

Shear layer instability in a highly diffusive stably stratified atmosphere

F. Lignières^{1,2}, F. Califano³, and A. Mangeney¹

¹ Département de Recherche Spatiale et Unité de Recherche associée au CNRS 264, Observatoire de Paris-Meudon, F-92195 Meudon Cedex, France

² Astronomy Unit, Queen Mary and Westfield College, Mile End Road, London E14NS, UK

³ Istituto Nazionale Fisica della Materia, Sez. A, Università di Pisa, Italy

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Abstract. The linear stability of a shear layer in a highly diffusive stably stratified atmosphere has been investigated. This study completes and extends previous works by Dudis (1974) and Jones (1977). We show that (i) an asymptotic regime is reached in the limit of large thermal diffusivity, (ii) there exists two different types of unstable modes. The slowest growing modes correspond to predominantly horizontal motions and the fast growing ones to isotropic motions, (iii) the physical interpretation of the stability properties in this asymptotic regime is simple. Applications to the dynamics of stellar radiative zone are discussed.

Key words: instabilities – hydrodynamics – stars: interiors

1. Introduction

The standard model describing stellar evolution neglects the role of macroscopic motions in the radiative layers of stars. During the past twenty years, however, determinations of surface chemical abundances and helioseismology data have shown that some mechanical mixing must be invoked to account for the observations. Current mixing theories involve either motions induced by the rotation or gravity waves generated at the boundary with the convective zones.

Hydrodynamical instabilities due to differential rotation play an essential role in this context. Turbulent motions arise through the non-linear development of hydrodynamical instabilities and whenever present, they ensure an efficient mixing of chemical and angular momentum. The study of instabilities is therefore necessary to determine the occurrence of turbulent mixing in stellar radiative zones. Moreover, in the absence of a better understanding of turbulent motions, the properties of the instability are often used to infer the properties of the subsequent turbulent motions.

In a rotating fluid of uniform density, a hydrodynamical instability occurs if the specific angular momentum decreases outwards the rotation axis or if the rotational velocity varies along the rotation axis. Instability can also be triggered by a

velocity shear, a typical example being the Kelvin-Helmholtz instability which develops at the interface between two streams of different velocities (see for example the reference books by Chandrasekhar 1961 and Drazin & Reid 1981). In principle, these different types of instability can also take place inside stellar radiative zones. But there, the stable stratification will strongly oppose to them. Indeed, the time scale characterizing the action of the restoring buoyancy force is generally much smaller than the dynamical time scale Ω^{-1} (the inverse of the Brunt-Väisälä frequency is of the order of 10^3 s whereas Ω^{-1} is equal to $10^{5.5}$ s for the sun and is of the order of 10^4 s for a fast rotating star with an equatorial velocity $v_{\text{eq}} = 200\text{km s}^{-1}$ and a radius $R = 3R_{\odot}$).

This does not mean that instabilities are systematically suppressed by stable stratification. But, as we shall see, this puts strong constraints on the type of motions which can arise from these instabilities.

A possibility to avoid the action of the buoyancy force is to consider displacements limited to the surfaces of constant entropy. Such purely two-dimensional motions are not affected by the buoyancy force and might be amplified if a differential rotation exists along these surfaces. However, even if the early development of the instability may not be affected by the stabilizing buoyancy force, the non-linear stage should produce three-dimensional motions which in turn will be affected by the stable stratification.

Another possibility is to consider three-dimensional perturbations whose vertical length scale is small enough to be significantly affected by the thermal diffusion. Thermal diffusion has indeed a destabilizing effect in stably stratified atmospheres as was first recognized in a geophysical context by Townsend (1958). When a fluid parcel is displaced vertically from its hydrostatic equilibrium position, thermal diffusion reduces the temperature departure between the fluid parcel and its environment. For incompressible motions, the density departure and so the amplitude of the restoring buoyancy force are reduced by the same factor. Thermal diffusion has therefore a destabilizing effect because it weakens the stabilizing effect of the stable stratification.

In the present paper, we investigate in details this destabilizing effect in the case of shear instabilities. Our work follows

Send offprint requests to: F. Lignières

Correspondence to: F.Lignieres@qmw.ac.uk

previous theoretical studies which we summarize now. Let us first recall that, in the absence of diffusive and viscous effects, there exists a criterion, derived in the context of the linear stability theory and valid for any parallel flows $\mathbf{U} = U(z)\mathbf{e}_x$ embedded in some vertically stably stratified Boussinesq atmosphere (z is the vertical coordinate and \mathbf{e}_x a horizontal unit vector). This criterion established by Miles (1961) states that the flow is stable to infinitesimal perturbations if the Richardson number defined as,

$$Ri = \left(\frac{N}{\frac{dU}{dz}} \right)^2, \quad (1)$$

is everywhere larger than $1/4$.

Such a criterion does not exist in the presence of thermal diffusion. Existing results consist either in general criteria based on phenomenological arguments or applications of the linear stability theory to a particular flow i.e. a particular velocity profile within a particular atmosphere.

The general criterion derived by Zahn (1974) states that the shear layer is stable if,

$$RiPe > \frac{1}{4}, \quad (2)$$

where $Pe = ul/\kappa$ is a Péclet number associated with an eddy of size l and velocity u . As expected, perturbations that would be stable in a perfect fluid because $Ri > 1/4$ can now be unstable provided $Pe < 1/(4Ri)$, i.e. if the thermal diffusion time scale of the eddy is small enough compared to its turnover time scale l/u . More recently, Maeder (1995) obtained a similar expression by extending to the diffusive case the energy considerations which allowed Chandrasekhar's derivation of Miles' criterion (Chandrasekhar 1961, see also Miles 1986).

In the context of the linear stability theory, thermal diffusion effects have been studied independently by Dudis (1974) and Jones (1977). Both authors have chosen the hyperbolic-tangent velocity profile, $U(z) = \Delta U \tanh(z/L)$, whose instability properties are well-known in the unstratified case (Drazin & Reid 1981). However they made different choices for the temperature profile characterizing the stable stratification.

Dudis considered a hyperbolic-tangent profile for the temperature $T(z) = T_0 + \Delta T \tanh(z/L)$. He found in particular that, for small enough Péclet numbers, the stability depends only on the product of the Richardson and Péclet number $RiPe$ and not on Ri and Pe separately, this result being consistent with the criterion proposed by Zahn (1974) (here $Ri = (NL)^2/\Delta U^2$ and $Pe = \Delta UL/\kappa$). However, it should be noticed that the thermal stratification chosen by Dudis is not adapted to investigate the small Péclet number regime. Actually, in this regime, the diffusion of the hyperbolic-tangent temperature profile is much faster than the growth of the instability, and this invalidates a starting assumption of this type of instability study, namely that the basic state is a stationary solution of the governing equations or, at least, can be considered as such during the development of the instability. The minimum growth time of the Kelvin-Helmholtz instability (i.e. the inverse of the maximum growth

rate) is known to be $\simeq 5L/\Delta U$ whereas the thermal diffusion time scale of the temperature profile is $t_\kappa = L^2/\kappa$. Thus, if the Péclet number $Pe = \Delta UL/\kappa$ is smaller than unity, the basic temperature profile would have been diffused much before the instability had time to grow.

To avoid this problem, one must consider a stationary solution of the heat equation and, for a Boussinesq fluid, this implies a temperature profile varying linearly with altitude. Jones (1977) chose such a profile and then studied the stability in the range $1/4 \leq Ri \leq 1$ as well as in the limit of very large horizontal wavelengths of the disturbance, k_x , and very large Richardson number. The stability criterion in this double limit is $k_x Pe Ri > 0.086$. However, the partial coverage of the parameter space (k_x, Pe, Ri) did not allow to investigate the small Péclet number regime, and in particular, the existence of the asymptotic state suggested by Dudis (1974).

Here we present a normal mode stability analysis of a hyperbolic-tangent shear layer embedded in a linearly stably stratified Boussinesq atmosphere, with the objective of completing the work of Dudis (1974) and Jones (1977). First, we determine the conditions of marginal stability for a much larger range of Richardson numbers, Péclet numbers and Reynolds numbers. Then, we concentrate on the small Péclet number regime and investigate for the first time the growth rates of the unstable modes as well as their spatial properties. We also provide simple physical interpretation of the stability properties by comparing the relative effects of shear, stable stratification and thermal diffusion. Lignières (1999) recently derived an asymptotic form of the Boussinesq equations in the limit of small Péclet number and showed that, in this asymptotic regime, thermal diffusion and stable stratification combine in a single process. We shall see that this property greatly simplifies the interpretation of our results.

The paper is organized as follows. The equations governing the evolution of perturbations are derived in Sect. 2.1 and the numerical method to solve the corresponding eigenvalue problem is presented in Sect. 2.2. Results are presented and interpreted in Sect. 3. In Sect. 4, they are summarized and discussed in an astrophysical context.

2. The mathematical model

2.1. Governing equations

From now on, dimensional quantities are denoted with an overbar. We consider a parallel flow $\bar{\mathbf{U}} = \bar{\Delta U} \tanh(\bar{z})\mathbf{e}_x$ in an unbounded vertically stratified atmosphere. The \bar{z} axis refers to the direction of gravity, while the \bar{x} axis corresponds to the stream direction. The equation of state is that of a Boussinesq fluid, $\bar{\rho} = \bar{\rho}_0(1 - \bar{\beta}(\bar{T} - \bar{T}_0))$ and the thermal stratification is given by the conductive profile $\bar{T}_{\text{eq}}(\bar{z}) = \bar{T}_0 + \bar{\Delta T}\bar{z}/\bar{L}$. Here $\bar{\beta}$ is the thermal expansion coefficient and, $\bar{\rho}_0$ and \bar{T}_0 are reference density and temperature.

Using, \bar{L} , $\bar{\Delta U}$ and $\bar{\Delta T}$ as units of length, velocity and temperature, respectively, the non-dimensional Boussinesq equations (Spiegel & Veronis 1960) read:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + Ri\theta \mathbf{e}_z + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad (3)$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta + w = \frac{1}{Pe} \nabla^2 \theta, \quad (4)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (5)$$

where, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$ is the velocity vector, p the pressure and $\theta(x, y, z) = T(x, y, z) - T_{\text{eq}}$ the temperature deviation from the temperature profile $T_{\text{eq}}(z)$. In the heat equation, the third term of the left hand side corresponds to the vertical advection of temperature against the mean temperature gradient $dT_{\text{eq}}(z)/dz$ which is equal to unity in our dimensionless units.

The system is governed by three non-dimensional numbers, the Richardson number, Ri , the Reynolds number, Re , and the Péclet number, Pe , respectively defined as

$$Ri = \left(\frac{\bar{N}\bar{L}}{\Delta\bar{U}} \right)^2, \quad Re = \frac{\Delta\bar{U}\bar{L}}{\bar{\nu}}, \quad Pe = \frac{\Delta\bar{U}\bar{L}}{\bar{\kappa}},$$

where, $N = (\bar{\beta}\bar{g}\Delta\bar{T}/\bar{L})^{1/2}$ is the Brunt-Väisälä frequency associated with the conductive temperature profile, $\bar{\nu}$ the molecular viscosity and $\bar{\kappa}$ the thermal diffusivity. These non-dimensional numbers compare the dimensional time scales associated with the dynamics $\bar{t}_S = \bar{L}/\Delta\bar{U}$, the stable stratification $\bar{t}_N = 1/\bar{N}$, the viscous dissipation $\bar{t}_\nu = \bar{L}^2/\bar{\nu}$, and the thermal diffusion $\bar{t}_\kappa = \bar{L}^2/\bar{\kappa}$.

Due to the horizontal homogeneity, perturbations are resolved into modes of the form,

$$w(x, y, z) = \Re(\hat{w}(z)e^{i(k_x x + k_y y - k_x c t)}), \quad (6)$$

where $\Re()$ denotes the real part of a complex number, $\hat{w}(z)$ is the vertical profile of the mode (here the vertical velocity), k_x and k_y are the horizontal wavenumbers, and $c = c_r + ic_i$ is a complex number which real part c_r is the phase velocity and which imaginary part c_i controls the growth or the decay of the mode. If c_i is positive, $\gamma = k_x c_i$ is the growth rate of the unstable mode. Since two-dimensional perturbations contained in the plane (x, z) are more unstable than three-dimensional ones (Gage & Reid 1968), the spanwise velocity and the wavenumber k_y can be taken equal to zero. Previous investigations on the stability of parallel shear flow indicate that the most unstable modes have symmetry properties according to the symmetry of the basic velocity profile. Here, due to the antisymmetry of the basic profiles with respect to $z = 0$, we can assume that $\hat{w}(z) = \hat{w}^*(-z)$ and $\hat{\phi}(z) = \hat{\phi}^*(-z)$, where $\hat{\phi}^*$ is the complex conjugate of $\hat{\phi}$. This implies that $c_r = 0$. After linearization and some simple manipulations, one gets the two equations,

$$\left[\frac{M^2}{Re} - ik_x(U - c)M + ik_x U'' \right] \hat{w} = Ri k_x^2 \hat{\phi}, \quad (7)$$

$$\left[\frac{M}{Pe} - ik_x(U - c) \right] \hat{\phi} = \hat{w}, \quad (8)$$

where $\hat{\phi}$ is the vertical profile of temperature disturbance and M denotes the operator $\frac{d^2}{dz^2} - k_x^2$. The functions U and U'' are

the basic velocity profile and its second derivative, respectively. Together with the boundary conditions

$$\hat{w}, \hat{\phi} \rightarrow 0 \text{ as } z \rightarrow \pm\infty, \quad (9)$$

Eqs. (7) and (8) form an eigenvalue problem, denoted (E1), which has been solved numerically.

The eigenvalue problem depends on five parameters k_x, c_i, Ri, Pe, Re and may be written as $\mathcal{F}(k_x, c_i, Ri, Re, Pe) = 0$. The hypersurface of marginal stability separating stable and unstable domains in the parameter space is given by the condition $c_i = 0$.

2.2. Numerical model

The linear system of ordinary differential equations (7) and (8) is solved with a numerical code that makes use of a finite-differences technique with deferred correction coupled to a Newton iteration. The step size can be dynamically refined in the regions where the eigenfunctions present strong spatial gradients.

Boundary conditions are applied at some distance z_{max} from the shear layer. Then, we increase z_{max} until the eigenvalues do no longer depend on the boundary conditions. Depending on the particular eigenfunction, $z_{\text{max}} \approx 20$ or 30 appeared to be sufficient to obtain an accurate solution. In the inviscid limit, the differential equations are singular at $z = 0$ for neutral modes ($c_i = 0$). Following Dudis (1974), we overcame this difficulty by solving the non-singular problem for $c_i \neq 0$ and then by decreasing c_i until the first two digits of the eigenvalue are not affected by the decay of c_i . These procedures have been successfully tested by recovering Dudis results.

3. Results

We first present an overall picture of the stability properties for a wide range of Richardson and Péclet numbers, restricting ourselves to the inviscid case (Sect. 3.1). Then we concentrate on the regime of small Péclet number (Sect. 3.2). The growth rates and the spatial structure of the unstable modes are studied in this regime (Sect. 3.3) as well as the effect of a large Reynolds number (Sect. 3.4).

3.1. Overall presentation of the stability domains in the inviscid case

The marginal stability curves are shown in Fig. 1 for a wide range of Péclet numbers: $Pe = +\infty$, $Pe = 1$, $Pe = 10^{-1}$, $Pe = 10^{-2}$ and $Pe = 10^{-3}$. For any fixed value of Pe , the right hand side of the marginal curve corresponds to stable perturbations whereas the left hand side corresponds to unstable ones. The $Pe = +\infty$ curve is the analytical solution $Ri_c = k_x^2(1 - k_x^2)$ determined by Drazin (1958) (see also Chandrasekhar, 1961) in the non-diffusive limit. Each of the other curves has been drawn from about thirty numerically computed triplets (k_x, Ri, Pe) between $k_x = 0.01$ and $k_x = 1$.

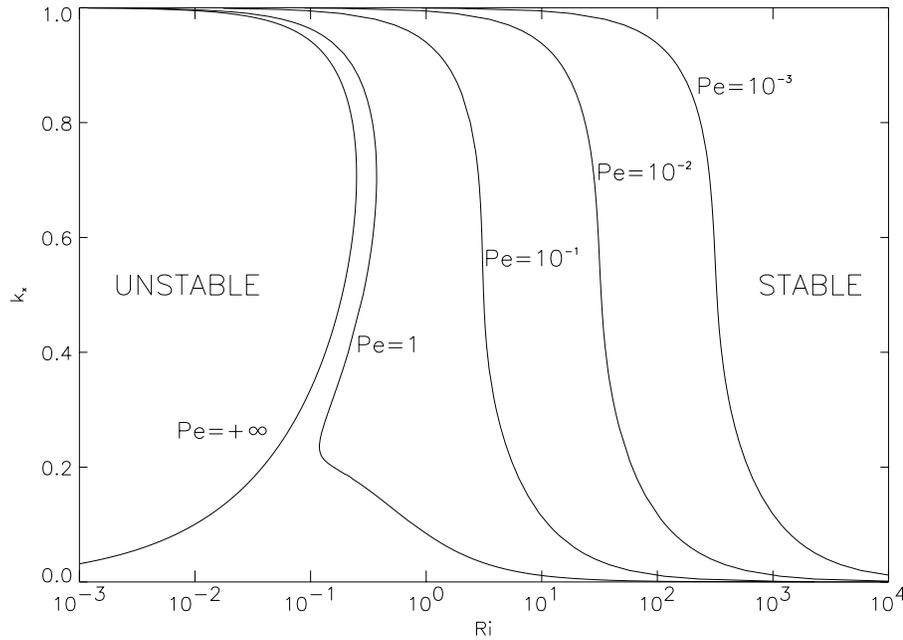


Fig. 1. Curves of marginal stability separating the stable and unstable domains in the plane (k_x, Ri) , for different values of the Péclet number

It can be seen in Fig. 1 that a large thermal diffusivity has a destabilizing effect since the unstable domain increases as the Péclet number decreases. This is easily explained as follows: When buoyancy is negligible, the temperature behaves like a passive scalar and thermal diffusion has no effect on the dynamics. Then, a first condition for a significant effect of thermal diffusion is that the flow be affected by the stable stratification, that is $\bar{t}_N < \bar{t}_S$ or equivalently $Ri > 1$. The second condition is that thermal diffusion reduces the stabilizing effect of the stratification. This will occur if thermal diffusion acts faster than stable stratification, that is if $\bar{t}_\kappa < \bar{t}_N$. Consequently the necessary conditions for a significant effect of the thermal diffusion are,

$$\bar{t}_\kappa < \bar{t}_N < \bar{t}_S. \quad (10)$$

If one considers a shear layer characterized by a Richardson number close to unity i.e. such that $\bar{t}_N \simeq \bar{t}_S$, condition (10) states that thermal diffusion will reduce the stabilizing effect of stable stratification if $\bar{t}_\kappa < \bar{t}_N \simeq \bar{t}_S$ that is if $Pe < 1$. Now, if the shear layer is very strongly stratified that is $Ri \gg 1$, or $\bar{t}_N \ll \bar{t}_S$, condition (10) requires that $\bar{t}_\kappa \ll \bar{t}_S$ or $Pe \ll 1$ to destabilize the layer. These simple arguments account for the mean displacement of the neutral curves towards larger Richardson number, starting from the $Pe = 1$ curve. Thus, condition (10) can be said to explain the global behaviour of the stability picture shown in Fig. 1.

However, another point apparent in Fig. 1 is that large horizontal wavelengths are stable for $Pe = +\infty$ but become unstable as Pe decreases. They become even the most unstable modes at fixed values of Pe ! This property is harder to understand because thermal diffusion acts primarily on small scale perturbations.

We give here a partial answer by showing how perturbations with large horizontal wavelength can be strongly affected by thermal diffusion: Again, in order to weaken the stable strat-

ification, thermal exchanges must act before stable stratification. This condition has been previously expressed by $\bar{t}_\kappa < \bar{t}_N$ but this inequality does not take into account the anisotropy of the buoyancy force. Actually, the effect of stable stratification tends to vanish when motions become more and more horizontal. By contrast, thermal diffusion remains efficient for these motions since predominantly horizontal motions must vary on a relatively small vertical scale to ensure mass conservation ($\nabla \cdot \mathbf{u} = 0$). Therefore, thermal diffusion tends to be more efficient than stable stratification for predominantly horizontal motions.

This reasoning can be formalized by considering infinitesimal perturbations of the form $w \propto \exp[i(k_x x + k_z z)]$ in a linearly stratified atmosphere at rest. The stable stratification time scale then corresponds to the inverse of gravity waves frequency,

$$t_N(k_x, k_z) = \sqrt{k_x^2 + k_z^2} / k_x \sqrt{Ri}, \quad (11)$$

whereas the thermal diffusion time scale is simply,

$$t_\kappa(k_x, k_z) = Pe / (k_x^2 + k_z^2). \quad (12)$$

Comparing both time scales shows that, whatever the value of the Péclet number, thermal diffusion acts much faster than stable stratification when $|k_x/k_z| = |w/u|$ goes to zero, that is when motions are predominantly horizontal.

To show that the motions associated with the marginal modes at large horizontal wavelengths are in this case, we calculated the square root of the ratio between the vertical and horizontal kinetic energy of the modes. The ratio is given by E_w/E_u where,

$$E_w = \frac{\int_{-\infty}^{+\infty} \int_0^{2\pi/k_x} w^2 dx dz}{\int_{-\infty}^{+\infty} \int_0^{2\pi/k_x} (w^2 + u^2) dx dz}, \quad (13)$$

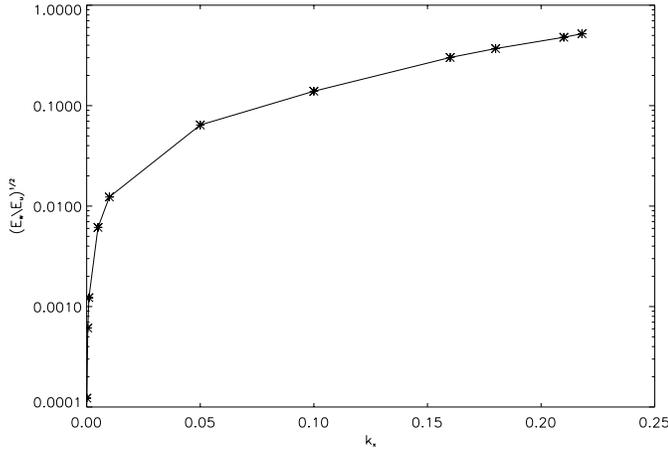


Fig. 2. Square root of the ratio between the vertical and horizontal kinetic energy of neutral modes taken along the $Pe = 1$ marginal stability curve. This figure shows that the motions associated with these modes become more and more horizontal as k_x goes to zero

and E_u is defined accordingly. It has been determined for marginal modes taken along the $Pe = 1$ neutral curve and the results are presented in Fig. 2. It clearly appears that disturbances with large horizontal wavelengths are associated with strongly anisotropic, predominantly horizontal motions. Actually, in the limit of small k_x , the eigenfunctions \hat{w} and $\hat{\phi}$ do not vary anymore so that the vertical scale height of the disturbance remains fixed while its horizontal wavelength increases. This explains why thermal diffusion can affect their stability, regardless of the value of the Péclet number. What is still unclear is why they are the most unstable disturbances. We shall come back to this point at the end of the next subsection.

3.2. The small Péclet number regime

In the following, we concentrate our analysis on the small Péclet number regime which is most relevant for stellar astrophysics. As suggested by Dudis' work, we plot once again the neutral curves corresponding to $Pe = 1, 10^{-1}, 10^{-2}, 10^{-3}$ but this time as a function of $RiPe$. The outcome is displayed in Fig. 3. We observe that three of the marginal stability curves nearly collapse into a single curve. While differences between the $Pe = 10^{-2}$ and $Pe = 10^{-3}$ curves are not distinguishable, the $Pe = 10^{-1}$ curve slightly deviates from those.

This remarkable property strongly suggests the existence of an asymptotic state for small Péclet numbers in which the governing equations only depend on the non-dimensional number, $RiPe$. Dudis (1974) had already found that the eigenvalue problem for neutral modes depends only on the product $RiPe$ if the mean flow advection is neglected in Eq. (8). Using asymptotic expansions and starting from the Boussinesq equations, Lignières (1999) recently obtained a general form of the asymptotic equations in the limit of small Péclet numbers. It is shown that, if \mathbf{u} and θ behave like Taylor series for small Péclet numbers ($\mathbf{u} = \mathbf{u}_0 + Pe\mathbf{u}_1 + \dots$, $\theta = \theta_0 + Pe\theta_1 + \dots$), the solutions

of the Boussinesq equations are identical to the solutions of the following system up to the first order in Pe :

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + R\psi \mathbf{e}_z + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad (14)$$

$$\nabla^2 \psi = w, \quad (15)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (16)$$

where $\psi = \theta/Pe$ are the temperature deviations scaled by Pe and

$$R = RiPe = \frac{\bar{t}_S \bar{t}_\kappa}{\bar{t}_N^2}. \quad (17)$$

In this context, the linear equations governing the shear layer stability become:

$$\left[\frac{M^2}{Re} - ik_x(U - c)M + ik_x U'' \right] \hat{w} = Rk_x^2 \hat{\psi}, \quad (18)$$

$$M \hat{\psi} = \hat{w}, \quad (19)$$

which, as expected, only depend on the non-dimensional numbers, $R = RiPe$ and Re . The eigenvalue problem formed by the above equations and the boundary conditions (9) has been solved numerically for neutral ($c = 0$) and inviscid ($Re = +\infty$) modes. We found that the resulting marginal stability curve cannot be distinguished from the $Pe = 10^{-2}$ and $Pe = 10^{-3}$ curves obtained in the context of the Boussinesq equations.

The existence of this asymptotic regime has various interests for the stability analysis. First, we can concentrate on the asymptotic limit and do not have to repeat the analysis for different values of the Péclet number. Second, it turns out that the stability is no longer governed by three processes (shear, stable stratification and thermal diffusion) but only by two, the shear destabilization and a stabilizing process combining the stable stratification and thermal diffusion effects.

To understand why, let us consider the differences between the heat conservation of the Boussinesq equations (Eq. (4)) and the simplified heat balance of the small-Péclet-number approximation (Eq. (15)). This simplified balance occurs because, unlike other advection terms, the vertical advection against the mean stratification does not vanish when the high thermal diffusion strongly reduces the temperature deviations. Indeed, as fluid parcels go up (or down) in a mean temperature gradient, the amplitude of the associated temperature deviations tends to increase continuously. While this process acts as a source of temperature deviations, thermal diffusion tends to smooth them out. Eq. (15) represents the balance between both processes. In this state, vertical advection against mean temperature gradient and thermal diffusion no longer act individually to determine the amplitude of the buoyancy force; the process which combines their action will be called the small-Péclet-number buoyancy in the following of the paper. Its elementary properties have been analyzed in Lignières (1999) and are briefly summarized here. It is purely dissipative in the sense that the work of the buoyancy force integrated over the whole domain is always negative, provided temperature deviations vanish on the outer boundaries. Its

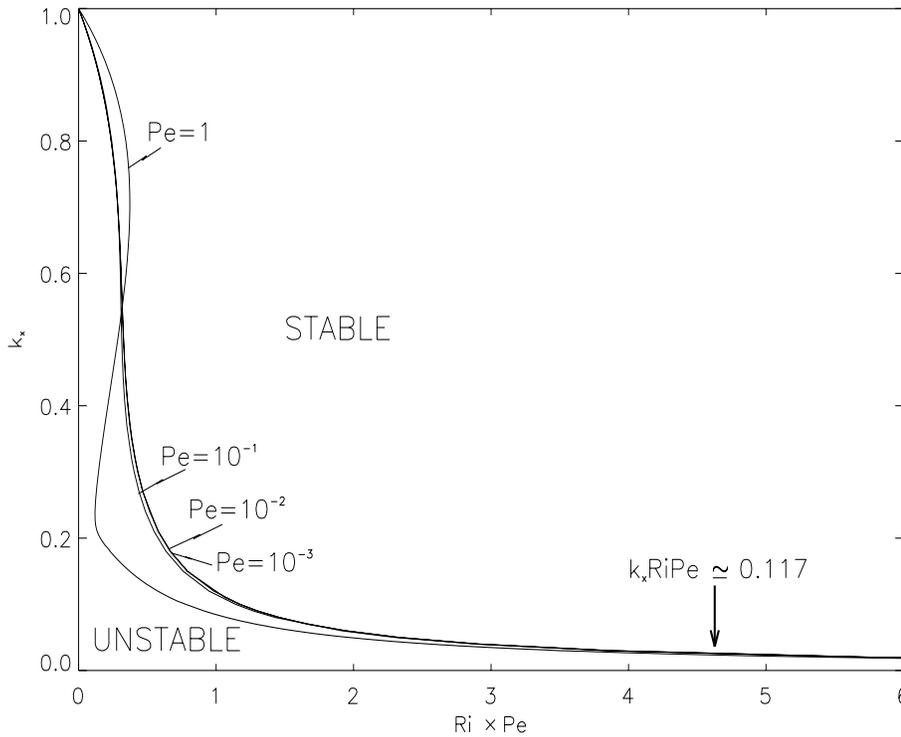


Fig. 3. The marginal stability curves shown in Fig. 1 are now plotted as a function of $RiPe$. The collapse of the $Pe = 10^{-2}$ and $Pe = 10^{-3}$ curves suggests the existence of an asymptotic regime in the limit of small Péclet number. Moreover, all the curves tend towards the curve $k_x RiPe \simeq 0.117$ as k_x goes to zero, suggesting the existence of another asymptotic regime in the limit of small horizontal wavenumber

time scale is $\bar{t}_B = \bar{t}_N^2 / \bar{t}_\kappa$, although the process being strongly anisotropic, the time scale of the small-Péclet-number buoyancy is better described by,

$$t_B(k_x, k_z) = \frac{(k_x^2 + k_z^2)^2}{Rk_x^2}, \quad (20)$$

which corresponds to the time scale necessary to dissipate an infinitesimal perturbation of the form $w \propto \exp[i(k_x x + k_z z)]$ in the atmosphere at rest.

Coming back to the stability analysis, we shall now compare the stabilizing effect of the small-Péclet-number buoyancy with the destabilizing one of the shear.

Let us consider the instability of large horizontal wavelength disturbances. In the previous subsection, we have shown that the vertical length scale of these disturbances remains fixed as their horizontal wavenumber goes to zero. Then, according to expression (20), the small-Péclet-number buoyancy time scale increases like $1/k_x^2$ in the same limit. By comparison, the time scale of the instability (the inverse of the growth rate in a fluid of constant density) is known to be proportional to $1/k_x$ for small k_x (Drazin & Reid 1981). Consequently, the stabilizing effect decreases faster than the destabilizing one as k_x vanishes. This explains why predominantly horizontal perturbations are unstable and why they are increasingly unstable as k_x tends to zero.

It is worth mentioning the difference with the non diffusive case ($Pe = +\infty$). In this case also, the stable stratification is inefficient for predominantly horizontal disturbances. But, by contrast with the diffusive situation, the stable stratification time scale increases as $1/k_x$ (see Eq. (11)), that is as fast as the instability time scale. The result is that distur-

bances with large horizontal wavelength are stable (see the $Pe = +\infty$ neutral-curve in Fig. 1). We therefore conclude that the shapes of the non-diffusive and highly diffusive marginal stability curves differ because the non-diffusive buoyancy has a stronger effect on predominantly horizontal motions than the small-Péclet-number buoyancy. In other words, the action of the small-Péclet-number buoyancy is more anisotropic than that of the non-diffusive buoyancy, and this triggers the instability of predominantly horizontal motions.

The small-Péclet-number buoyancy allows also to understand why, as indicated in Fig. 3, the marginal stability curves converge towards $k_x R = 0.117$ when k_x goes to zero. Since stabilizing and destabilizing effects balance each other for marginal modes, the marginal curve can be characterized by an equality between the dynamical time scale and the small-Péclet-number buoyancy time scale. In the limit of vanishing k_x , this reads:

$$\frac{1}{k_x} = \frac{k_z^4}{k_x^2 R}, \quad (21)$$

that is,

$$k_x R = \text{constant}, \quad (22)$$

since the vertical scale height of these modes remains fixed.

Although outside the regime of small Péclet numbers, we observe that the $Pe = 1$ marginal curve also converges towards $k_x R = 0.117$ for small k_x . This is due to the fact that modes associated with predominantly horizontal motions satisfy the simplified heat equation (15) whatever the value of the Péclet number. In the limit of small k_x , the advective term $U\partial\theta/\partial x$ vanishes naturally while the temporal variation $\partial\theta/\partial t$ is also negligible because, as we have seen before, thermal diffusion

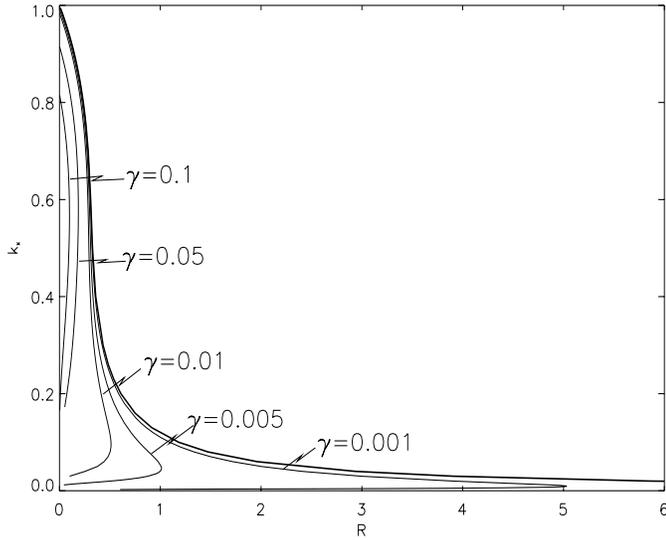


Fig. 4. Growth rate isocontours in the limit of small Péclet numbers. The bold curve corresponds to marginal stability ($\gamma = 0$). Note that the maximum growth rate in the unstratified case is $\gamma_0 \simeq 0.189$

dominates stable stratification for any value of the Péclet number provided the motions are sufficiently horizontal.

From a mathematical point of view, the convergence towards $k_x R = 0.117$ means that the eigenvalue problem has reached an asymptotic regime in the limit of small k_x . Effectively, when terms proportional to k_x^2 are neglected in Eqs. (18) and (19), the asymptotic eigenvalue problem depends only on the parameter $k_x R$. Jones (1977) had already obtained this asymptotic eigenvalue problem, starting from Eqs. (7) and (8) and considering the double limit $k_x \rightarrow 0$ and $Ri \rightarrow +\infty$. He found a slightly smaller value, $k_x R = 0.086$, most probably because of the difference in the boundary conditions. In his study, the requirement that perturbations vanish at $z = \pm 4$ decreases a little bit the domain of instability.

3.3. Growth rates in the small Péclet number regime

In the unstratified case, it is well known that the growth rate of the instability is maximum for a particular mode ($\gamma_0 \simeq 0.189$ at $k_x \simeq 0.447$, see for example Drazin & Reid 1981). In the same way, one would like to know whether, for a fixed value of $R = RiPe$, a particular mode will grow faster than the others. In that case, a relevant issue will be to quantify the reduction of the maximum growth rate due to the small-Péclet-number buoyancy.

Contours of constant growth rate have been determined inside the unstable domain and are presented in Fig. 4. Unsurprisingly, the growth rates decrease when $RiPe$ increases, that is when the stabilizing effect of the buoyancy force becomes stronger. For example at $RiPe \simeq 5$, the maximum growth rate is $\simeq 0.001$, that is smaller by more than two order of magnitude than the maximum growth rate of the unstratified case.

It can be inferred by looking at Fig. 4 that a maximum growth rate exists for each fixed value of $R = RiPe$. This maximum

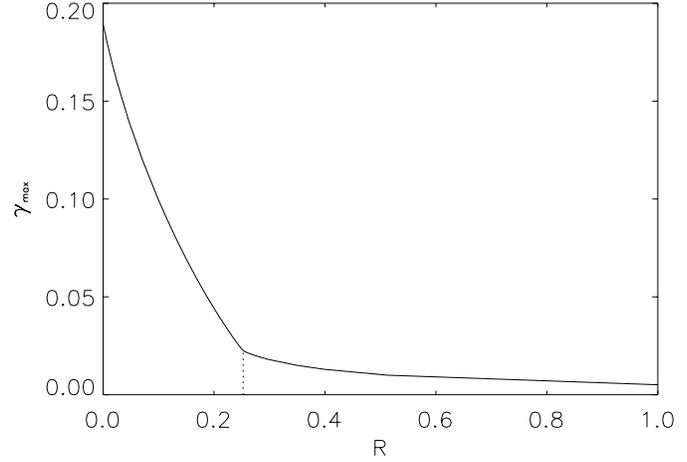


Fig. 5. Maximum growth rate as a function of $R = RiPe$. Only the range $0 < R < 1$ is represented here

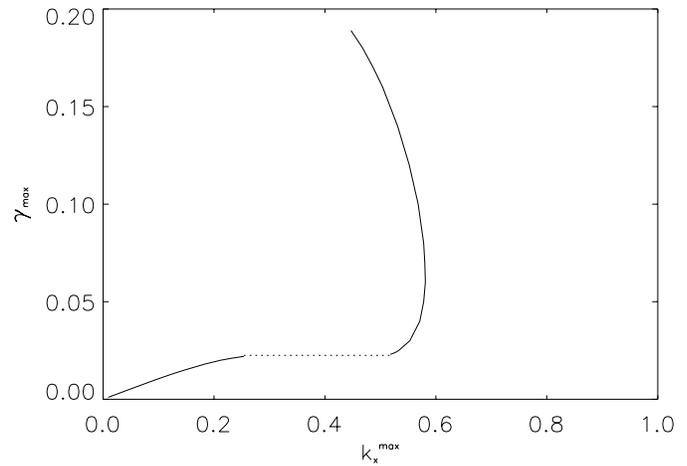


Fig. 6. Horizontal wavenumber of the most unstable modes

value denoted γ_{\max} is shown in Fig. 5 as a function of R . It appears that a rapid decrease of the maximum growth rate is followed by a much slower one after an abrupt change of slope at $R \simeq 0.253$. This change of behaviour is due to the existence of two different types of modes as supported by Fig. 6 where, k_x^{\max} , the horizontal wave number of the most amplified disturbance is plotted against γ_{\max} .

Figs. 5 and 6 show that a first type of mode characterized by horizontal wavenumbers close to 0.5 is dominant in the regime $R < 0.253$ whereas a second type of mode with much smaller wavenumbers becomes dominant once $R > 0.253$. The “jump” between both types of modes occurs at $R \simeq 0.253$ when the curve $\gamma = f(k_x)$ has two equal local maxima.

The vertical structure of both types of modes has been analyzed in order to characterize their geometrical properties. For the modes which dominate at small values of R , the vertical decay outside the shear layer is quite rapid since their vertical scale height varies between \bar{L} and $2\bar{L}$. These modes can be said to be isotropic as their vertical and horizontal length scale are of the same order.

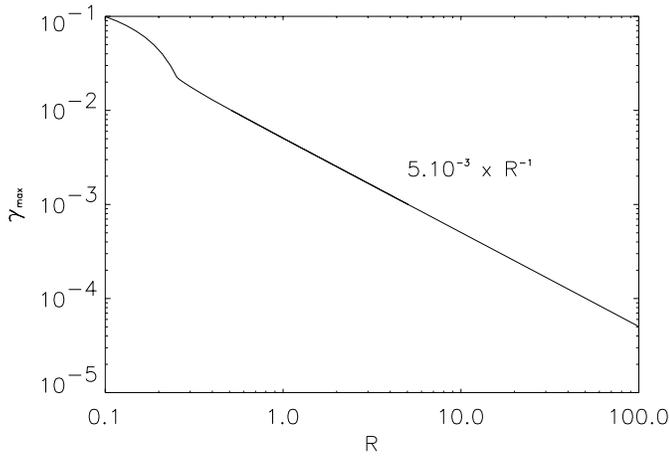


Fig. 7. Maximum growth rate as a function of $R = RiPe$ in log-log coordinates

By contrast, the modes which dominate for large values of R are strongly anisotropic and become more and more so as R increases. Effectively, we found that their vertical scale height remains fixed to $\simeq 5.56\bar{L}$ while their horizontal wavelength ranges from $24\bar{L}$ to $+\infty$ as R increases. Another property of this type of modes is that their growth rate and horizontal wavenumbers are related to R through scaling laws. This can be seen in Fig. 7 where the relation between γ_{\max} and R is plotted again using log-log coordinates and including large values of R . The scaling law,

$$\gamma_{\max} \simeq 5. \times 10^{-3} R^{-1}, \quad (23)$$

is rapidly established as R increases above 0.253. In the same way, it is found that the wavenumber of the most amplified mode and R are related by,

$$k_x^{\max} \simeq 4.86 \times 10^{-2} R^{-1}. \quad (24)$$

These scaling laws are due to the fact that the eigenvalue problem reaches an asymptotic regime for small horizontal wavenumbers. Indeed, in the limit $k_x \rightarrow 0$, the eigenvalue problem for non-neutral modes ($k_x c_i = \gamma \neq 0$) is governed by two parameters only, $a = k_x R$ and $b = \gamma/k_x$. Relations (23) and (24) then show that the most unstable mode corresponds to a particular solution of the asymptotic eigenvalue problem.

3.4. Effect of a large Reynolds number in the small Péclet number regime

For a constant density fluid, the Kelvin-Helmholtz instability is said to be inviscid because viscosity does not play any role in the mechanism which triggers the instability. Viscous dissipation can still influence the stability but, at least for the hyperbolic-tangent shear layer, its effect is rapidly negligible as the Reynolds number increases. This is not the case in a highly diffusive atmosphere since we found that viscous dissipation has a significant influence even if the Reynolds number is very large. This point is illustrated by Fig. 8, where the inviscid neutral curve is compared with the neutral curve calculated at

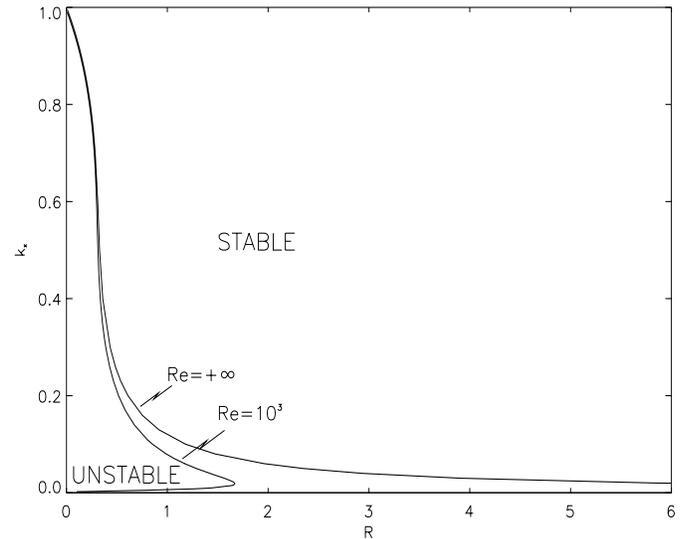


Fig. 8. Curves of marginal stability for $Re = 1000$ and $Re = +\infty$ in the small-Péclet-number regime

$Re = 1000$. The figure shows that viscosity essentially comes into play by stabilizing the large scale predominantly horizontal disturbances, its influence on the other type of perturbations being much weaker.

This can be explained as follows: for wavelengths of the order of the shear layer thickness, the viscous time scale is much larger than the other time scales. However, as k_x goes to zero, both the time scale of the small-Péclet-number buoyancy and the time scale of the instability become infinite while the viscous time scale tends towards a constant value,

$$t_\nu = H_z^2 Re, \quad (25)$$

where H_z denotes the vertical length scale of the predominantly horizontal modes. Then, at some point, viscous dissipation becomes dominant and stabilizes the perturbations.

The critical values of R and k_x have been determined for two other Reynolds numbers, $Re = 1100$ and $Re = 1500$. They satisfy the following relations,

$$\frac{R_{\text{crit}}}{Re} = 1.67 \times 10^{-3} \quad Re k_x^{\text{crit}} = 19, \quad (26)$$

so that the stability criterion is,

$$RiPr = \frac{R}{Re} > 1.67 \times 10^{-3}, \quad (27)$$

for the three Reynolds numbers considered.

There are two different ways of recovering the relations (26) by simple arguments. First, by saying that all perturbations are stable when viscous damping balances the growth of the most unstable mode in the inviscid case. Equalizing the viscous time scale t_ν with the inverse of the maximum growth rate $1/\gamma_{\max}$ yields $R/Re = \text{constant}$. The second scaling law (23) shows that the corresponding wavenumber satisfies $k_x Re = \text{constant}$.

The same relations can be obtained without knowing the expression of the maximum growth rate. We first define a stabi-

lizing time scale which includes the effects of the small-Péclet-number buoyancy and the viscous dissipation as,

$$t_{\text{stab}} = \frac{(k_x^2 + k_z^2)^2}{Rk_x^2 + \frac{(k_x^2 + k_z^2)^3}{Re}}. \quad (28)$$

It corresponds to the damping time of an infinitesimal perturbation proportional to $\exp[i(k_x x + k_z z)]$ in a quiescent atmosphere described by the small-Péclet-number approximation. Then, the marginal stability curve can be characterized by an equality between this time scale and the instability time scale. In the limit of small k_x , it turns out that the function $R(k_x)$ possesses a maximum $R_{\text{max}} = Re k_z^2/4$, reached at $k_x Re = k_z^2$. The above relations follow since the vertical scale height of these modes remains fixed.

For smaller Reynolds numbers ($25 \leq Re \leq 150$), Jones (1977) found a criterion similar to (27), although the constant is somewhat higher $RiPr > 7 \times 10^{-3}$. This difference can be attributed to the boundary conditions.

4. Summary and discussion

The linear stability of a hyperbolic-tangent shear layer in a stably stratified atmosphere has been investigated for a wide range of Richardson, Péclet and Reynolds numbers. Emphasis has been put on the regime of small Péclet numbers (i.e. high thermal diffusivity) relevant for stellar interior dynamics. Note that this regime is also important for the upper atmosphere of the Earth and Venus as Townsend (1958) and Dudis (1974) pointed out.

In the inviscid case, we found that the shear layer is always unstable provided the horizontal wavelength of the perturbations is not bounded. It should be specified that the growth rate goes to zero as the horizontal wavelength goes to infinity. Viscous dissipation stabilizes this type of perturbation and so introduces a critical value of $R = RiPe$ above which the shear layer is stable.

The analysis of the growth rates in the inviscid small Péclet number regime revealed the existence of two different types of modes. A first type, isotropic and relatively fast growing, is dominant in the range $0 < R = RiPe < 0.253$. By relatively fast, we mean that the growth rate is always larger than $0.1\gamma_0$, where γ_0 is the maximum growth rate in the unstratified case. For larger values of R , the most unstable modes are predominantly horizontal. Their growth rates continue to decrease, following the scaling law $\bar{\gamma}_{\text{max}} = 5. \times 10^{-3} Ri^{-1} \bar{t}_\kappa^{-1}$, in dimensional units.

Perhaps, the most striking result is the existence of an asymptotic regime where the mathematical and physical analysis of the stability is considerably simplified. The physical simplification comes from the fact that the buoyancy force is no longer determined by two independent processes, temperature advection in the stratified atmosphere and thermal diffusion, but rather by a unique process combining both. The effect of this so-called small-Péclet-number buoyancy turns out to be purely dissipative and strongly anisotropic. Comparing its stabilizing effect to the destabilizing Kelvin-Helmholtz mechanism leads to a straightforward interpretation of the stability properties.

It should be noted that the small-Péclet-number buoyancy characterizes the response of a highly diffusive stably stratified atmosphere to any sufficiently slow motions. As such, it can potentially be applied to other types of motions than those driven by a parallel shear flow. This includes for example other types of double diffusive instabilities like the Goldreich-Schubert instability or the thermohaline instability. We note in particular that the most unstable modes of the latter correspond to long thin columns. This is fully compatible with the properties of the small-Péclet-number buoyancy since the associated stabilizing time scale (20) becomes infinite for this type of motions ($k_x \rightarrow +\infty$ and k_z remaining fixed). Work is in progress to assess the usefulness of the small-Péclet-approximation in this context.

Strictly speaking, the results presented here are only valid for the particular flow considered and assuming that the perturbations are infinitesimal. We expect nevertheless that similar results would be obtained for any shear layer subject to the Kelvin-Helmholtz instability, that is when the vorticity profile possesses an extremum (Drazin & Reid 1981). But, in any case, our results cannot be directly extended to the other velocity profiles and more importantly if the basic flow is perturbed by finite amplitude disturbances. Effectively, since Reynolds' experiments on flows in a pipe (Reynolds 1883), it is well known that finite amplitude perturbations are able to trigger shear layer instabilities when infinitesimal ones cannot. In this context the criterion we obtained appears as a sufficient - but not necessary - condition for instability.

In principle, it can be applied as such, to shear layers inside stellar radiative zones. However, taking typical solar values for the thermal diffusivity and the Brunt-Väisälä frequency ($\kappa = 10^7 \text{cm}^2 \text{s}^{-1}$ and $N = 10^{-3} \text{s}^{-1}$), it is found that shear layers unstable according to this criterion would have vertical extent much smaller than the vertical resolution of helioseismology data ($L/R_\odot < 10^{-5}$). To give an example, if one assumes that the shear across the solar tachocline can be modeled by a hyperbolic-tangent profile entirely embedded in the radiative zone, the associated Richardson number would be about 2.5×10^3 and the Péclet number about 10^6 (we took 5 percent of the solar radius for the tachocline thickness and $\Delta\Omega = 10^{-6} \text{s}^{-1}$ for the differential rotation across the tachocline). Then, our criterion would predict stability because the smallest unstable horizontal wavelength according to $k_x = 0.117/RiPe$ is much larger than the solar radius. Note again that shears across the tachocline could be subjected to finite amplitude instabilities as discussed in Michaud & Zahn (1998).

For the time being, one relies on the criterion established by Zahn (1974) to decide about the non-linear stability of a shear layer. This criterion is not in contradiction with our results. But this consistency check is not sufficient to prove its validity. Interestingly enough, the energetic consideration made by Maeder (1995) have shown that the criterion is consistent energetically. However, in the derivation by Zahn (1974) and by Maeder (1995), perturbations with a velocity scale u and a length scale l are introduced ab initio and it is not known whether such perturbations correspond to possible solutions of the equa-

tions of motion. Therefore, this criterion can be interpreted as a necessary condition for instability.

Further investigations are clearly needed to better constrain this criterion. In a geophysical context, numerical simulations of stably stratified sheared turbulence have been able to find critical Richardson numbers for which turbulence neither grows nor decays (see a review by Schumann, 1996). Such simulations should be performed at smaller Péclet numbers to approach stellar conditions and, as pointed out by Lignières (1999), the small-Péclet-number approximation would be very useful in this context. Another possible approach is to assume that finite amplitude disturbances provoke a defect of the basic profile and then to analyze the linear stability of the perturbed velocity profile as a function of the defect amplitude. This method has been used by Dubrulle & Zahn (1991) for the unbounded Couette flow in a constant density fluid and could be extended to the stably stratified case at small Péclet number.

As quoted in the introduction, beyond the stability of shear layer, the final objective is to estimate the angular momentum and chemical element transport driven by shear layer turbulence. Assuming that small scale turbulence behaves like a diffusion process, Zahn (1992) proposed an expression for the turbulent viscosity on the basis of arguments used to derive the stability criterion. This expression reads,

$$\nu_t = \text{constant} \times \bar{\kappa} \left(\frac{\bar{r}}{\bar{N}} \frac{d\bar{\Omega}}{d\bar{r}} \right)^2 \propto \bar{\kappa} Ri_g^{-1}, \quad (29)$$

where Ri_g is the Richardson number formed with the local velocity gradient $\bar{r}d\bar{\Omega}/d\bar{r}$ and the local Brunt-Väisälä frequency. Clearly our linear stability analysis gives no clues about the non-linear processes governing turbulent transport. Nevertheless, as the instability mechanism controls the energy injection into turbulence, it is natural to suppose that changes in the stability properties have a direct impact on turbulent transport. On dimensional grounds, we can form a vertical transport coefficient using the growth rate and the vertical length scale characterizing the most unstable mode is $\bar{\gamma}_{\max} \bar{H}_z^2$. For large values of $R = RiPe$ ($R > 0.253$), γ_{\max} follows the scaling law (23)

whereas the typical vertical length of the most unstable mode remains fixed, so that $\bar{\gamma}_{\max} \bar{H}_z^2 \propto \bar{\kappa} Ri^{-1}$. This exactly corresponds to the scaling of the turbulent viscosity (29). Such a close agreement is not found in the regime of small $RiPe$. However, the variation of the maximum growth with $RiPe$ and the expression of the turbulent viscosity are still compatible. Actually, the main difference comes from the very existence of two different regimes which are not contained in the expression of the turbulent viscosity (29). One corresponds to rapid isotropic motions on the length scale of the shear layer, the other to slow nearly horizontal motions.

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