

# Extending the scope of models for large-scale structure formation in the universe

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Received 23 November 1998 / Accepted 29 March 1999

**Abstract.** We propose a phenomenological generalization of the models of large-scale structure formation in the Universe by gravitational instability in two ways: we include pressure forces to model multi-streaming, and noise to model fluctuations due to neglected short-scale physical processes. We show that pressure gives rise to a viscous-like force of the same character as the one introduced in the *adhesion model*, while noise leads to a roughening of the density field, yielding a scaling behavior of its correlations.

**Key words:** gravitation – hydrodynamics – instabilities – methods: analytical – cosmology: theory – cosmology: large-scale structure of Universe

## 1. Introduction

Analytical approximations for the evolution of large-scale structure (LSS) are based on the paradigm that small initial perturbations grow by gravitational instability, which is in turn implemented in the simplest matter model, *dust* (Peebles 1980, Zel'dovich & Novikov 1983, Sahní & Coles 1995, and ref. therein). However, this approximation has some limitations: one has to restrict its application to the early stages of structure formation and when the effects on the evolution of physical processes different from gravitational instability are negligible. In this paper we purport to generalize this matter model in order to overcome some of these limitations.

One of the problems at the later stages of LSS evolution is the formation of multi-stream regions, i.e., regions where particles of dust come together with very different velocities. This fact manifests itself as the emergence of caustics in the density field, where the velocity field is *vertical* (i.e., where it acquires an infinite derivative at a point) and later on multiply valued. This problem arises from insisting on following the trajectory of each particle of dust. We therefore propose a set of hydrodynamic-like equations for the *coarse-grained* fields, which trace the average motion rather than that of individual

particles. A substantial ingredient of this approach is that the coarse-grained velocity evolves under the combined action of gravity and *pressure-like* forces due to velocity dispersion (i.e., because particles of dust do not move exactly with the coarse-grained velocity). We cannot resort to hydrodynamic considerations of local equilibrium but make instead use of “equations of state” as phenomenological matter models without any further justification. In these models the pressure is assumed isotropic and may only depend on the (coarse-grained) density, that is,  $p = p(\varrho)$ . This assumption makes the problem accessible to analytical study and helps to illuminate our major argument that the presence of forces which counteract the gravitational attraction is a basic step in understanding the dynamics of self-gravitating matter; a detailed study of the origin and properties of pressure forces is carried out elsewhere (Buchert & Domínguez 1998).

That the putative problem is far more complex than the dust case already becomes clear in the investigation of the one-dimensional Euler-Newton system with the simple matter model  $p \propto \varrho$  (Götz 1988). Götz has shown that solutions to the one-dimensional problem can be generated by solutions of the Sine-Gordon equation. This well-studied equation has a rich spectrum of solutions that includes solitons. Götz also pointed out that an asymptotic N-soliton state is generic, i.e., will be realized almost independently of the initial data. We see already in this comparatively simple case, that we are faced with a generic picture which is completely different from what emerges in a cosmology based on dust matter: special nonlinear features build up structures at large times which are *absent* in the dust cosmology. This illustrates that the complexity introduced by a pressure term could bear far-reaching surprises. We also want to stress that the introduction of a pressure term is not in contradiction with our present understanding of LSS formation, since it partly arises from N-body simulations of the structure formation process which capture multi-streaming effects, and hence are not constrained by the analytical approximation to dust matter<sup>1</sup> which, in its simplest realizations (e.g., Zel'dovich's approximation (Zel'dovich 1970)), features immediate decay of structures after their formation.

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<sup>1</sup> We here imply “single-streamed dust” as opposed to models which also cover multi-dust regions.

We also consider an extension of the model to include *stochastic* effects. This “stochasticity” arises from the effect on the dynamical evolution of physical processes occurring on time- and/or length-scales much smaller than those directly associated with LSS formation, thus allowing to model them by means of a stochastic source (a noise). Possible sources are deviations from the mean field approximation, fluctuations inherent to the hydrodynamic (i.e., coarse-grained) description, and non-gravitational processes in baryonic matter. We shall use the simplest model of a Gaussian-distributed forcing on the coarse-grained evolution.

As with pressure-like forces, we want to stress that a noisy forcing could be relevant to LSS formation, but a detailed consideration of its origin and properties is beyond the scope of the present paper. In fact, application of the Renormalization Group shows that noise is relevant in non-exceptional conditions (Barbero et al. 1997; Domínguez et al. 1999), implying that even if it is very weak (apparently negligible), its effects are amplified and can have a non-negligible impact on the dynamical evolution.

This paper is structured as follows: in Sect. 2 we begin by presenting the basic system of equations in the Newtonian regime, and then proceed to a discussion of restrictive assumptions for the weakly nonlinear regime. In Sect. 3 we discuss the role of the pressure-like force for some particular choices of the equation of state  $p = p(\varrho)$  and the connection with Burgers’ equation. In Sect. 4 we consider the role of noise and provide a detailed description of the relationship between the cosmological equations and the Kardar–Parisi–Zhang (KPZ) equation. In Sect. 5 we study the linear regime in the presence of pressure and noise. We finally conclude in Sect. 6. Some technicalities have been left to two appendices, one devoted to the exploration of the validity of what we call in Sect. 4 the “adiabatic approximation”, and another to a more detailed discussion of the linear regime.

## 2. Basic equations and restrictive assumptions

We are interested in discussing LSS formation in the non-relativistic regime and therefore consider the Newtonian cosmological equations for a self-gravitating fluid in a standard Friedmann–Lemaître (FL) cosmological background dominated by non-relativistic matter (Peebles 1980). The cosmological background is characterized by the cosmic expansion factor  $a(t)$  and the homogeneous background matter density  $\varrho_b(t)$ , which obey (Hubble’s function is defined as  $H = \dot{a}/a$ )

$$H^2 = \frac{8\pi G}{3}\varrho_b - \frac{K}{a^2} + \frac{\Lambda}{3}, \quad \varrho_b = \varrho_0 a^{-3}, \quad (1)$$

where the constant  $K$  determines the sign of the spatial curvature (treated as an integration constant in the Newtonian framework considered throughout this paper),  $\varrho_0$  is the background density at some time  $t_0$  when  $a(t_0) = 1$ , and  $\Lambda$  is the cosmological constant. Without loss of generality, one can choose  $t_0$  to correspond to the present epoch.

It is convenient to work in comoving coordinates  $\mathbf{x} \equiv a^{-1}\mathbf{r}$ , where  $\mathbf{r}$  are the standard non-rotating Eulerian coordinates. The

fundamental fields will be as follows: the density  $\varrho$  (or equivalently, the density contrast  $\delta := (\varrho/\varrho_b) - 1$ ), the peculiar-velocity  $\mathbf{u} \equiv \mathbf{u}_{phys} - H\mathbf{r}$ , where  $\mathbf{u}_{phys}$  is the physical velocity and  $H\mathbf{r}$  is the Hubble flow, and the gravitational peculiar-acceleration  $\mathbf{g} = \mathbf{g}_{phys} + \frac{1}{3}(4\pi G\varrho_b - \Lambda)\mathbf{r}$ , where  $\mathbf{g}_{phys}$  is the physical gravitational acceleration and  $-\frac{1}{3}(4\pi G\varrho_b - \Lambda)\mathbf{r}$  is the Newtonian counterpart of the gravitational acceleration associated to the background expansion. We subject  $\delta$ ,  $\mathbf{u}$  and  $\mathbf{g}$  to periodic boundary conditions on some large scale to assure uniqueness of the cosmological solutions, in which case  $\varrho_b$  is equal to the spatially averaged density (see Ehlers & Buchert 1997).

The fields  $\varrho$  and  $\mathbf{u}$  obey a set of hydrodynamic equations expressing the conservation of mass and momentum in an expanding background. These equations are similar to those of the standard dust model (Peebles 1980), except for two forcing terms in Euler’s equation that model the multi-streaming and stochastic effects discussed in the Introduction<sup>2</sup>:

- Continuity equation:

$$\frac{\partial \varrho}{\partial t} + 3H\varrho + \frac{1}{a}\nabla \cdot (\varrho\mathbf{u}) = 0; \quad (2)$$

- Euler’s equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{a}(\mathbf{u} \cdot \nabla)\mathbf{u} + H\mathbf{u} = \mathbf{g} - \frac{1}{a\varrho}\nabla p + \mathbf{s}; \quad (3)$$

- Newtonian field equations:

$$\nabla \cdot \mathbf{g} = -4\pi G a(\varrho - \varrho_b), \quad \nabla \times \mathbf{g} = 0. \quad (4)$$

We emphasize that the integral curves of the peculiar-velocity field  $\mathbf{u}$  are not associated to trajectories of *individual* particles, rather  $\varrho$ ,  $\mathbf{u}$  and  $\mathbf{g}$  are considered as coarse-grained fields. This coarse-graining is the origin of the two new terms on the right-hand-side of Euler’s equation (3). One of these new terms is the pressure force  $\nabla p$ , which accounts for the isotropic part of the multi-stream force, and therefore models velocity dispersion (that is, the fact that in any infinitesimal cell there are particles with different velocities). Because of this, the integral curves of  $\mathbf{u}$  represent trajectories of the *mean* (possibly multi-streamed) flow after averaging over velocity space. This pressure term is *not* related to thermal pressure, which can be indeed neglected on the scales we are interested in. It is a model of the velocity dispersion generated by gravitational instability (see Buchert & Domínguez 1998 and, for a recent generalization to general relativity, Maartens et al. 1999).

The other new term is the stochastic force represented by the noise  $\mathbf{s}$ ; it accounts for processes hidden by the coarse-grained description of the fluid and whose typical time- and length-scales are much shorter than those explicitly considered for LSS formation. We have resorted to modelling these processes as a stochastic forcing and include (but are not restricted to): (a) the effects of small-scale degrees of freedom whose physics is also

<sup>2</sup> From now on, time derivatives are taken at constant  $\mathbf{x}$  and gradients refer to comoving coordinates.

governed by non-gravitational processes, (b) deviations from mean field behavior, manifested as random forces acting on the particles of the gravitational gas as a consequence of independent impulses of random size and amplitude arising from “sling-like” processes in encounters (Kandrup 1980), (c) deviations of the density and velocity fields from the values prescribed by the deterministic version of Eqs. (2-4) due to the graininess of the underlying physical system of particles (Lifshitz & Pitaevskii 1980).

To close the system of Eqs. (2-4) a relation is needed between the dynamical pressure  $p$  and the two independent fields  $\varrho$  and  $\mathbf{u}$ , as well as a specification of the statistical properties of the stochastic force  $\mathbf{s}$ . As for the former, we assume the local relationship  $p = p(\varrho)$ . There does not seem to be any *a priori* reason for this “slaving” of pressure to density (one cannot resort to the hypothesis of local equilibrium as is done for the thermodynamical pressure in fluids), so the success of this *phenomenological* assumption must be judged according to the conclusions following from it. In fact, a detailed study of the origin of pressure forces in Eq. (3) provides  $p \propto \varrho^{5/3}$  under the assumption of small velocity dispersion (Buchert & Domínguez 1998) and therefore  $p = p(\varrho)$  is the most straightforward phenomenological generalization. We also require  $p'(\varrho) > 0$ , that is, pressure opposes gravitational collapse.

As for the noise, we make the assumption of Gaussian distributed noise. Since noise is due to the short-scale degrees of freedom, this assumption could be justified by the central limit theorem. As is well known, Gaussian noise can be characterized by just two moments: its mean, which we require to vanish,  $\langle \mathbf{s} \rangle = \mathbf{0}$  (since any systematic forcing should be made explicit in Eq. (3)), and its two-point correlations

$$\langle s_i(\mathbf{x}, t) s_j(\mathbf{x}', t') \rangle = 2D_{ij}(\mathbf{x}, \mathbf{x}', t, t'), \quad (5)$$

where  $D_{ij}(\mathbf{x}, \mathbf{x}', t, t')$  is the covariance matrix with mixed discrete and continuous indices (Gardiner 1994; Van Kampen 1992). We ignore the possibility that  $D_{ij}$  depends on  $\varrho$  or  $\mathbf{u}$  (quenched noise), since this renders the analysis of Eqs. (2-4) too difficult. As with pressure, this must be seen as a *phenomenological* assumption, whose merit will be judged from the final results. Notice however that we consider the possibility of *colored* Gaussian noise, i.e., the noise  $\mathbf{s}$  is correlated over space and time, as expressed by the (in general) non-trivial dependence of  $D$  on position and time in Eq. (5). Colored noise represents a richer texture of physical effects than white noise and the particular case of power-law correlated noise is still amenable to analytical study by means of the Renormalization Group (Barbero et al. 1997; Domínguez et al. 1999). We later restrict the generality further by choosing a curl-free noise and characterize it statistically by a single function  $D$  instead of  $D_{ij}$ .

The analytical study of the system of Eqs. (2-4) is very difficult in general. Inspired by the deterministic dust case (i.e.,  $p = 0$ ,  $\mathbf{s} = \mathbf{0}$ ), we simplify the problem and put forward the assumption of parallelism: we impose the condition that the peculiar-velocity is a potential field and remains parallel to the gravitational peculiar-acceleration field:

$$\mathbf{g} = F(t)\mathbf{u}, \quad (6)$$

where  $F(t) > 0$  is a proportionality coefficient which follows from the deterministic linear theory for dust (see Sect. 5 and in particular the discussion after Eqs. (B14-B15) in Appendix B). This assumption implicitly requires that both pressure and the intensity of the noise are small when compared with the dominant self-gravity in Eq. (3), which restricts our considerations to spatial scale regimes close to the validity limit of the dust model; noise and pressure will typically dominate on scales small compared to this limit. The assumption of parallelism underlies the well-known Zel’dovich’s approximation (Zel’dovich 1970), and is well-justified for deterministic dust models in the linear as well as in the weakly nonlinear regimes (see Bildhauer & Buchert 1991; Kofman 1991; Buchert 1992; Hui & Bertschinger 1996; Susperregi & Buchert 1997). This (indeed oversimplifying assumption) will be very useful to analytically access the problem, and to define “local” approximations.

Furthermore, we study that assumption first because popular models like the adhesion approximation (Gurbatov et al. 1989) can be *derived* on the basis of this assumption. As we shall see, this assumption is consistent with the picture that, coming from large scales, the mean motion is ruled by the dust model, but incorporation of the effects of pressure and noise adds several interesting aspects to it. At the smaller scales, the parallelism assumption requires that the “backreaction” of these effects on the trajectories of the mean dust flow be neglected. In the dust case it can be shown that this assumption admits a class of 3D solutions (Buchert 1989); this class is highly restrictive, but using it as the basis for approximation schemes turns out to be very successful (e.g. Buchert 1996 and ref. therein) - a subclass of these schemes is Zel’dovich’s approximation. However, in the present case we cannot disprove that the assumption (6) is too restrictive to allow for the existence of a class of exact solutions. We will learn more about the justification of this assumption in Sect. 5.

Under the assumption (6), Eq. (3) reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{a}(\mathbf{u} \cdot \nabla)\mathbf{u} + (H - F)\mathbf{u} = -\frac{p'(\varrho)}{a\varrho}\nabla\varrho + \mathbf{s}, \quad (7)$$

with constraints following from the field equations (4): the velocity field is irrotational,  $\nabla \times \mathbf{u} = \mathbf{0}$ , and its divergence satisfies  $F\nabla \cdot \mathbf{u} + 4\pi Ga(\varrho - \varrho_b) = 0$ .

The pressure force term  $\nabla\varrho$  can be computed from this last expression, which gives

$$\nabla\varrho = -\frac{F}{4\pi Ga}\nabla^2\mathbf{u}, \quad (9)$$

and Euler’s equation may now be finally written as:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{a}(\mathbf{u} \cdot \nabla)\mathbf{u} + (H - F)\mathbf{u} = \nu\nabla^2\mathbf{u} + \mathbf{s}, \quad (10)$$

where we have defined a coefficient  $\nu$  which behaves as a kinematic viscosity and which we call *gravitational multi-stream* (GM) coefficient (because it has its origin in the interplay between self-gravitation and multi-streaming flow); it is given by

$$\nu := \frac{F(t)p'(\varrho)}{4\pi Ga(t)^2\varrho} > 0, \quad (11)$$

and depends on the density, and explicitly on time through  $F(t)$  and  $a(t)$ . As we see from Eq. (10) and (11), the pressure behaves in Euler's equation *effectively* as a viscous force that prevents caustics in the velocity field, a behavior that essentially requires the participation of self-gravity. The interplay between self-gravity and pressure leads to the stabilization of the structures, and holds them together. We also want to stress that the GM coefficient *does not* generate dissipation, since our starting set of Eqs. (2-4) lacks any sort of dissipative term (see Buchert & Domínguez 1998 for further details about this apparently paradoxical point).

The last step is the elimination of the density  $\varrho$  in the expression of the GM coefficient in favor of  $\nabla \cdot \mathbf{u}$  by means of the constraint (8), thus reducing Eqs. (10) to a *closed* equation for the velocity field  $\mathbf{u}$ . This equation will be explored in the next sections.

### 3. The role of pressure: Burgers-like equations

In this section we consider the role of pressure in the dynamical evolution and thus for the time being we drop the noise term,  $\mathbf{s}(\mathbf{x}, t)$ . But before going into the details, it is worthwhile to get an intuitive picture of how the pressure term affects the dynamical evolution. As remarked above, pressure has in Eq. (10) a stabilizing effect opposing collapse. But this effect holds even in regimes where the assumptions leading to Eq. (10) (especially the parallelism assumption) break down: Fig. 1 shows the evolution of coarsening cells in a simulation and how velocity dispersion influences their motion.

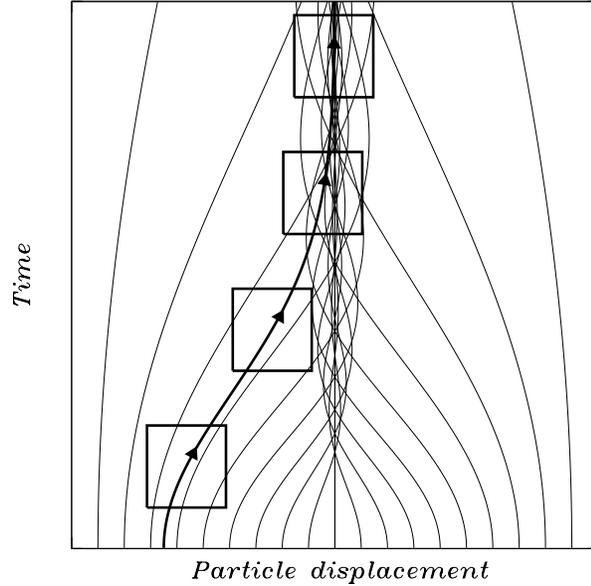
Different relationships between pressure and density yield different  $\varrho$ -dependences of the GM coefficient. In this section we will concentrate on polytropic models  $p = \kappa \varrho^\gamma$ , where  $\kappa$  is constant and  $\gamma$  is the polytropic index. The prefactor  $\kappa$  should be small enough so that the parallelism assumption (6) is approximately satisfied on the length scales we are interested in, and it should also satisfy  $\kappa\gamma > 0$  in order to fulfill the condition  $p'(\varrho) > 0$ . The choice of polytropic models is interesting because we are able to recover well-known approximations in cosmology (Zel'dovich's approximation, the *sticky-particle* model and the adhesion model) as well as other interesting models, and we can understand them within a single equation. With this form of  $p(\varrho)$ , Eq. (10) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{a}(\mathbf{u} \cdot \nabla)\mathbf{u} + (H - F)\mathbf{u} = \frac{\gamma\kappa F}{4\pi G a^2} \varrho^{\gamma-2} \nabla^2 \mathbf{u}. \quad (12)$$

We use now Eq. (8) to eliminate  $\varrho$  in place of  $\nabla \cdot \mathbf{u}$ , and since  $\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u})$  (because  $\nabla \times \mathbf{u} = \mathbf{0}$ ) one gets<sup>3</sup>:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{a}(\mathbf{u} \cdot \nabla)\mathbf{u} + (H - F)\mathbf{u} = -\tilde{\nu} \nabla[1 - \beta \nabla \cdot \mathbf{u}]^{\gamma-1}, \quad (13)$$

where  $\tilde{\nu} = \gamma\kappa\varrho_b^{\gamma-1}/(\gamma-1)a$  and  $\beta = F/4\pi G a \varrho_b$  are functions which depend only on time. This equation can be further simplified as follows: making explicit use of  $b(t)$ , the growing



**Fig. 1.** The coarse-graining idea is exemplified in a schematic way based on a 2D tree-code simulation (Buchert 1996): the particles are attracted towards the pancake, a multi-stream region containing many streams (velocities) at a given Eulerian position. The coarsening cell follows initially a trajectory similar to that of individual particles. But as the cell moves into the pancake, the kinetic energy of the bulk motion is gradually transformed into internal kinetic energy, and its trajectory becomes qualitatively different from that of a particle.

mode of the density field in the linear regime of the deterministic dust case, one has  $F = 4\pi G \varrho_b b/b$  (see Eqs. (B14-B15) in Appendix B and discussion thereafter). Hence, defining a new velocity field  $\mathbf{v} = \mathbf{u}/ab$  and using  $b$  to define a new time variable, Eq. (13) can be written as:

$$\frac{\partial \mathbf{v}}{\partial b} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\mu \nabla[1 - b \nabla \cdot \mathbf{v}]^{\gamma-1}, \quad \mu := \frac{\tilde{\nu}}{a b^2}. \quad (14)$$

When  $\gamma = 2$ , this equation becomes the 3D Burgers' equation (Burgers 1974) (except for the time-dependence of the GM coefficient  $\mu$ ). In the cases  $\gamma \neq 2$ , we are dealing with generalizations of Burgers' equation. We now discuss several cases for the choice of  $\gamma$ <sup>4</sup>.

#### Zel'dovich's approximation

If one chooses  $\gamma = 0$ , then  $\mu = 0$  and Eq. (14) reduces to

$$\frac{\partial \mathbf{v}}{\partial b} + (\mathbf{v} \cdot \nabla)\mathbf{v} = 0, \quad (15)$$

which corresponds to the well-known Zel'dovich's approximation; that approximation is equivalent to a subclass of Lagrangian first-order solutions, the subclass being just defined by our assumption (6) (see Buchert 1989, 1992). The dynamical evolution governed by this equation generically leads to

<sup>3</sup> Eq. (13) is valid for  $\gamma \neq 1$ . If  $\gamma = 1$ , one has a logarithmic function instead of a power on the right-hand-side of Eq. (13), see Eq. (17).

<sup>4</sup> The names given below to some of the models stem from thermodynamical notions; we kept those names but do not imply that we describe situations in thermodynamical equilibrium.

the formation of singularities where the velocity field is multi-valued (Arnol'd et al. 1982).

#### Adhesion approximation

With the choice  $\gamma = 2$ , the GM coefficient is independent of the density, and Eq. (14) becomes Burgers' equation with a time-dependent "viscosity":

$$\frac{\partial \mathbf{v}}{\partial b} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \mu b \nabla^2 \mathbf{v}. \quad (16)$$

This equation is formally equivalent to the so-called "adhesion approximation" (Gurbatov et al. 1989; Weinberg & Gunn 1989), aside from the time-dependence of the GM coefficient<sup>5</sup>.

Letting  $\kappa \rightarrow 0$  (which implies  $\mu \rightarrow 0$ ) in this model, we get maximal adhesion and the large-scale structure is built from a skeleton of the "honeycomb-type": the evolution is governed by Zel'dovich's approximation, Eq. (15), everywhere except at the singularities, which become shock fronts. This limiting model is known as *sticky-particle* model, for which geometrical construction methods have been developed (Pogosyan 1989; Kofman et al. 1992; for review see Sahni & Coles 1995). In our derivation this limit implies  $p \rightarrow 0$ , i.e., vanishing velocity dispersion. It must be noticed, however, that this limit is singular: if  $p = 0$  *strictly*, then we would recover Zel'dovich's approximation, which is qualitatively different from the sticky-particle model. Therefore, a non-vanishing, though very small, velocity dispersion is *required* to recover typical properties of the adhesion model.

#### Isothermal model

This model corresponds to  $\gamma = 1$  and is mostly studied in connection with the linear theory of gravitational instability (since linearization of the pressure term in Eq. (3) amounts to choosing  $\gamma = 1$ ). Eq. (12) yields:

$$\frac{\partial \mathbf{v}}{\partial b} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\kappa}{a^2 b^2} \nabla \ln(1 - b \nabla \cdot \mathbf{v}). \quad (17)$$

It is worth mentioning that Eq. (12) particularized to  $\gamma = 1$  resembles the Navier–Stokes equation with respect to the density dependence of the GM coefficient.

As pointed out in the introduction, Götz (1988) has shown that the asymptotic state of this model in one-dimensional space and neglecting the expanding background is a collection of solitons. On the other hand, parallelism is not an approximation but an exact relationship in one dimension (although the proportionality coefficient will be in general a function of time *and* position). Hence, one can expect that Eq. (17) leads to a similar picture to that obtained from the adhesion model, which is in turn consistent with that offered by Götz: the shock fronts, stabilized by velocity dispersion, will play the role of solitons.

<sup>5</sup> In an Einstein–de Sitter background cosmology  $b\mu \propto b^{-3}$  and thus the GM coefficient in Eq. (16) approaches zero as time goes by. Remember also that in the adhesion approximation the constant coefficient was introduced *ad hoc* to phenomenologically model gravitational sticking of particles as observed in N-body simulations.

#### Adiabatic model

For this model  $\gamma = 5/3$  and Eq. (14) becomes

$$\frac{\partial \mathbf{v}}{\partial b} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\mu \nabla [1 - b \nabla \cdot \mathbf{v}]^{2/3}. \quad (18)$$

The interesting point about this choice is that there are physical reasons to prefer this model over any other one in the early stages of LSS formation: Buchert & Domínguez (1998) show how this form of  $p(\varrho)$  can be *derived* from dynamical considerations.

#### Cosmogenetic model

By choosing  $\gamma = 4/3$  we recover the "cosmogenetic" model (Chandrasekhar 1967):

$$\frac{\partial \mathbf{v}}{\partial b} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\mu \nabla [1 - b \nabla \cdot \mathbf{v}]^{1/3}. \quad (19)$$

This model is of interest because it is the only polytropic model compatible with comoving hydrostatic equilibrium, i.e., with the solutions of Eqs. (2-4) corresponding to  $\mathbf{u} \equiv \mathbf{0}$ , so that the temporal evolution of the gas simply follows the expansion: Eq. (2) yields  $\varrho(\mathbf{x}, t) = a^{-3}(t)\varrho_0(\mathbf{x})$ , Eq. (4) then becomes  $\mathbf{g}(\mathbf{x}, t) = a^{-2}(t)\mathbf{g}_0(\mathbf{x})$ . The hydrostatic equilibrium condition follows from Eq. (3) (dropping the noise):  $\nabla p = a\varrho\mathbf{g}$ . Combining this condition with the polytropic relationship yields

$$\gamma\kappa a^{-3\gamma}\varrho_0^{\gamma-1}\nabla\varrho_0 = a^{-4}\varrho_0\mathbf{g}_0, \quad (20)$$

which can be satisfied only if  $\gamma = 4/3$ . (Obviously, the state of comoving hydrostatic equilibrium itself is incompatible with the parallelism assumption).

To conclude this section, one can conjecture that Burgers-like equations (14) lead quite generically in the limit  $\kappa \rightarrow 0$  to the same picture: the dynamical evolution would be governed almost everywhere by Zel'dovich's approximation except at the caustics, where the right-hand-side of Eq. (14) would dominate the evolution and a shock structure would be formed (known as a "pancake" in the cosmological literature). Only the details of the density and velocity profiles in the neighborhood of pancakes would depend on  $\gamma$ . It is an open question whether and in which cases velocity dispersion and self-gravity could balance to form stable soliton-like configurations.

## 4. The role of noise: emergence of a KPZ-like equation

In this section we consider the dynamical effect of noise. We come back to Eq. (7) and introduce a velocity potential  $\psi(\mathbf{x}, t)$  through  $\mathbf{u} \equiv -\nabla\psi$  (since the parallelism assumption (6) implies that  $\mathbf{u}$  is irrotational). Then

$$\begin{aligned} \nabla \left[ \frac{\partial \psi}{\partial t} - \frac{1}{2a} (\nabla \psi)^2 + (H - F)\psi \right] &= \\ &= \frac{1}{a} \nabla \int_{\varrho_b}^{\varrho} dy \frac{p'(y)}{y} - \mathbf{s}. \end{aligned} \quad (21)$$

We now make the assumption that the noise  $\mathbf{s}$  is a potential forcing, i.e., that there exists a (stochastic) potential  $\eta$  such that

$s \equiv -\nabla\eta$ . This stochastic potential will be chosen Gaussian–distributed with zero mean,  $\langle\eta\rangle = 0$ , and correlations given by

$$\langle\eta(\mathbf{x}, t)\eta(\mathbf{x}', t')\rangle = 2D(\mathbf{x}, \mathbf{x}', t, t'). \quad (22)$$

The assumption of potential noise can be motivated by the fact that in the linear regime only the potential component of the noise contributes to the growth of perturbations by gravitational instability (see Sect. 5 and Eq. (B7) in Appendix B). If the noise had a non–potential component, it would generate vorticity in the velocity field  $\mathbf{u}$  and would invalidate the parallelism assumption.

After inserting this stochastic force into Eq. (21), integrating, dropping an irrelevant additive function of time and changing from  $\varrho$  to  $\delta$  in the integral, one obtains:

$$\begin{aligned} \frac{\partial\psi}{\partial t} - \frac{1}{2a}(\nabla\psi)^2 + (H - F)\psi &= \\ &= \frac{1}{a} \int_0^\delta dy \frac{p'(\varrho_b(1+y))}{1+y} + \eta. \end{aligned} \quad (23)$$

On the other hand, the constraint (8) can be converted into an equation relating the velocity potential  $\psi$  and the density contrast  $\delta$ , that is

$$\delta(\mathbf{x}, t) = \frac{F(t)}{4\pi G a(t)\varrho_b(t)} \nabla^2\psi(\mathbf{x}, t). \quad (24)$$

Inserting this equation into Eq. (23) above then yields a non–linear partial differential equation for the velocity potential (after specifying the matter model  $p(\varrho)$ ) given by:

$$\begin{aligned} \frac{\partial\psi}{\partial t} - \frac{1}{2a}(\nabla\psi)^2 + (H - F)\psi &= \\ &= \frac{1}{a} \int_0^{\frac{F}{4\pi G a \varrho_b} \nabla^2\psi} dy \frac{p'(\varrho_b(1+y))}{1+y} + \eta. \end{aligned} \quad (25)$$

There are two sources of non–linearities: the convective (also known in the literature as advective) term  $(\nabla\psi)^2$  and the integral arising from the pressure force. To proceed further we need an explicit expression for  $p(\varrho)$  to evaluate the integral. For a polytropic model,  $p = \kappa\varrho^\gamma$ , one recovers Eq. (13), expressed in terms of the potential  $\psi$ , plus the noise term. As discussed at the end of the previous section, one may expect that the gross features of the LSS emerging from these models are insensitive to the particular equation of state  $p = p(\varrho)$  in the limit of small pressure and that only the fine details depend on it. We therefore simplify this equation further by expanding the integral in a Taylor series and keeping only the lowest order term in  $\delta$ . The resulting equation is *exact* for the particular choice  $p = \kappa\varrho^2$ :

$$\frac{\partial\psi}{\partial t} - \frac{1}{2a}(\nabla\psi)^2 + (H - F)\psi = \frac{Fp'(\varrho_b)}{4\pi G a^2 \varrho_b} \nabla^2\psi + \eta. \quad (26)$$

As remarked in the previous section, this choice leads to the adhesion approximation, which is in fact postulated on a similar reasoning: it is the simplest way of modelling sticking particles (Gurbatov et al. 1989). Hence, the adhesion approximation can

be viewed as the first term in a Taylor series that approximates the integral expression.

Eq. (26) is the simplest equation for  $\psi$  that we can write (still containing the main ingredients that enter into the physics of the self–gravitating gas we are describing), although as we will see, it already entails a considerable degree of complexity.

#### 4.1. Discussion of Eq. (26)

This stochastic partial differential equation is first order in time and second order in position; it is also non–linear in  $\psi$  and the coefficients of the different terms are functions of time. The physical meaning of each term is transparent:

(a) The term proportional to  $\psi$  encompasses the competition between damping of perturbations due to the cosmological expansion,  $(H\psi)$ , and enhancing of perturbations due to gravitational collapse,  $(-F\psi)$ . This term introduces a time–dependent time scale  $|H - F|^{-1}$ , which is the time scale for the damping (or enhancing) of perturbations in regimes where the nonlinearity is negligible.

(b) The term proportional to the Laplacian describes the damping of perturbations due to velocity dispersion. This term defines a time–dependent (comoving) length scale, Jeans' length,

$$L_J := \left[ \frac{p'(\varrho_b)}{4\pi G a^2 \varrho_b} \right]^{1/2}, \quad (27)$$

which is discussed in Appendix B, after Eq. (B12).

(c) The non–linear term is the convective term in Euler's equation (3). The effect of this term is to broaden the peaks in the field  $\psi$ . (An intuitive picture of how a term like  $(\nabla\psi)^2$  behaves can be found in Barabási & Stanley 1995, Fig. 6.2).

(d) The noise term incorporates the effects of fluctuations due to different sources and leads to a roughening of the field  $\psi$  in space and in time as it evolves.

Eq. (26) describes what is known in the condensed matter literature as interface growth phenomena, but with two added ingredients which are substantive to cosmology: the presence of time–dependent coefficients due to cosmological expansion, and the presence of a term proportional to  $\psi$ , which in the context of condensed matter physics is interpreted as a finite, albeit time–dependent, correlation length. This equation (Berera & Fang 1994; Barbero et al. 1997) is then a generalization to cosmological settings of the KPZ equation (Kardar et al. 1986, Barabási & Stanley 1995) for surface growth. In principle, one could use techniques similar to those used there to study this equation, but it turns out that complications arise as a consequence of the time dependence of the coefficients. Because of this, the first thing one thinks of is to perform changes of variables that will bring the equation into an equation with constant coefficients from which one can later on proceed with the analysis. With this in mind we rewrite Eq. (26) as

$$\frac{\partial\psi}{\partial t} = f_1(t)\nu\nabla^2\psi + \frac{1}{2}f_2(t)\lambda(\nabla\psi)^2 + \frac{f_3(t)}{T}\psi + \eta, \quad (28)$$

where we have defined the following dimensionless functions of time:

$$f_1(t) = \frac{1}{\nu} \frac{F(t)p'(\varrho_b(t))}{4\pi G a^2(t)\varrho_b(t)}, \quad (29)$$

$$f_2(t) = \frac{1}{\lambda a(t)}, \quad (30)$$

$$f_3(t) = [F(t) - H(t)]T. \quad (31)$$

The dimensional parameters  $\nu, \lambda, T$  are (in principle) introduced as bookkeeping quantities to carry the dimensions.

Defining a new time coordinate  $\tau$ , a new velocity potential  $\Psi(\mathbf{x}, \tau)$  and a new noise  $\xi(\mathbf{x}, \tau)$  via

$$\tau(t) = \int_{t_0}^t dy f_1(y), \quad \Psi = \frac{f_2}{f_1} \psi, \quad \xi = \frac{f_2}{f_1^2} \eta, \quad (32)$$

allows one to recast this equation into:

$$\frac{\partial \Psi}{\partial \tau} = \nu \nabla^2 \Psi + \frac{\lambda}{2} (\nabla \Psi)^2 - r(\tau) \Psi + \xi, \quad (33)$$

where

$$r(\tau) = \frac{1}{f_1(t(\tau))} \frac{df_1(t(\tau))}{d\tau} - \frac{1}{f_2(t(\tau))} \frac{df_2(t(\tau))}{d\tau} - \frac{1}{T} \frac{f_3(t(\tau))}{f_1(t(\tau))}. \quad (34)$$

The above equation is a “massive” KPZ equation, but with the peculiarity that the coefficient of the term proportional to  $\psi$  (the “mass” term) depends on time: one has a standard KPZ equation if  $r(\tau) = 0$ , time-dependent damping of the surface growth if  $r(\tau) > 0$ , or explosive unstable behavior for  $r(\tau) < 0$ .

Due to the noise term in Eq. (33), the field  $\Psi$  will develop dynamical correlations in addition to those due to the initial conditions. If the nonlinearity were neglected in Eq. (33), the effect of noise would simply be to superimpose Gaussian fluctuations on the deterministic evolution. But the nonlinearity *couples different length scales*, so that the evolution at any given scale also receives a contribution from fluctuations on other scales. Eq. (33) implicitly involves a coarse-grained description and thus a smoothing length scale, so that the nonlinear coupling promotes the phenomenological coefficients  $\nu, \lambda, r$  (and the noise itself) into scale-dependent quantities.

The correlations induced by noise can be obtained by means of the Renormalization Group (Gell-Mann & Low 1954, Binney et al. 1993, Weinberg 1996), via the computation of the scale-dependence of the coefficients. The fact that  $r(\tau)$  depends on time again complicates the application of the Renormalization Group. However, if  $r(\tau)$  happens to be independent of  $\tau$  or if an adiabatic approximation is justified (i.e., if  $r(\tau)$  varies very slowly on the time-scales associated with the dynamical evolution prescribed by (33)), then one can to some degree of approximation neglect the  $\tau$ -dependence of  $r(\tau)$  when applying the Renormalization Group. In Appendix A a detailed study of this question is carried out.

Under the adiabatic assumption, application of the Renormalization Group is straightforward albeit analytically cumbersome, and has been carried out elsewhere (Barbero et al. 1997;

Domínguez et al. 1999). Here we content ourselves by quoting the conclusions. The KPZ equation exhibits self-affine (i.e., invariant under rescaling of position, time, and field) correlations in the large-distance, long-time regime, and the massive KPZ equation, Eq. (33), can also exhibit this behavior. In particular, the equal-time two-point density correlation function obeys in such case the scaling  $\langle \delta(\mathbf{x}, t) \delta(\mathbf{x}', t) \rangle \sim |\mathbf{x} - \mathbf{x}'|^{2\chi-4}$ , where  $\chi$  is the *roughness* exponent, which can be computed by means of the Renormalization Group. This result shows that noise may be relevant in that it can induce the generation of self-affine correlations in a self-gravitating collisionless gas, even if the initial conditions are not self-affine.

## 5. The linear theory revisited

The strongest assumption that we have made has been the parallelism hypothesis (6), motivated by the well-studied dust case, where it is justified in the linear and weakly nonlinear regimes. In this section we will study the linearized version of the system of Eqs. (2-4) to find out how well this assumption is justified in the presence of pressure and noise.

The linearized Newtonian cosmological equations around the unperturbed FL background (i.e.,  $\delta = 0, \mathbf{u} = \mathbf{0}, \mathbf{g} = \mathbf{0}$ ) read:

- Continuity equation:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot \mathbf{u} = 0; \quad (35)$$

- Euler’s equation:

$$\frac{\partial \mathbf{u}}{\partial t} + H \mathbf{u} = \mathbf{g} - \frac{p'(\varrho_b)}{a} \nabla \delta + \mathbf{s}; \quad (36)$$

- Newtonian field equations:

$$\nabla \cdot \mathbf{g} = -4\pi G a \varrho_b \delta, \quad \nabla \times \mathbf{g} = \mathbf{0}. \quad (37)$$

Since the noise  $\mathbf{s}(\mathbf{x}, t)$  has vanishing mean and the equations are linear, the general solution can be split into the sum of the averaged fields  $\langle \delta \rangle, \langle \mathbf{u} \rangle$  and  $\langle \mathbf{g} \rangle$ , obeying the deterministic version of the Eqs. (35-37) (i.e., setting  $\mathbf{s} = \mathbf{0}$ ), plus a fluctuating part which is a linear functional of the noise  $\mathbf{s}$ . The dynamical evolution thus follows a deterministic trajectory with Gaussian fluctuations superimposed on it, but these corrections will be small if the noise intensity is small. The detailed study of this system of equations is performed in Appendix B. The main conclusions for our purposes are the following:

(i) The average vorticity of the peculiar-velocity is damped by the background expansion, even if  $p \neq 0$ , and thus  $\langle \mathbf{u} \rangle$  becomes asymptotically a potential field as  $t \rightarrow +\infty$ .

(ii) The parallelism assumption (6) gets modified due to pressure. When the lowest order correction is included, we find that<sup>6</sup>

$$\langle \mathbf{g} \rangle \approx F_0(t) \langle \mathbf{u} \rangle - \frac{p'(\varrho_i) F_1(t)}{4\pi G a_i^2 \varrho_i} \nabla^2 \langle \mathbf{u} \rangle. \quad (38)$$

<sup>6</sup> The notation and the assumptions leading to this expression are explained in Appendix B.

This equation shows that corrections to parallelism in the average motion are negligible in the limit of small pressure far from caustics (i.e., where  $\langle \mathbf{u} \rangle$  is a smooth field and  $\nabla^2 \langle \mathbf{u} \rangle$  does not diverge). Thus, pressure becomes important in pancakes, where parallelism no longer holds and the assumptions of small and isotropic pressure break down.

## 6. Conclusions

In this paper we have put forward the set of Eqs. (2-4) as a new, phenomenological approach to the problem of LSS formation. The difference with regard to similar approaches lies in the interpretation of these equations as describing the dynamical evolution of the *coarse-grained* fields  $\delta$ ,  $\mathbf{u}$  and  $\mathbf{g}$ , which is the origin of the pressure-like force and of the fluctuations (noise).

We have shown that under the parallelism assumption (6), the pressure force gives rise to a viscous-like force of the same character as that of the adhesion model, which is a successful model on large scales. As is also known in the context of the adhesion model, the limit of vanishing pressure is singular: models with no pressure ( $p = 0$  exactly) are *qualitatively* different from models with  $p \neq 0$ . Therefore, even if pressure is very small (seemingly negligible), one should not set  $p = 0$ .

We have also considered the effects of fluctuations. Under the parallelism assumption, the problem can be cast into a massive KPZ model of surface growth phenomena, which can exhibit self-affine correlations because the noise can be relevant, even if it is vanishingly small. Hence, the limit of vanishing noise is also singular and the same warning as with pressure applies.

Finally, we have explored the plausibility of the parallelism assumption (6) by studying the linearized equations and reached the conclusion that it is justified far from caustics and close to the limit of vanishing pressure and noise. In fact, given the number of assumptions we have made, there is plenty of room to generalize the results presented here: relaxing the parallelism assumption, generalizing the equation of state  $p = p(\varrho)$ , modelling multi-streaming by an anisotropic stress-tensor  $\Pi_{ij}$  in Eq. (3) rather than by a pressure force (see, e.g., Maartens et al. 1999), as well as performing a full Renormalization Group analysis with time-dependent coefficients.

Furthermore, we have given the formal hydrodynamical basis on which to anchor future studies leading to an understanding of various scaling relations found in the context of LSS.

*Acknowledgements.* We would like to thank Herbert Spohn for valuable discussions. Thomas Buchert acknowledges support by the ‘‘Sonderforschungsbereich SFB 375 f ur Astro-Teilchenphysik der Deutschen Forschungsgemeinschaft’’ and thanks LAEFF for hospitality and generous support during working visits in Madrid. Alvaro Dom nguez thanks ‘‘Sonderforschungsbereich SFB 375’’ for support during working visits at the Ludwig-Maximilians-Universit t.

## Appendix A: adiabatic approximation

In this appendix we explore under what conditions the coefficient  $r(\tau)$  in Eq. (33) can be considered approximately time-

independent. Our starting point is Eq. (34):

$$r(\tau) = \frac{1}{f_1(t(\tau))} \frac{df_1(t(\tau))}{d\tau} - \frac{1}{f_2(t(\tau))} \frac{df_2(t(\tau))}{d\tau} - \frac{f_3(t(\tau))}{T f_1(t(\tau))}. \quad (\text{A1})$$

The function  $F$  in the definitions (29-31) obeys Eq. (B14) of Appendix B. Therefore:

$$\frac{1}{f_1} \frac{df_1}{dt} = \frac{4\pi G \varrho_b}{F} - F - 3H \varrho_b \frac{p''(\varrho_b)}{p'(\varrho_b)}, \quad (\text{A2})$$

$$\frac{1}{f_2} \frac{df_2}{dt} = -H. \quad (\text{A3})$$

Inserting these results into Eq. (A1) yields

$$r = \frac{4\pi G \varrho_b a^2}{F p'(\varrho_b)} \left[ \frac{4\pi G \varrho_b}{F} + 2(H - F) - 3H \varrho_b \frac{p''(\varrho_b)}{p'(\varrho_b)} \right]. \quad (\text{A4})$$

On the other hand, one can identify two time scales in Eq. (33): a time scale due to expansion

$$T_{exp} := a \left( \frac{da}{d\tau} \right)^{-1} = \frac{f_1}{H}, \quad (\text{A5})$$

and an intrinsic time scale  $T_{int} := |r|^{-1}$ . If the *relative* variation of the coefficient  $r(\tau)$  on these time scales is small, then an adiabatic approximation is justified and  $r$  can be considered to be a slowly varying function of time. To quantify this statement we define *adiabatic indices* as

$$E_{exp} := T_{exp} \left| \frac{1}{r} \frac{dr}{d\tau} \right| = \left| \frac{1}{Hr} \frac{dr}{dt} \right|, \quad (\text{A6})$$

$$E_{int} := T_{int} \left| \frac{1}{r} \frac{dr}{d\tau} \right| = \left| \frac{1}{f_1 r^2} \frac{dr}{dt} \right|. \quad (\text{A7})$$

The adiabatic approximation then requires  $E_{exp}, E_{int} \ll 1$ . We now discuss it for various background cosmologies.

### A.1. Einstein-de Sitter background ( $K = 0, \Lambda = 0$ )

We first study the simplest case, for which

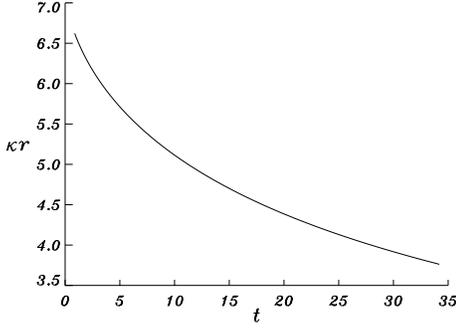
$$a(t) = \left( \frac{t}{t_0} \right)^{2/3}, \quad H(t) = \frac{2}{3t}, \quad \varrho_b(t) = \frac{1}{6\pi G t^2}. \quad (\text{A8})$$

From Eq. (B17) of Appendix B we find that  $F = t^{-1}$ . Taking these expressions into Eq. (A4) then yields

$$r(t) = -\frac{2}{9\pi G t_0^{4/3}} \frac{p''(\varrho_b(t))}{p'(\varrho_b(t))^2} t^{-8/3}. \quad (\text{A9})$$

We are interested in polytropic models  $p = \kappa \varrho^\gamma$ , for which the previous expression becomes

$$r(t) = \frac{4(6\pi G)^\gamma (1 - \gamma)}{3\kappa \gamma t_0^{4/3}} t^{\frac{2}{3}(3\gamma - 4)}. \quad (\text{A10})$$



**Fig. A1.**  $\kappa r$  versus time in an undercritical FL background ( $\Omega_0 < 1$ ).  $\kappa r$  is measured in units of  $10^{-3}(1 - \Omega_0)h^2 \text{ Gyr}^{-2}$ , and time is measured in units of  $\Omega_0(1 - \Omega_0)^{-3/2}h^{-1} \text{ Gyr}$ .

We extract some interesting conclusions: for the isothermal model,  $\gamma = 1$ , we get  $r = 0$  exactly and Eq. (33) reduces to a *bona fide* KPZ equation. For the cosmogenetic model  $\gamma = 4/3$ , we get  $r(t) = r_0 < 0$  independent of time. For other matter models we have

$$E_{exp} = |3\gamma - 4|, \quad E_{int} = \frac{|3\gamma - 4|}{3|\gamma - 1|}. \quad (\text{A11})$$

Then the adiabatic approximation will be justified when  $\gamma$  is close to the cosmogenetic value of  $4/3$ , in which case  $r < 0$ .

#### A.2. Background cosmologies with $K \neq 0$ , $\Lambda = 0$

In this case we can also obtain explicit expressions for the coefficient  $r$  and the indices  $E_{exp}$ ,  $E_{int}$  analytically, but the algebra is rather involved and the final expressions are best studied numerically. We therefore skip the algebra and go right to the final results, particularized to the isothermal model  $p = \kappa \varrho$ , because this leads to the simplest expression in Eq. (A4).

In an undercritical universe ( $K < 0$ ) the coefficient  $r$  is a positive, monotonically decreasing function of time. In Fig. A1 we plot  $\kappa r$  versus  $t^7$ . The adiabatic indices are monotonically increasing functions of time but they are bounded:

$$0 \leq E_{exp} \leq 1, \quad \frac{32}{75} \leq E_{int} \leq \frac{1}{2}. \quad (\text{A12})$$

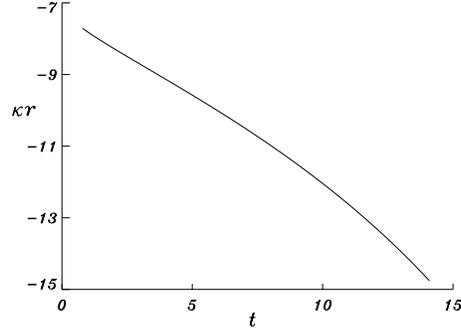
In the range  $0.2 \lesssim \Omega_0 \leq 1$  and at epochs  $t \lesssim 10h^{-1} \text{ Gyr}$  one gets  $0 \leq E_{exp} \lesssim 0.35$ . Although the adiabatic indices are not much smaller than one, one can still work under the adiabatic assumption to a first approximation.

In an overcritical universe ( $K > 0$ ) the coefficient  $r$  is a negative, decreasing function of time. In Fig. A2 we plot  $\kappa r$  versus  $t$ . The index  $E_{int}$  decreases in time and is bounded:

$$\frac{2\pi^2(9\pi^2 - 64)}{(3\pi^2 - 64)^2} \leq E_{int} \leq \frac{32}{75}. \quad (\text{A13})$$

The index  $E_{exp}$  increases with time and is unbounded: it diverges at the epoch of maximum expansion because then the

<sup>7</sup> We have introduced Hubble's constant at the present epoch,  $H_0 = 0.1h \text{ Gyr}^{-1}$ , and the density parameter,  $\Omega_0 = 8\pi G \varrho_0 / 3H_0^2$ .



**Fig. A2.**  $\kappa r$  versus time in an overcritical FL background ( $\Omega_0 > 1$ ) up to the epoch of maximum expansion.  $\kappa r$  is measured in units of  $10^{-3}(\Omega_0 - 1)h^2 \text{ Gyr}^{-2}$ , and time is measured in units of  $\Omega_0(\Omega_0 - 1)^{-3/2}h^{-1} \text{ Gyr}$ .

expansion rate  $H$  vanishes. This is, however, physically uninteresting. At epochs  $t \lesssim 10h^{-1} \text{ Gyr}$  and with the conservative bound  $\Omega_0 \lesssim 2$ , one finds  $E_{int} \lesssim 0.5$  and the same remarks apply as in the undercritical universe.

Hence we conclude that within the physically interesting range of values of the parameters, an adiabatic approximation for the coefficient  $r$  can be justified.

#### Appendix B: linear regime

In this appendix we study the linear regime when pressure is taken into account. The linearized system of Eqs. (35-37) can be more easily studied in Fourier space. We therefore define the Fourier transformation (assuming a box of volume  $\mathcal{V}$ ):

$$\begin{aligned} \delta(\mathbf{x}, t) &= \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} \tilde{\delta}(\mathbf{k}, t) e^{-i\mathbf{k} \cdot \mathbf{x}} \Rightarrow \\ \Rightarrow \tilde{\delta}(\mathbf{k}, t) &= \int_{\mathcal{V}} d^3\mathbf{x} \delta(\mathbf{x}, t) e^{i\mathbf{k} \cdot \mathbf{x}}, \end{aligned} \quad (\text{B1})$$

and similar definitions for  $\tilde{\mathbf{u}}(\mathbf{k}, t)$ ,  $\tilde{\mathbf{g}}(\mathbf{k}, t)$  and  $\tilde{\mathbf{s}}(\mathbf{k}, t)$ . In Fourier space the linearized Eqs. (35-37) read:

$$\frac{\partial \tilde{\delta}}{\partial t} - \frac{i}{a} \mathbf{k} \cdot \tilde{\mathbf{u}} = 0; \quad (\text{B2})$$

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + H \tilde{\mathbf{u}} = \tilde{\mathbf{g}} + i \frac{p'(\varrho_b)}{a} \tilde{\delta} \mathbf{k} + \tilde{\mathbf{s}}; \quad (\text{B3})$$

$$i\mathbf{k} \cdot \tilde{\mathbf{g}} = 4\pi G a \varrho_b \tilde{\delta}, \quad \mathbf{k} \times \tilde{\mathbf{g}} = \mathbf{0}. \quad (\text{B4})$$

We decompose the velocity field into transversal  $\tilde{\mathbf{u}}_{\perp}$  (satisfying  $\mathbf{k} \cdot \tilde{\mathbf{u}}_{\perp} = 0$ ) and longitudinal components  $\tilde{\mathbf{u}}_{\parallel}$  (satisfying  $\mathbf{k} \times \tilde{\mathbf{u}}_{\parallel} = \mathbf{0}$ ), and do the same with the fields  $\tilde{\mathbf{g}}$  and  $\tilde{\mathbf{s}}$ . From Eq. (B4) one finds that  $\tilde{\mathbf{g}}_{\perp} = \mathbf{0}$  and then from Euler's equation (B3) that:

$$\begin{aligned} \frac{\partial \tilde{\mathbf{u}}_{\perp}}{\partial t} + H \tilde{\mathbf{u}}_{\perp} &= \tilde{\mathbf{s}}_{\perp} \Rightarrow \\ \Rightarrow \tilde{\mathbf{u}}_{\perp}(\mathbf{k}, t) &= \frac{1}{a(t)} [\tilde{\mathbf{u}}_{\perp}(\mathbf{k}, t_0) + \int_{t_0}^t d\tau a(\tau) \tilde{\mathbf{s}}(\mathbf{k}, \tau)] \end{aligned} \quad (\text{B5})$$

Therefore, the average transversal component of the peculiar-velocity is damped by the background expansion even in the presence of pressure (this is not surprising, since pressure does not generate vorticity for barotropic fluids, i.e.,  $p = p(\varrho)$ ), and only its fluctuations grow due to the noise. The longitudinal component is obtained from the continuity equation (B2), while the gravitational acceleration is found from Poisson's equation (B4):

$$\tilde{\mathbf{u}}_{\parallel} = -\frac{ia}{k^2} \frac{\partial \tilde{\delta}}{\partial t} \mathbf{k}, \quad \tilde{\mathbf{g}} = -\frac{4\pi G i a \varrho_b}{k^2} \tilde{\delta} \mathbf{k}. \quad (\text{B6})$$

The elimination of the velocity and gravitational acceleration fields in favor of the density contrast field by means of these expressions yields a closed equation for each mode  $\tilde{\delta}(\mathbf{k}, t)$  of the density contrast:

$$\frac{\partial^2 \tilde{\delta}(\mathbf{k}, t)}{\partial t^2} + 2H \frac{\partial \tilde{\delta}(\mathbf{k}, t)}{\partial t} - U_k \tilde{\delta}(\mathbf{k}, t) = \frac{1}{a(t)} i\mathbf{k} \cdot \tilde{\mathbf{s}}_{\parallel}(\mathbf{k}, t), \quad (\text{B7})$$

with

$$U_k := 4\pi G \varrho_b - \frac{k^2}{a^2} p'(\varrho_b), \quad (\text{B8})$$

It is clear from this equation that the noise will induce correlations on the density field on top of those that may already be present in the initial condition. In the linear regime, this superposition is linear and weak noise will only induce small Gaussian fluctuations of  $\tilde{\delta}$  about the deterministic solution. Hence, we can restrict our study to the evolution of the average field,  $\langle \tilde{\delta} \rangle$ , which follows the deterministic evolution (i.e., Eq. (B7) dropping the noise). This evolution is, however, different from the dust case because of the pressure term. In the rest of the appendix we drop the noise source from this equation and understand that  $\tilde{\delta}$  means  $\langle \tilde{\delta} \rangle$ , and the same with the other fields.

We are particularly interested in the justification of the parallelism condition, Eq. (6). From Eqs. (B6) one can easily find a relationship between  $\tilde{\mathbf{u}}_{\parallel}$  and  $\tilde{\mathbf{g}}$ :

$$\tilde{\mathbf{g}}(\mathbf{k}, t) = \tilde{F}(\mathbf{k}, t) \tilde{\mathbf{u}}_{\parallel}, \quad (\text{B9})$$

where

$$\tilde{F}(\mathbf{k}, t) = 4\pi G \varrho_b(t) \tilde{\delta}(\mathbf{k}, t) \left( \frac{\partial \tilde{\delta}(\mathbf{k}, t)}{\partial t} \right)^{-1}. \quad (\text{B10})$$

The equation satisfied by  $\tilde{F}$  can be easily found from Eq. (B7):

$$\frac{\partial \tilde{F}}{\partial t} = 4\pi G \varrho_b - H \tilde{F} - \frac{U_k}{4\pi G \varrho_b} \tilde{F}^2, \quad (\text{B11})$$

(i.e., a Riccati equation). There is parallelism in position space between  $\mathbf{u}$  and  $\mathbf{g}$  if  $\tilde{F}$  does not depend on  $\mathbf{k}$ ; a necessary condition is that  $U_k$  be also  $k$ -independent (i.e., there is no pressure). Therefore, not surprisingly, pressure destroys parallelism, but the deviations decrease as one goes to larger scales.

The exact solution to Eq. (B11) is difficult to obtain. For polytropic models in an Einstein–de Sitter background ( $K = 0$ ,  $\Lambda = 0$ ) one can solve Eq. (B7) in terms of Meijer's G-functions

(Haubold et al. 1991) and then compute  $\tilde{F}$  from its definition, Eq. (B10). This procedure is, however, algebraically cumbersome and not very illuminating from the physical point of view (for some particular values of the polytropic index  $\gamma$  however the solution can be written in terms of elementary functions).

For the purposes of the work presented here, we adopt a different approach. We first define a (comoving) Jeans' length  $L_J$  by the condition  $U_{L_J^{-1}} = 0$ , yielding

$$L_J := \left[ \frac{p'(\varrho_b)}{4\pi G a^2 \varrho_b} \right]^{1/2}. \quad (\text{B12})$$

From Eq. (B7) one can easily grasp the physical meaning of this quantity: density perturbations with  $k > L_J^{-1}$  are damped in the linear regime by both pressure and expansion, while those with  $k < L_J^{-1}$  are damped only by expansion (self-gravity dominates over pressure).

In the limit of small pressure,  $L_J \rightarrow 0$ , and the pressure can be considered a perturbation on scales  $k \ll L_J^{-1}$ . Defining the small parameter  $\varepsilon := L_i^2 k^2$  (where  $L_i$  is Eq. (B12) evaluated at some initial time  $t_i$ ), we can write

$$\frac{U_k}{4\pi G \varrho_b} = 1 - \varepsilon S(t), \quad S(t) := \frac{L_J^2}{L_i^2} = \frac{a}{a_i} \frac{p'(\varrho_b)}{p'(\varrho_i)}, \quad (\text{B13})$$

and then carry out a perturbative expansion in powers of  $\varepsilon$ . In particular, for polytropic models one has  $S(t) = [a(t)/a_i]^{4-3\gamma}$ , that is, the perturbative expansion is better at later times if  $\gamma > 4/3$ . It should also be noticed that the correction due to pressure at a fixed time becomes smaller as the polytropic index  $\gamma$  grows.

We therefore write  $\tilde{F}(\mathbf{k}, t) = \sum_{n=0}^{\infty} \varepsilon^n F_n(\mathbf{k}, t)$  with the initial conditions  $F_0(\mathbf{k}, t_i) = \tilde{F}(\mathbf{k}, t_i)$  and  $F_n(\mathbf{k}, t_i) = 0$ ,  $n > 0$ . From Eq. (B11) one finds

$$\frac{\partial F_0}{\partial t} = 4\pi G \varrho_b - H F_0 - F_0^2, \quad (\text{B14})$$

$$\frac{\partial F_1}{\partial t} = -(H + 2F_0)F_1 + S F_0^2, \quad (\text{B15})$$

⋮

$F_0$  obeys the equation of the well-studied pressureless case (Peebles 1980). As  $t \rightarrow \infty$  it becomes independent of  $k$  (i.e., it forgets the initial conditions) and there is parallelism in position space. The asymptotic form of  $F_0$  is to be identified with  $F(t)$  in the parallelism assumption (Eq. [6] in the main text), and turns out to be given as  $F_0 = 4\pi G \varrho_b b/\dot{b}$ , where  $b(t)$  is the *growing mode* solution of Eq. (B7) when particularized to the dust case,  $p = 0$ .  $F_1$  can be computed once  $F_0$  is known:

$$F_1(\mathbf{k}, t) = \int_{t_i}^t d\tau S(\tau) F_0^2(\mathbf{k}, \tau) e^{-\int_{\tau}^t d\theta [H(\theta) + 2F_0(\mathbf{k}, \theta)]}. \quad (\text{B16})$$

Notice that the whole dependence of  $F_1$  on  $\mathbf{k}$  stems from  $F_0$  (and hence from initial conditions, which is also true for every  $F_n$ ). The study of this expression can be carried out analytically for polytropic models,  $p = \kappa \varrho^\gamma$ , in an Einstein–de Sitter background. In this case the function  $F_0$  is

$$F_0(\mathbf{k}, t) = \frac{t^{5/3} + 2A(\mathbf{k})}{t^{8/3} - 3A(\mathbf{k})t}, \quad (\text{B17})$$

with  $A(\mathbf{k})$  given by the initial conditions. For polytropic models the function  $S(t)$  comes down to the simple expression  $S(t) = (t/t_i)^{8/3-2\gamma}$ . Inserting these results into Eq. (B16) then yields

$$F_1(\mathbf{k}, t) = \frac{t_i^{-\frac{8}{3}+2\gamma} t^{2/3}}{[t^{5/3} - 3A(\mathbf{k})]^2} \left[ \frac{3}{13 - 6\gamma} (t^{\frac{13}{3}-2\gamma} - t_i^{\frac{13}{3}-2\gamma}) + \frac{12A(\mathbf{k})}{8 - 6\gamma} \left( t^{\frac{8}{3}-2\gamma} - t_i^{\frac{8}{3}-2\gamma} \right) + \frac{A(\mathbf{k})^2}{1 - 2\gamma} (t^{1-2\gamma} - t_i^{1-2\gamma}) \right], \quad (\text{B18})$$

with the understanding that terms like  $(t^a - t_i^a)/a$  should be substituted by  $\log(t/t_i)$  when  $a = 0$ .

As  $t \rightarrow +\infty$ , one has the asymptotic behaviors  $F_0(\mathbf{k}, t) \rightarrow t^{-1}$  and

$$F_1(\mathbf{k}, t) \rightarrow \frac{3}{(13 - 6\gamma)t_i} \left( \frac{t}{t_i} \right)^{\frac{5}{3}-2\gamma}, \quad (\gamma \neq \frac{13}{6}), \quad (\text{B19})$$

$$F_1(\mathbf{k}, t) \rightarrow \frac{1}{t_i} \left( \frac{t}{t_i} \right)^{-\frac{8}{3}} \log \left( \frac{t}{t_i} \right), \quad (\gamma = \frac{13}{6}). \quad (\text{B20})$$

It must be noted that both  $F_0$  and  $F_1$  forget initial conditions and thus become asymptotically  $k$ -independent, so that Eq. (B9) yields

$$\tilde{\mathbf{g}}(\mathbf{k}, t) = [F_0(t) + k^2 L_i^2 F_1(t) + o(k^2 L_i^2)] \tilde{\mathbf{u}}_{\parallel}(\mathbf{k}, t). \quad (\text{B21})$$

Assuming that the velocity field  $\mathbf{u}$  is smooth (and in particular not vertical or multi-valued), such that  $\tilde{\mathbf{u}}$  decays fast enough when  $k \rightarrow \infty$ , we can Fourier transform this expression back to position space and thus obtain the following relationship between gravitational acceleration and the potential component of the velocity,

$$\mathbf{g} \approx F_0(t) \mathbf{u}_{\parallel} - L_i^2 F_1(t) \nabla^2 \mathbf{u}_{\parallel}, \quad (\text{B22})$$

which is the result quoted in Sect. 5.

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