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Partial frequency redistribution with Hanle and Zeeman effects

Non-perturbative classical theory

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Abstract. A theory for the scattering of polarized radiation with partial frequency redistribution and coherence effects in the presence of magnetic fields of arbitrary strength and direction is developed within a classical framework. The time-dependent equation for a classical oscillator is solved. While the oscillator is being excited, it is also damped by emission of radiation and subject to phase-destroying collisions. Fourier transformation of the emitted wave train with phase-scrambling collisions leads to the partial-redistribution expressions for the relation between the polarization and frequencies of the incident and scattered radiation. While previous treatments of partial redistribution have been based on quantum perburbation theory, the classical theory has the advantage of being fully non-perturbative. It is therefore conceptually more transparent and leads itself to direct physical interpretation. The classical and quantum theories give identical results for a $J = 0 \rightarrow 1 \rightarrow 0$ transition.

Key words: atomic processes – line: formation – scattering – polarization – magnetic fields – radiative transfer

1. Introduction

As the scattering polarization of the "second solar spectrum" has recently become accessible to systematic exploration (Stenflo & Keller 1996, 1997), a new diagnostic window for spectroscopy and for solar studies has opened up. Due to the Hanle effect the second solar spectrum is a sensitive function of the strength and small-scale geometry of the spatially unresolved magnetic fields (Stenflo et al. 1998) in a parameter domain that is inaccessible to ordinary Zeeman-effect observations. This may potentially allow for major advances in our understanding of solar magnetism. The Hanle effect is also beginning to find its applications in non-solar astrophysics (cf. Ignace et al. 1997). However, the theory needed to interpret the wealth of new observed phenomena in scattering physics on the Sun is not yet fully developed or understood, although much progress has been made in recent years (cf. Stenflo & Nagendra 1996; Nagendra & Stenflo 1999). It is now a challenge to establish a good theoretical foundation and develop the adequate theoretical tools that may be used in a polarized radiative transfer formalism.

Recently Bommier (1997a,b) has developed a general theory for partial frequency redistribution of polarized radiation in the presence of arbitrary magnetic fields. Her treatment is based on a quantum-mechanical perturbation expansion, which has been carried out to all orders to bring out the effects of partial redistribution and collisional line broadening. Since the physical interpretations of the various mathematical terms in the perturbation expansion are not straightforward, the formalism lacks transparency and is not well suited for an intuitive physical understanding. Still it is the theory that provides us with the most general description of the scattering processes.

The classical oscillator theory on the other hand is applicable to $J = 0 \rightarrow 1 \rightarrow 0$ transitions (normal Zeeman triplets), but it cannot treat general quantized systems directly. It however readily lends itself to phenomenological extension from the $J = 0 \rightarrow 1 \rightarrow 0$ case to general Raman scattering for atomic systems with arbitrary quantum numbers (cf. Stenflo 1998, and Sect. 12 below). A major advantage of the classical approach is that it is non-perturbative and therefore physically transparent, and it gives us excellent guidance for an intuitive grasp of the physics. In contrast, no non-perturbative quantum-mechanical scattering theory is available.

Previous treatments of classical scattering (Stenflo 1994, 1996, 1998) have used stationary solutions of the classical oscillator equation to derive a scattering Mueller matrix that includes the effects of arbitrary magnetic-fields (Hanle and Zeeman effects). In addition, a classical model for collisions that scramble the phase of a damped oscillator has been used to derive how collisions enter into the Hanle effect (Stenflo 1994, Chapter 10). However, stationary solutions always lead to frequency coherence and provide no information on the redistribution between the incident and scattered frequencies. To obtain such information one has to solve the time-dependent oscillator equation, which we will do here in combination with the previously mentioned classical model for the effect of collisions. This will give us self-consistent and non-perturbative expressions for the polarized partial frequency redistribution line profiles in the presence of magnetic fields of arbitrary strengths.

These expressions are found to be identical to those of the general perturbative quantum theory of Bommier (1997b) for the case of a $J = 0 \rightarrow 1 \rightarrow 0$ transition with an unbroadened ground state.

The theory of partial frequency redistribution has been developed within a quantum-mechanical framework by Weisskopf (1933), Wooley (1938), Hummer (1962), Omont et al. (1972, 1973), Cooper et al. (1982), Domke & Hubeny (1988), and Bommier (1997a,b), while Zanstra (1941) has used a classical model to consider the redistribution effects of collisions. The present work represents the first attempt at a more comprehensive treatment of partial redistribution (with collisions, polarization, and arbitrary magnetic fields) within a classical framework.

2. General formulation of classical scattering

The classical oscillator equation, which describes the motion of a particle with charge $-e$ and mass m in a central Coulomb potential (due to an atomic nucleus), subject to an external magnetic field $\boldsymbol{B}$ and an external, oscillating electric field $\boldsymbol{E}'$, is given by

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{e}{m}(\boldsymbol{v} \times \boldsymbol{B}) + \gamma \boldsymbol{v} + \omega_0^2 \boldsymbol{r} = -\frac{e}{m}\boldsymbol{E}' \,. \tag{1}$$

γ is the damping constant, the value of which can easily be derived classically from the radiative reaction force on the accelerated electric charge (cf. Stenflo 1994, pp. 44–47). ω_0 is the resonant frequency of the oscillating charge.

To decouple the component equations one introduces complex spherical vectors $\boldsymbol{e}_q$, $q = 0, \pm 1$, where

$$\begin{aligned} \boldsymbol{e}_0 &= \boldsymbol{e}_z \,, \\ \boldsymbol{e}_\pm &= \mp(\boldsymbol{e}_x \pm i\boldsymbol{e}_y)/\sqrt{2} \,. \end{aligned} \tag{2}$$

$\boldsymbol{e}_{x,y,z}$ are Cartesian unit vectors with the z axis along the direction of the magnetic field. We now obtain the three decoupled component equations

$$\frac{\mathrm{d}^2 r_q}{\mathrm{d}t^2} - (2qi\omega_L - \gamma)\frac{\mathrm{d}r_q}{\mathrm{d}t} + \omega_0^2 r_q = -\frac{e}{m}E'_q \,, \tag{3}$$

where $\omega_L = eB/(2m)$ is the Larmor frequency.

Usually one uses the stationary solutions of these equations to find the complex dispersion properties of the medium. For the frequency-redistribution problem, however, it is necessary to solve the time-dependent problem.

Let us assume that we have found a solution $\boldsymbol{r} = \boldsymbol{r}(t)$ to the time-dependent problem. This solution describes the trajectory of a moving charge, whose electric field will fluctuate due to the motions of the charge. These fluctuations propagate (as retarded electric potentials) with the speed of light radially away from the moving charge. In a plane perpendicular to the direction of propagation the transverse electric vector can simply be described by the movements of the charge as projected onto this plane and retarded by the time of propagation from the charge to the chosen plane.

For problems in polarimetry it is usually most convenient to represent the light in terms of a linear polarization basis. Thus we use linear unit vectors $\boldsymbol{e}_\beta$, $\beta = 1, 2$, oriented perpendicular to the incident radiation (which is represented by the external electric field $\boldsymbol{E}'$ in Eq. (1)), while for the emitted radiation we use linear unit vectors $\boldsymbol{e}_\alpha$, $\alpha = 1, 2$, which are oriented perpendicular to the direction of scattering. We denote the scalar product between a linear and a complex spherical unit vector by

$$\varepsilon_q^\alpha = \boldsymbol{e}_q \cdot \boldsymbol{e}_\alpha \,. \tag{4}$$

It describes the normalized projection of the vectors for the classical oscillator on the vector system for the scattered radiation. We can then express the scattered electric vector as

$$E_\alpha(t) \sim \sum_q \varepsilon_q^{\alpha *}\, r_q(t) \tag{5}$$

($*$ means complex conjugation). Similarly we can project the linear vector components of the incident radiation on the complex spherical vectors of the classical oscillator:

$$E'_q = \sum_\beta \varepsilon_q^\beta \, E'_\beta \,. \tag{6}$$

As r_q according to Eq. (3) scales with E'_q, we can combine Eqs. (5) and (6) to get

$$E_\alpha \sim \sum_\beta w_{\alpha\beta}\, E'_\beta \,, \tag{7}$$

where

$$w_{\alpha\beta} = \sum_q [\, r_q(t, \omega')/E'_{q,0} \,]\, \varepsilon_q^{\alpha *}\, \varepsilon_q^\beta \tag{8}$$

is the *Jones scattering matrix* (cf. Stenflo 1994, p. 57), and $E'_{q,0}$ is the amplitude of the incoming monochromatic plane wave

$$E'_q = E'_{q,0}\, e^{-i\omega' t} \,. \tag{9}$$

The oscillating phase factor has been omitted in Eq. (8), since it vanishes when we form the bilinear tensor product $\boldsymbol{w} \otimes \boldsymbol{w}^*$ of the Jones matrices $\boldsymbol{w}$ to form the coherency matrix. The Mueller scattering matrix $\boldsymbol{M}$ that describes scattering of the Stokes vector is readily obtained from the coherency matrix through

$$\boldsymbol{M} \sim \boldsymbol{T}(\boldsymbol{w} \otimes \boldsymbol{w}^*)\boldsymbol{T}^{-1} \tag{10}$$

(cf. Stenflo 1994, p. 73; Stenflo 1998). For explicit expressions of $\boldsymbol{w} \otimes \boldsymbol{w}^*$ and the purely mathematical transformation matrices $\boldsymbol{T}$ and $\boldsymbol{T}^{-1}$, see Stenflo (1998).

The εs in Eq. (8) contain purely geometrical information in the form of trigonometric expressions in terms of the angles that the external magnetic field makes with the incident and scattered radiation. They have been given explicitly in Stenflo (1994, p. 57) for a coordinate system that has the magnetic field along the polar axis. r_q contains all the physics describing the time variation and its dependence on the frequency ω' of the incident radiation. It is obtained as the time-dependent solution of the oscillator equations (3).

3. Spectral domain and the coherency matrix

What we want are the spectral properties of the scattered radiation. They can be found by performing a Fourier transformation of the fluctuating scattered electric vector $E_\alpha(t)$ to obtain the transform $\tilde{E}_\alpha(\omega)$, where ω represents the scattered frequencies:

$$\tilde{E}_\alpha(\omega) = \int_{-\infty}^{+\infty} E_\alpha(t)\, e^{i\omega t}\, \mathrm{d}t\,. \tag{11}$$

This implies according to Eqs. (7) and (8) that we need to compute the Fourier transform

$$\tilde{r}_q(\omega,\omega') = \int_{-\infty}^{+\infty} r_q(t,\omega')\, e^{i\omega t}\, \mathrm{d}t\,, \tag{12}$$

where the time-dependent solution $r_q(t,\omega')$ of the oscillator equation (3) is assumed to implicitly also include the effects of phase-scrambling collisions. The explicit procedure for treating the collisions will be presented in Sect. 5.

Due to random phase shifts when we have random, phase-destroying collisions, we need to perform ensemble averages of the bilinear products $\tilde{r}_q\,\tilde{r}^*_{q'}$ for many initial conditions for the oscillator. In addition, as we will see in Sect. 5, we need to average over the random lengths of the time intervals between collisions. We write such an ensemble average of the coherency matrix as $\langle \tilde{r}_q\,\tilde{r}^*_{q'}\rangle$. It contains all the frequency information, including the partial redistribution expressions that relate the incident and scattered frequencies to each other.

4. Solution of the time-dependent classical oscillator loequation

With standard mathematical techniques for solving second-order differential equations we arrive at the following general solution of Eq. (1):

$$r_q(t) = C_1 e^{iw^*_{-q}t} + C_2 e^{-iw_q t} - \frac{e}{2im\omega_0}\int_{t_0}^{t} E'_q(z)\left[e^{iw^*_{-q}(t-z)} - e^{-iw_q(t-z)}\right]\mathrm{d}z\,, \tag{13}$$

where

$$w_q = \omega_0 - q\omega_L - i\gamma/2\,, \tag{14}$$

and $C_{1,2}$ are integration constants determined by the initial conditions.

The incident radiation is in the form of a monochromatic plane wave with frequency ω', given by Eq. (9). The state of the oscillator at time t_0 is an initial condition determined by the constants $C_{1,2}$. Due to the randomness of the collisions (see next section), the constants $C_{1,2}$ and the time t_0 may be considered as random.

With the plane wave expression (9) for E'_q, Eq. (13) becomes

$$r_q(t) = C_1 e^{iw^*_{-q}t} + C_2 e^{-iw_q t} + \frac{eE'_{q,0}}{2m\omega_0}\cdot\left(\frac{e^{iw^*_{-q}(t-t_0)-i\omega' t_0} - e^{-i\omega' t}}{\omega' + w^*_{-q}} - \frac{e^{-iw_q(t-t_0)-i\omega' t_0} - e^{-i\omega' t}}{\omega' - w_q}\right). \tag{15}$$

Since the non-resonant terms are small in comparison with the resonant ones, they may be neglected. We may therefore write

$$r_q(t) = r_{q,\,\mathrm{stat}}(t) + C r_{q,\,\mathrm{trans}}(t)\, e^{i\delta}\,, \tag{16}$$

where, if we for convenience neglect a proportionality factor that includes the incoming wave amplitude $E'_{q,0}$,

$$r_{q,\,\mathrm{stat}}(t) = \frac{e^{-i\omega' t}}{\omega' - w_q} \tag{17}$$

represents the stationary solution (cf. Stenflo 1994, p. 49), which is independent of t_0 and $C_{1,2}$, while

$$r_{q,\,\mathrm{trans}}(t) = \frac{e^{-iw_q t}}{\omega' - w_q} \tag{18}$$

represents the transitory solution for a free, damped oscillator. It would be obtained from the homogeneous version of the oscillator equation (1) without driving term on the right hand side. The phase δ and amplitude factor C in Eq. (16) depend on the random value of t_0 and on the phase and amplitude of C_2 in Eq. (15).

Other types of scenarios will in general lead to the same structure for the solution of the time-dependent equation as in Eqs. (16)–(18), namely as the sum of a stationary and a transitory solution, with an arbitrary phase relation between them. The factor $1/(\omega' - w_q)$ that appears in both the stationary and transitory solutions represents the absorption profile. As we will see the stationary solution is the source of the frequency coherence ($\boldsymbol{R}_{\mathrm{II}}$) in the redistribution matrix, while the transitory solution is the source of complete redistribution ($\boldsymbol{R}_{\mathrm{III}}$), since its oscillation frequencies are decoupled from the driving frequency ω'.

5. Collisional destruction of the phase coherence

The collisions affect the scattered radiation in two different ways: (a) They broaden the polarized line profiles (collisional "damping"). (b) They change the magnetic-field dependence of the Hanle effect (destruction of atomic polarization). The advantage of the collisional model described here is that it explains in a unified way both (a) and (b) as a consequence of one single mechanism or parameter, the damping parameter γ_c. It produces the correct ratio (=2) between the line broadening rate and the destruction rate for the atomic polarization. The theory is the same as introduced in Stenflo (1994, p. 211), but here we integrate it into a consistent and comprehensive treatment of the partial redistribution problem.

Although in principle the Fourier integrals in Eqs. (11) and (12) extend to infinity in both directions, a random collision introduces an impulsive force (briefly dominating the right-hand side of the oscillator equation (1)), which randomly scrambles the phase of the oscillator. This scrambling statistically randomizes the oscillators and destroys the phase coherence. Mathematically it corresponds to a truncation of the Fourier transform

at the time interval between two collisions. Since the phase of the stationary solution is determined by the driving electromagnetic wave E'_q, which is not affected by collisions, it is only the transitory solution that is subject to the phase scrambling.

In terms of the classical scenario of the preceding section, t_0 represents the time of a collision (abrupt phase change). During the subsequent period t_c the oscillator is undisturbed by collisions, so that the coherency is preserved, until another collision occurs at time $t_0 + t_c$. The Fourier integral therefore only gives non-zero contributions over the interval $[t_0, t_0+t_c]$, which means that Eq. (12) becomes

$$\tilde{r}_{q,\,\mathrm{trans}}(\omega,\omega') = \int_{t_0}^{t_0+t_c} r_{q,\,\mathrm{trans}}(t,\omega')\, e^{i\omega t}\,\mathrm{d}t \tag{19}$$

for the transitory solution.

Due to the randomness of the phase δ in Eq. (16), the cross terms in the coherency matrix $\tilde{r}_q(\omega,\omega')\,\tilde{r}^*_{q'}(\omega,\omega')$ between the stationary and transitory solutions vanish when we perform an ensemble average over the collisions. Thus

$$\tilde{r}_q\,\tilde{r}^*_{q'} = \tilde{r}_{q,\,\mathrm{stat}}\,\tilde{r}^*_{q',\,\mathrm{stat}} + CC'\tilde{r}_{q,\,\mathrm{trans}}\,\tilde{r}^*_{q',\,\mathrm{trans}}\,. \tag{20}$$

Therefore the two solutions can be treated independently of each other. Although the amplitude factors C and C' may differ for each individual transient period t_c, the ensemble averages $\langle C\rangle$ and $\langle C'\rangle$ will be the same. The actual value of $\langle CC'\rangle$ is determined by the normalization, which will be done in Sect. 10.

The coherency matrix $\tilde{r}_q\,\tilde{r}^*_{q'}$, which is thus independently formed for the stationary and transitory solutions, does not depend on t_0 or the random phase δ of Eq. (16). However, for the transitory solution it depends on t_c, the time during which the absorption-emission process is undisturbed by a collision. Since the collisions occur randomly, t_c can be assumed to obey Poisson statistics. The probability that the interval between two successive collisions is in the range $[t_c,\, t_c + \mathrm{d}t_c]$ is then $e^{-t_c/\tau_c}\,\mathrm{d}t_c/\tau_c$, where τ_c is the average time between collisions. $\tilde{r}_q\,\tilde{r}^*_{q'}$ needs to be averaged over this distribution:

$$\langle\tilde{r}_q\,\tilde{r}^*_{q'}\rangle = {\tau_c}^{-1}\int_0^\infty \tilde{r}_q\,\tilde{r}^*_{q'}\,e^{-t_c/\tau_c}\,\mathrm{d}t_c\,. \tag{21}$$

For comparison with line-profile expressions in the literature we introduce a collisional damping constant γ_c, *defined* by

$$\gamma_c/2 = 1/\tau_c\,. \tag{22}$$

6. Coherency matrix for the transitory solution

To simplify the following expressions we introduce the profile functions

$$\begin{aligned}\Phi_\gamma(\nu_q-\nu) &= \frac{2/i}{\omega_0 - q\omega_L - \omega - i\gamma/2}\,,\\ \Phi'_\gamma(\nu_{q'}-\nu') &= \frac{2/i}{\omega_0 - q'\omega_L - \omega' - i\gamma/2}\,,\end{aligned} \tag{23}$$

where

$$\nu_q = \nu_0 - q\nu_L\,, \tag{24}$$

$\nu_{0,L} = \omega_{0,L}/2\pi$, and the normalization factor $2/i$ is introduced to make the integral of Φ over all ν unity. Note that the primed Φ' (which contains the incident frequency ν') represents the absorption profile, the unprimed Φ the emission profile.

We will disregard a t_0-dependent phase factor, which vanishes when we form the coherency matrix and take into account the t_0-dependent terms in Eq. (15). The truncated Fourier transform of Eq. (19) then becomes, if we also disregard the proportionality factor 4 that comes from the $2/i$ factors in Eq. (23),

$$\tilde{r}_{q,\mathrm{trans}}(\omega,\omega') \sim \Phi'_\gamma(\nu_q-\nu')\Phi_\gamma(\nu_q-\nu)[1-e^{-i(w_q-\omega)t_c}]. \tag{25}$$

After some straightforward algebra the average coherency matrix becomes

$$\begin{aligned}\langle\tilde{r}_q\,\tilde{r}^*_{q'}\rangle_{\mathrm{trans}} \;\sim\; & \Phi'_\gamma(\nu_q-\nu')\,\Phi'^*_\gamma(\nu_{q'}-\nu')\\ & \Phi_{\gamma+\gamma_c}(\nu_q-\nu)\,\Phi^*_{\gamma+\gamma_c}(\nu_{q'}-\nu)\\ & \frac{(q-q')\,\omega_L + i\,(\gamma+\gamma_c)}{(q-q')\,\omega_L + i\,(\gamma+\gamma_c/2)}\,.\end{aligned} \tag{26}$$

Next we introduce the generalized profile function

$$\Phi^\gamma_{q,q'}(\nu) = \tfrac{1}{2}\,[\,\Phi_\gamma(\nu_q-\nu) + \Phi^*_\gamma(\nu_{q'}-\nu)\,] \tag{27}$$

and the Hanle angle $\alpha_{q-q'}$, where

$$\tan\alpha_{q-q'} = \frac{(q-q')\,\omega_L}{\gamma+\gamma_c/2}\,. \tag{28}$$

It is easy to verify that the products $\Phi_\gamma\,\Phi^*_\gamma$ of ordinary profile functions can be expressed in terms of the general profile function via

$$\Phi_\gamma\,\Phi^*_\gamma = \frac{4}{i\,(q-q')\,\omega_L - \gamma}\;\Phi^\gamma_{q,q'}(\nu)\,. \tag{29}$$

We may therefore write

$$\langle\tilde{r}_q\,\tilde{r}^*_{q'}\rangle_{\mathrm{trans}} \sim f_{\mathrm{abs}}(\nu')\cos\alpha_{q-q'}\,e^{i\alpha_{q-q'}}\,\Phi^{\gamma+\gamma_c}_{q,q'}(\nu)\,, \tag{30}$$

where the absorption profile $f_{\mathrm{abs}}(\nu')$, which according to Eq. (26) is proportional to $\Phi'_\gamma\Phi'^*_\gamma$, will be discussed in Sect. 8 below.

7. Comparison with quantum collision theory

Let us at this point stress the important result that the total collisional rate that enters in the Hanle effect (in the Hanle angle $\alpha_{q-q'}$ in Eq. (28)) is $\gamma+\gamma_c/2$, while the total collisional line broadening rate that determines the width of the profile function $\Phi^{\gamma+\gamma_c}_{q,q'}$ is $\gamma+\gamma_c$. This finding is to be compared with the results of the quantum theory, according to which the line broadening rate is $\Gamma+\Gamma_I+\Gamma_E$, where Γ is the radiative rate, Γ_I and Γ_E the rates of inelastic and elastic collisions, respectively. The decay rate that appears in the Hanle angle $\alpha_{q-q'}$ is according to quantum theory $\Gamma+\Gamma_I+D^{(K)}$, $K=1,2$, where $D^{(K)}$ is the rate of destruction of the $2K$-multipole (the atomic polarization). Extensive calculations in quantum collision theory give the result that $D^{(K)} \approx \frac{1}{2}\Gamma_E$ (Spielfiedel et al. 1991; Faurobert-Scholl et al. 1995).

When comparing the classical and quantum theories we may make the identifications

$$\begin{aligned} \gamma &= \Gamma + \Gamma_I \,, \\ \gamma_c &= \Gamma_E \,. \end{aligned} \tag{31}$$

It then automatically follows that

$$D^{(K)} = \gamma_c/2 \,, \tag{32}$$

and therefore

$$D^{(K)} = \Gamma_E/2 \,. \tag{33}$$

We have thus retrieved the correct order of magnitude of the result of quantum collision theory. In the classical theory Eq. (33) follows exclusively from the collisional truncation of the Fourier integral. There is no need to consider special types of collisions that would selectively destroy the atomic polarization. This elegant feature of the classical collision theory was previously demonstrated in Stenflo (1994, pp. 212–213).

8. Absorption profile

Our treatment of collisions through truncation of the emission process has induced collisional broadening, so that γ in the profile function gets replaced by $\gamma + \gamma_c$. The absorption profile $f_{\rm abs}$ is unaffected by this procedure, because it has been determined when solving the time-dependent classical oscillator equation, *before* the operation of a truncated Fourier transform is performed to compute the spectral properties of the emitted radiation.

The absorption profile must however be similarly broadened by collisions. Due to time-reversal symmetry of the scattering process we may infer that the absorption profile should also be broadened as described by the total damping constant $\gamma + \gamma_c$ rather than by γ alone. Thus

$$f_{\rm abs}(\nu') \sim \Phi'_{\gamma+\gamma_c}(\nu_q - \nu')\, \Phi'^{*}_{\gamma+\gamma_c}(\nu_{q'} - \nu') \,, \tag{34}$$

or, in terms of the generalized profile function of Eq. (27),

$$f_{\rm abs}(\nu') \sim \cos\beta_{q-q'}\, e^{i\beta_{q-q'}}\, \Phi^{\gamma+\gamma_c}_{q,q'}(\nu') \,, \tag{35}$$

where a second Hanle angle $\beta_{q-q'}$ has been introduced, defined by

$$\tan\beta_{q-q'} = \frac{(q-q')\,\omega_L}{\gamma+\gamma_c} \,. \tag{36}$$

Although it may seem that the Hanle effect now enters the coherency matrix in two different ways, in the form of the Hanle angle $\beta_{q-q'}$ in the absorption profile, and in the form of the Hanle angle $\alpha_{q-q'}$ in the transitory portion of the emission expression, it turns out that only the effects due to the Hanle angle $\alpha_{q-q'}$ survive when the branching ratios (see Sect. 10 below) are taken into account (cf. Bommier 1997b). In the line core the effects of $\beta_{q-q'}$ vanish due to cancellation between the contributions from the stationary and transitory solutions, while in the line wings the Hanle effect (from both $\alpha_{q-q'}$ and $\beta_{q-q'}$) vanishes as shown in Stenflo (1998).

9. Coherency matrix for the stationary solution

As we see from Eqs. (17) and (18), the results for the stationary solution can be obtained from the transitory one if we in the time-dependent exponential factor replace w_q by ω'. This is achieved by making the substitutions $\omega_0 \to \omega'$, $\omega_L \to 0$, and $\gamma \to 0$. Further, since the stationary solution, being driven by the external electromagnetic field, is unaffected by collisions, we may use the results from the transitory solution if we in Eqs. (19) and (21) let $t_c \to \infty$ and therefore $\gamma_c \to 0$. These substitutions then imply that

$$\Phi^{\gamma+\gamma_c}_{q,q'}(\nu) \to \delta(\nu - \nu') \,, \tag{37}$$

where $\delta(\nu - \nu')$ is the Dirac delta function. The coherency matrix for the stationary solution is then

$$\langle \tilde{r}_q\, \tilde{r}^{*}_{q'} \rangle_{\rm stat} \sim f_{\rm abs}(\nu')\, \delta(\nu - \nu') \,. \tag{38}$$

10. Branching ratios

Combining Eqs. (30) and (38) we have

$$\begin{aligned} \langle \tilde{r}_q\, \tilde{r}^{*}_{q'} \rangle \,/ f_{\rm abs}(\nu') \;\sim\; & A\, \delta(\nu - \nu') \\ & + \; B\, \cos\alpha_{q-q'}\, e^{i\alpha_{q-q'}}\, \Phi^{\gamma+\gamma_c}_{q,q'}(\nu) \,, \end{aligned} \tag{39}$$

where A and B are branching ratios between the stationary and transitory solutions, determined by probability arguments and normalization.

As in Sect. 7 we make the identifications of Eq. (31), from which the value of $D^{(K)}$ to be used in the expression for the Hanle angle $\alpha_{q-q'}$ follows according to Eq. (32). For simplicity we will here use the quantum concept of transition rates to derive expressions for the branching ratios.

It is only the fraction $\Gamma/(\gamma + \gamma_c)$ of the scattering processes that represent radiative transitions undisturbed by collisions, which corresponds to frequency-coherent scattering. The branching ratio A for frequency coherence is therefore

$$A = \frac{\Gamma}{\Gamma + \Gamma_I + \Gamma_E} \,. \tag{40}$$

The second branching ratio B represents the fraction of the scattering processes that are subject to elastic collisions that destroy the frequency coherence but not the atomic polarization (the $2K$-multipole). The rate of such elastic collisions is $\Gamma_E - D^{(K)}$ (note that $D^{(0)} = 0$). We then get

$$B = \frac{\Gamma_E - D^{(K)}}{\Gamma + \Gamma_I + \Gamma_E}\; \frac{\Gamma}{\Gamma + \Gamma_I + D^{(K)}} \,. \tag{41}$$

Here the first factor on the right-hand side represents the probability that an elastic collision that does not destroy the atomic polarization occurs during the life time of the excited state. The second factor represents the probability that spontaneous emission occurs before the atomic polarization is destroyed. The product is the joint probability for radiative decay of the excited state to occur while the frequency coherence but not the atomic polarization is destroyed.

11. Redistribution matrix

The redistribution matrix $\boldsymbol{R}$ is defined by the equation

$$\boldsymbol{j} = \sigma \int \frac{\mathrm{d}\Omega'}{4\pi} \int \mathrm{d}\nu' \, \boldsymbol{R} \boldsymbol{I}_{\nu'} \,, \tag{42}$$

where $\boldsymbol{j}$ is the Stokes emission vector, σ the scattering coefficient, and $\boldsymbol{I}_{\nu'}$ is the incident Stokes vector. The rest frame redistribution matrix $\boldsymbol{R}_0$ is

$$\boldsymbol{R}_0 \sim \boldsymbol{M} \,, \tag{43}$$

where the Mueller scattering matrix $\boldsymbol{M}$ is given by Eq. (10), and the constant of proportionality is determined by the normalization condition that the integral of $\boldsymbol{R}$ over all incident and scattered directions and frequencies should be $\Gamma/(\Gamma + \Gamma_I)$ (the fraction of radiated energy) for the first matrix component, R_{11}. From Eqs. (8), (10), and (43) we see that $\boldsymbol{R}_0$ contains the coherency matrix $\tilde{r}_q \, \tilde{r}^*_{q'}$, properly averaged over the collisional processes as in Eq. (21).

In terms of commonly used terminology in partial redistribution theory (cf. Mihalas 1978; Frisch 1996) we may write

$$\boldsymbol{R}_0 = A\boldsymbol{R}_{\mathrm{II},\,0} + B\boldsymbol{R}_{\mathrm{III},\,0} \,, \tag{44}$$

where A and B are the two branching ratios of Eqs. (40) and (41), and $\boldsymbol{R}_{\mathrm{II},\,0}$ and $\boldsymbol{R}_{\mathrm{III},\,0}$ are the redistribution matrices for frequency coherence and complete frequency redistribution, respectively. They contain the combined redistribution in both frequency and polarization.

Only in the limit of zero magnetic field is it possible to factorize out the frequency redistribution from the polarization redistribution, such that we get scalar frequency functions $R_{\mathrm{II},\,0}$ and $R_{\mathrm{III},\,0}$ times frequency-independent phase matrices (cf. Stenflo 1994, pp. 216–217). The scalar functions are the same as those that appear in unpolarized partial redistribution theory.

In the general case of arbitrary magnetic fields such a factorization is not possible, since the different matrix elements have different frequency profiles due to the Zeeman splitting, and the Hanle angles $\alpha_{q-q'}$ and $\beta_{q-q'}$ are coupled to the generalized profile functions $\Phi^{\gamma}_{q,q'}(\nu)$ that depend on the Zeeman effect, through the two indices q and q'. Due to this coupling, the general case may be called the mixed Hanle-Zeeman regime.

Explicit expressions for the redistribution matrix $\boldsymbol{R}_{\mathrm{II},\,0}$ (the frequency-coherent case) have been given in the semi-classical treatment of Stenflo (1998) for the general, mixed Hanle-Zeeman regime of arbitrary magnetic fields. It could be shown (as was also demonstrated in the quantum treatment of Bommier 1997b) how various limiting cases could be retrieved from the general case, like the weak-field Hanle phase matrix, the polarization of coronal forbidden lines (the strong-field limit), "thermal" radiation (emission vector in LTE), and incoherent scattering. It could also be shown how and why there is a transition of the Hanle effect from being present in the line core to being absent in the line wings.

When we go from the rest frame to the observer's frame we have to introduce Doppler shifts for both the incident and scattered frequencies and integrate over a Maxwellian velocity distribution (cf. Stenflo 1994, pp. 75).

In a manner similar to the derivation of the general matrix $\boldsymbol{R}_{\mathrm{II},\,0}$ in Stenflo (1998) it is straightforward to use Eqs. (30) and (35) to derive the complete frequency redistribution matrix $\boldsymbol{R}_{\mathrm{III},\,0}$ for the general mixed Hanle-Zeeman case. It is outside the scope of the present paper to do this explicitly here.

12. Concluding remarks

The classical theory that we have presented here gives a self-consistent treatment of the general case of partial frequency redistribution for polarized radiation in the mixed Hanle-Zeeman regime. The results obtained are identical to those of quantum perturbation theory (Bommier 1997a,b) for the case of a $J = 0 \rightarrow 1 \rightarrow 0$ scattering transition. The advantage of the classical theory is that it is, in contrast to the quantum theory, non-perturbative and therefore conceptually more transparent, allowing us to see how the various physical effects enter in the various expressions.

Although it is not clear whether it is possible or meaningful to construct a classical analog for other types of atomic transitions with other quantum numbers, it is straightforward to make a phenomenological extension of the classical theory to enable it to cover scattering transitions with arbitrary quantum numbers. This is possible, because the quantum-mechanical scattering amplitudes given by the Kramers-Heisenberg formula have the same frequency dependence as r_q in the classical theory. The main additional features of the quantum theory are the relative line strengths determined by the oscillator strengths and 3-j symbols, and the transition frequency between the Zeeman components, which replaces $\omega_0 - q\omega_L$ in the classical theory (cf. Stenflo 1998). By replacing the r_qs in the classical theory with the corresponding Kramers-Heisenberg expressions, it is possible to develop a partial-redistribution theory for arbitrary multi-level atomic systems for the general case of Raman scattering (when the initial and final states are allowed to be different) in arbitrary magnetic fields. It is outside the scope of the present paper to do this extension explicitly here.

The remaining limitation of such an extended theory is the implicit assumption made here that the initial state has zero atomic polarization. To treat the still more general case when the initial, lower level is polarized by optical depopulation pumping one has to solve the statistical equilibrium equations for polarized radiation, as has been done by Trujillo Bueno and Landi Degl'Innocenti (1997) and Landi Degl'Innocenti (1998).

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