*}\right)^2 \tag{8}$$

The specific energy and angular momentum corresponding to r_{ms} and r_{mb} are expressed as follows (W98a, Abramowicz & Lasota 1980):

$$E_{ms} = (4\chi_{ms} - 3a_*)\Big/(\sqrt{3}\chi_{ms}^2) \tag{9}$$

$$L_{ms} = 2M\,(3\chi_{ms} - 2a_*)\Big/(\sqrt{3}\chi_{ms}) \tag{10}$$

$$E_{mb} = 1 \tag{11}$$

$$L_{mb} = 2M\left(1+\sqrt{1-a_*}\right) \tag{12}$$

The above equations are applicable to both prograde ($a_* > 0$) and retrograde ($a_* < 0$) accretion. r_{ms} is a function of M and a_*, and its rate of change can be expressed as

$$\begin{aligned} dr_{ms}/dt \;=\; & (\partial r_{ms}/\partial M)(dM/dt)_{ms} \\ & +(\partial r_{ms}/\partial a_*)(da_*/dt)_{ms} \end{aligned} \tag{13}$$

Hereafter, the subscript "ms" indicates the quantities involving thin disks with inner edge radius r_{ms}. Incorporating Eqs. (5), (9), (13), and the expressions for $(dM/dt)_{ms}$ and $(da_*/dt)_{ms}$(W98a), we have the following relations involving r_{ms}:

$$dr_{ms}/dt = -\frac{(4\chi_{ms} - 3a_*)}{\sqrt{3}}dM_0/dt \tag{14}$$

$$Mr_{ms} = 6\,(M)_0^2 = const \tag{15}$$

Hereafter, $(M)_0$ represents the mass of the CBH corresponding to $a_* = 0$. Similarly, the rate of change of r_{mb} can be expressed as

$$\begin{aligned} dr_{mb}/dt \;=\; & (\partial r_{mb}/\partial M)(dM/dt)_{mb} \\ & +(\partial r_{mb}/\partial a_*)(\partial a_*/dt)_{mb} \end{aligned} \tag{16}$$

Hereafter, the subscript "mb" indicates the quantities involving thick disks with the inner edge radius r_{mb}. Incorporating Eqs. (8), (11), (16), and the expressions for $(dM/dt)_{mb}$ and $(da_*/dt)_{mb}$ (W98b), we have the following relations involving r_{mb}:

$$dr_{mb}/dt = -\left(1+\sqrt{1-a_*}\right)^2 dM_0/dt \tag{17}$$

$$Mr_{mb} = 4\,(M)_0^2 = const \tag{18}$$

From Eqs. (14), (15), (17) and (18), we conclude that both r_{ms} and r_{mb} decrease monotonously as time, being inversely proportional to the mass of CBH.

3. Evolution of specific energy and specific angular momentum corresponding to inner edge orbit

From Eq. (11), the specific energy E_{mb} is a constant. The rate of change of L_{mb} can be expressed as

$$\begin{aligned} dL_{mb}/dt \;=\; & (\partial L_{mb}/\partial M)(dM/dt)_{mb} \\ & +(\partial L_{mb}/\partial a_*)(da_*/dt)_{mb} \end{aligned} \tag{19}$$

Substituting Eq. (12) and the expressions for $(dM/dt)_{mb}$ and $(da_*/dt)_{mb}$ into Eq. (19), we have

$$dL_{mb}/dt = 0 \tag{20}$$

which means L_{mb} is also a constant in evolving process. Taking $a_* = 0$, we have

$$L_{mb} = 4\,(M)_0 = const \tag{21}$$

The rate of change of specific energy E_{ms} corresponding to r_{ms} can be expressed as

$$\frac{dE_{ms}}{dt} = \left(\frac{\partial E_{ms}}{\partial \chi_{ms}}\frac{d\chi_{ms}}{da_*} + \frac{\partial E_{ms}}{\partial a_*}\right)(da_*/dt)_{ms} \tag{22}$$

Combining Eqs. (5), (9) and the expression for $(da_*/dt)_{ms}$ with Eq. (22), we have the following relations involving E_{ms}:

$$dE_{ms}/dt = -\frac{2}{3r_{ms}}dM_0/dt \tag{23}$$

and

$$\frac{M^2}{9\,(M)_0^2} + E_{ms}^2 = 1 \tag{24}$$

Similarly, the rate of change of the specific angular momentum L_{ms} can be expressed as:

$$dL_{ms}/dt = -\frac{2a_*}{3\chi_{ms}^2}dM_0/dt = -\frac{2a_*M}{3r_{ms}}dM_0/dt \tag{25}$$

Eq. (25) shows that $dL_{ms}/dt > 0$ in the case $a_* < 0$, while $dL_{ms}/dt < 0$ in the case $a_* > 0$. So there is a maximum of L_{ms} corresponding to $a_* = 0$, and we have

$$(L_{ms})_{\max} = 2\sqrt{3}(M)_0 \tag{26}$$

Comparing Eq. (21) with Eq. (26), we have

$$L_{ms} \le (L_{ms})_{\max} < L_{mb} \tag{27}$$

Dividing Eq. (2) by Eq. (25), and taking $L_{in} = L_{ms}$ in Eq. (2), we have

$$dJ/dL_{ms} = -\frac{3r_{ms}L_{ms}}{2a_*M} = -\frac{9\,(M)_0^2\,L_{ms}}{J}$$

where Eq. (15) and $a_* \equiv J/M^2$are used in deriving the last equation. Integrating the above equation and taking Eq. (26) into account, we have

$$\frac{J^2}{108\,(M)_0^4} + \frac{L_{ms}^2}{12\,(M)_0^2} = 1 \tag{28}$$

From Eqs. (24) and (28) we find that both E_{ms} and L_{ms} evolve along elliptical orbits in the corresponding parameter space.

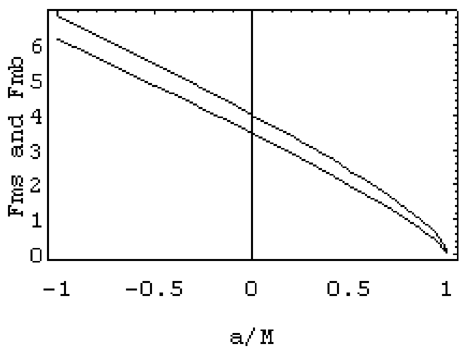

Fig. 1. The curves representing F_{mb} versus a_* (upper curve) and F_{ms} versus a_* (lower curve), for $-1 < a_* < 1$

4. Evolution of the dimensionless angular momentum of the CBH

Recently a lot of work has been done on the evolution of black hole accretion disk by considering the Blandford-Znajek (BZ) effects (Moderski & Sikora 1996; Moderski et al. 1997; Lu et al. 1996; W98a; W98b). In these work the evolution of a_* was investigated under different assumptions on accretion status in the inner region of the disk with an inner edge radius r_{ms} or r_{mb}, and a_* turns out to evolve to a stable value less than unity. Thus a question is whether there is any change in the above results if the inner edge radius r_{in} lies between r_{mb} and r_{ms}? In this case the rate of change of a_* can be expressed as

$$\begin{aligned}(da_*/dt)_{in} &= M^{-2}(dJ/dt)_{in} - 2M^{-1}a_*(dM/dt)_{in} \\ &= M^{-1}dM_0/dt\left(M^{-1}L_{in} - 2a_*E_{in}\right) \end{aligned} \quad (29)$$

Substituting Eqs. (3) and (4) into Eq. (29), we have

$$(da_*/dt)_{in} = \frac{\left(\chi^4 + 2a_*\chi - a_*^2 - 2a_*\chi^3\right)}{\chi\left(\chi^4 - 3\chi^2 + 2a_*\chi\right)^{1/2}} M^{-1}dM_0/dt \quad (30)$$

Starting from Eq. (30), we can derive (see Appendix).

$$\frac{\partial}{\partial r}(da_*/dt)_{in} = \frac{1}{2M\chi}\frac{\partial}{\partial \chi}(da_*/dt)_{in} \le 0 \quad (31)$$

Eq. (31) is valid in the value range: $r_{mb} \le r \le r_{ms}$, and it shows that $(da_*/dt)_{in}$ decreases monotonously as r for the given dM_0/dt, M and a_*. It follows that

$$0 \le (da_*/dt)_{ms} \le (da_*/dt)_{in} \le (da_*/dt)_{mb} \quad (32)$$

Eq. (32) shows that $(da_*/dt)_{in} \ge 0$ always holds in a pure accretion, and the equality holds for the extreme Kerr black hole. Therefore the conclusion that a_* will increase monotonously as time until it evolves to unity is still valid for r_{in} lying between r_{mb} and r_{ms}. The curves representing $F_{mb} \equiv M(dM_0/dt)^{-1}(da_*/dt)_{mb}$ versus a_* and $F_{ms} \equiv M(dM_0/dt)^{-1}(da_*/dt)_{ms}$ versus a_* are shown by the upper and lower curve, respectively in Fig. 1. Obviously, the curve indicating $M(dM_0/dt)^{-1}(da_*/dt)_{in}$ versus a_* should be intermediate between these two curves.

Considering the BZ effects on the evolution of a_*, we derived the rate of change of a_* as follows (W98a):

$$\begin{aligned}&(da_*/dt)_{BZ} \\ &= (da_*/dt)_{in} - (2P/Ma_*)(1+q)\left(k^{-1} - 1 + q\right)\end{aligned} \quad (33)$$

where $q = \sqrt{1 - a_*^2}$, and the BZ power P extracting from the spinning black hole reads

$$P = \lambda k(1-k)a_*^2 E_{in} dM_0/dt \quad (34)$$

where in Eqs. (33) and (34), $k = \Omega_F/\Omega_H$ is the ratio of the angular velocity of the magnetic field lines to that of horizon of CBH. λ is a parameter indicating the strength of the BZ process.

As is well known, the strength of the BZ process depends crucially on the strength of the magnetic field, $B_\perp$, normal to the horizon of the CBH. Very recently, some authors pointed out that the values of $B_\perp$ were overestimated substantially in previous works, and the BZ power P is not as strong as imagined previously (GA; Livio et al. 1999; Meier 1999). The overestimate of $B_\perp$ arises from the incomplete argument on the BZ process, in which the Maxwell pressure, $B_\perp^2/8\pi$, was assumed to correspond to an equilibrium with the maximum pressure, $p_{\max}$, in the inner parts of the accretion disk. In fact, according to a reasonable consideration on the continuity of the magnetic field between the horizon and r_{in}, $B_\perp^2/8\pi$ should be determined by the Maxwell pressure near r_{in}, rather than $p_{\max}$. The ratio of the former to the latter has been proved to have typical values of a few percent from a series of numerical simulations of the magnetohydrodynamic behavior of accretion disks (GA and the references therein). By using a rather reliable argument GA proposed the expression of the BZ power as follows:

$$\begin{cases} P_{RPD}\left(erg\cdot s^{-1}\right) = 2\times 10^{44} M_8 a_*^2 \\ P_{GPD}\left(erg\cdot s^{-1}\right) = 8\times 10^{42} M_8^{11/10} m_{-4}^{4/5} a_*^2 \end{cases} \quad (35)$$

where the upper and the lower expression, in Eq. (35), are applicable to radiation-pressure dominated (RPD) and gas- pressure dominated (GPD) inner regions, respectively. In order to determine the parameter λ according the strength given by Eq. (35), we set the ratio of Eq. (34) to the upper expression of Eq. (35) equal to unity as follows

$$\begin{aligned}P/P_{RPD} &= \lambda k(1-k)\dot{M}c^2/\left(2\times 10^{44}M_8\right) \\ &\approx \lambda k(1-k)L_{Edd}/\left(2\times 10^{44}M_8\right) = 1,\end{aligned} \quad (36)$$

where in Eq. (36), $\dot{M} = E_{in}dM_0/dt$ is assumed at the Eddington rate, and $L_{Edd} = 1.3\times 10^{46} M_8\, erg\cdot s^{-1}$. Thus λ is determined as $\lambda \approx 4/65 \approx 0.06$ for $k = 0.5$ (corresponding to a maximum of P). Incorporating Eqs. (3), (30), (33) and (34), we have

$$(da_*/dt)_{BZ} = M^{-1}(dM_0/dt)\frac{f(a_*,\chi)}{\chi\left(\chi^4 - 3\chi^2 + 2a_*\chi\right)^{1/2}} \quad (37)$$

where

$$\begin{aligned}f(a_*,\chi) &= \chi^3(\chi - a_*) - a_*\left(\chi^3 - 2\chi + a_*\right) \\ &\quad \times \left[1 + (\lambda/2)(1+q)^2\right]\end{aligned} \quad (38)$$

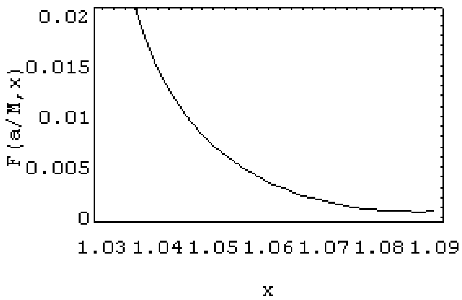

Fig. 2. The variation of $F(a_*, \chi)$ with χ, $a_* = 0.9990$, $1.032 < \chi < 1.087$

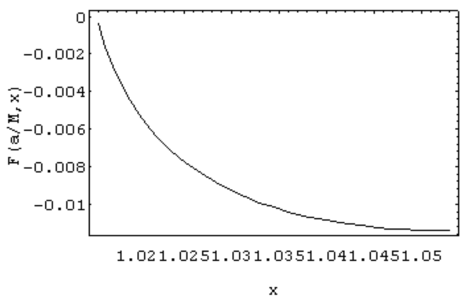

Fig. 3. The variation of $F(a_*, \chi)$ with χ, $a_* = 0.9998$, $1.016 < \chi < 1.053$

Setting $f(a_*, \chi) = 0$ with $\lambda \approx 4/65$, and substituting $\chi = \chi_{ms}$ and $\chi = \chi_{mb}$, respectively, into Eq. (38), we can derive the stable values of a_* : $\left(a_*^{stable}\right)_{ms} \approx 0.9990$ and $\left(a_*^{stable}\right)_{mb} \approx 0.9998$, which are closer to unity than our previous results: $\left(a_*^{stable}\right)_{ms} \approx 0.9831$ and $\left(a_*^{stable}\right)_{mb} \approx 0.9921$ (W98a; W98b). The latter two values were derived by setting $f(a_*, \chi) = 0$ with $\lambda \approx 4/13$, and the BZ power was overestimated 5 times previously in this case. Using Eqs. (37) and (38), we obtain the variation of $F(a_*, \chi) \equiv M(dM_0/dt)^{-1}(da_*/dt)_{BZ}$ with χ shown in Figs. 2 and 3:

Inspecting Figs. 2 and 3, we find the following distribution features of $(da_*/dt)_{BZ}$:

– (1) For $\left(a_*^{stable}\right)_{ms} \approx 0.9990$ and the corresponding value range of χ: $1.032 < \chi < 1.087$, $(da_*/dt)_{BZ} > 0$ holds, and $(da_*/dt)_{BZ}$ decreases monotonously as χ;
– (2) For $\left(a_*^{stable}\right)_{mb} \approx 0.9998$ and the corresponding value range of χ: $1.016 < \chi < 1.053$, $(da_*/dt)_{BZ} < 0$ holds, and $(da_*/dt)_{BZ}$ still decreases monotonously as χ (absolute value increases).

Thus the stable value of a_*, $\left(a_*^{stable}\right)_{in}$, satisfies the following inequality:

$$\left(a_*^{stable}\right)_{ms} < \left(a_*^{stable}\right)_{in} < \left(a_*^{stable}\right)_{mb} \tag{39}$$

where r_{in} lies between r_{mb} and r_{ms}. Using Eq. (38) for $\lambda = 4/65$, we can derive a series of stable values of a_* corresponding to different values of r_{in}, of which the value range is $1.016 < r_{in} < 1.087$, as shown in Table 1.

From Table 1, we find that inequality (39) is indeed satisfied, and the third law of black hole thermodynamics is also valid in the case involving the inner edge radius r_{in}. These results make an extension to our previous work (W98a, W98b).

Table 1. Stable values of a_* corresponding to different values of r_{in}

r_{in}	1.03M	1.06M	1.09M	1.12M	1.15M	1.18M
χ	1.016	1.030	1.044	1.058	1.072	1.087
$\left(a_*^{stable}\right)_{in}$	0.9998	0.9995	0.9993	0.9992	0.9991	0.9990

It is worth pointing out that these stable values of a_* in Table 1 are all less than the upper limit $a_* \approx 0.998$, which was derived by considering the capture effects of a black hole on the photons emitted by the surrounding accretion disk (Thorne 1974). As far as the upper limit of a_* is concerned, the BZ effects, as an anti-accretion, seems no stronger than the mechanism proposed by Thorne (1974). This result is contrary to the previous ones (Lu et al. 1996; W98a; W98b).

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Appendix: proof of $\partial E_{in}/\partial r < 0$, $\frac{\partial L_{in}}{\partial r} < 0$ and $\frac{\partial}{\partial r}(da_*/dt)_{in} < 0$

Defining $r \equiv M\chi^2$, we can write the partial derivative of L_{in} with respect to r as

$$\frac{1}{E_{in}}\frac{\partial E_{in}}{\partial r} = \frac{1}{2M\chi E_{in}}\frac{\partial E_{in}}{\partial \chi} \tag{A1}$$

To facilitate calculations, Eq. (3) can be rewritten as

$$E_{in} = \frac{\left(\chi^3 - 2\chi + a_*\right)}{\chi\left(\chi^4 - 3\chi^2 + 2a_*\chi\right)^{1/2}}$$

It follows that

$$\frac{1}{E_{in}}\frac{\partial E_{in}}{\partial \chi} = \frac{D_1}{\chi\left(\chi^3 - 2\chi + a_*\right)\left(\chi^3 - 3\chi + 2a_*\right)} \tag{A2}$$

where

$$D_1 = \chi^4 - 6\chi^2 + 8a_*\chi - 3a_*^2 \tag{A3}$$

Inspecting the deriving process of the radius, r_{ms}, of the last stable circular (Shapiro & Teukolsky 1983), we know that the circular orbit with $r < r_{ms}$ is unstable, and the corresponding χ satisfies

$$\chi^4 - 6\chi^2 + 8a_*\chi - 3a_*^2 < 0, \qquad (\chi < \chi_{ms}) \tag{A4}$$

It is easy to prove that the denominator of RHS of Eq. (A2) is positive. Incorporating Eqs. (A1)–(A4), we have $\partial E_{in}/\partial r < 0$.

Similarly, the partial derivative of L_{in} with respect to r can be expressed as

$$\frac{1}{L_{in}}\frac{\partial L_{in}}{\partial r} = \frac{1}{2M\chi L_{in}}\frac{\partial L_{in}}{\partial \chi} \tag{A5}$$

Eq. (4) can be rewritten as

$$L_{in} = \frac{M\left(\chi^4 - 2a_*\chi + a_*^2\right)}{\chi\left(\chi^4 - 3\chi^2 + 2a_*\chi\right)^{1/2}}$$

It follows that

$$\frac{1}{L_{in}}\frac{\partial L_{in}}{\partial \chi} = \frac{D_2}{\chi\left(\chi^4 - 2a_*\chi + a_*^2\right)\left(\chi^3 - 3\chi + 2a_*\right)} \quad \text{(A6)}$$

where

$$D_2 = \left(\chi^3 + a_*\right)\left(\chi^4 - 6\chi^2 + 8a_*\chi^2 - 3a_*^2\right) \quad \text{(A7)}$$

It is easy to prove that the denominator of RHS of Eq. (A6) is positive. Incorporating Eqs. (A4)–(A7), we have $\partial L_{in}/\partial r < 0$.

Similarly, the partial derivative of $(da_*/dt)_{in}$ with respect to can be expressed as

$$\left[(da_*/dt)_{in}\right]^{-1}\frac{\partial}{\partial r}(da_*/dt)_{in} = \frac{1}{2M\chi}\left[(da_*/dt)_{in}\right]^{-1}\frac{\partial}{\partial \chi}(da_*/dt)_{in} \quad \text{(A8)}$$

Using Eq. (39), we have

$$\left[(da_*/dt)_{in}\right]^{-1}\frac{\partial}{\partial \chi}(da_*/dt)_{in} = \frac{D_3}{\chi\left(\chi^4 - 2a_*\chi^3 + 2a_*\chi - a_*^2\right)\left(\chi^3 - 3\chi + 2a_*\right)} \quad \text{(A9)}$$

where

$$D_3 = \left(\chi^3 - a_*\right)\left(\chi^4 - 6\chi^2 + 8a_*\chi^2 - 3a_*^2\right) \quad \text{(A10)}$$

It is easy to prove that the denominator of RHS of Eq. (A9) is positive. Incorporating Eqs. (A4), and (A8)–(A10), we have. $\frac{\partial}{\partial r}(da_*/dt)_{in} < 0$.

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