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A class of self-gravitating accretion disks

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Abstract. We consider a class of steady-state self-gravitating accretion disks for which efficient cooling mechanisms are assumed to operate so that the disk is self-regulated at a condition of approximate marginal Jeans stability. In an earlier paper, this scenario had been shown to lead naturally, in the absence of a central point mass, to a self-similar solution characterized by a flat rotation curve. In this article we investigate the entire parameter space available for such self-regulated accretion disks and provide two non-trivial extensions of the model. The first extension is that of a bimodal disk, obtained by partially relaxing the self-regulation constraint, so that full matching with an inner "standard" Keplerian accretion disk takes place. The second extension is the construction of self-regulated accretion disks embedded in a diffuse spherical "halo". The analysis is further strengthened by a careful discussion of the vertical structure of the disk, in such a way that the transition from self-gravity dominated to non-gravitating disks is covered uniformly.

Key words: accretion, accretion disks – gravitation – hydrodynamics

1. Introduction

For simplicity, traditional models of accretion disks ignore the self-gravity associated with the accreting material and consider the case where the disk is approximately Keplerian (e.g., see Pringle 1981). On the other hand, several studies have been carried out in order to investigate the effects related to the disk self-gravity. Studies of this type have addressed three different levels. The first stage is that where the disk self-gravity is incorporated in the study of the vertical structure of the disk. Here we may recall, among others, the pioneering work by Paczyński (1978) and the very recent analysis by Bardou et al. (1998). The second important level is that where one considers the effects of the disk self-gravity on waves and on transport processes in the disk. Here we may recall a number of interesting analyses, in particular the studies by Lin & Pringle (1987), by Adams et al. (1989), with the modified viscosity prescription proposed by Lin & Pringle (1990) (see also Andalib et al. 1997 and references therein). Recent hydrodynamical simulations (Laughlin & Bodenheimer 1994, Laughlin & Różyczka 1996, Laughlin et al. 1997) have focused on one key aspect of the modeling of accretion disks. i.e. they have tried to assess whether the standard constant α viscosity prescription (Shakura & Sunyaev 1973) is justified in systems where non-axisymmetric instabilities driven by the disk self-gravity play a major role. The third level where the self-gravity of the disk can be considered is that where, because of the gravitational field contributed by the disk matter, the rotation itself of the accreting material is affected and is no longer kept to be Keplerian. In this direction very few attempts have been made (see, e.g., Bodo & Curir 1992).

The need for investigations where the self-gravity of the disk is properly taken into account, is further stimulated by recent accurate observations that point to significant deviations from Keplerian rotation in objects that are naturally interpreted as accretion disks (e.g., see Greenhill et al. 1996, Moran 1998) and by the finding that in some contexts, especially in the dynamics of protostellar disks, there are empirical indications that the amount of mass in the disk is large (see Hillenbrand et al. 1992, Drimmel 1996). From the theoretical point of view, to address the dynamics of a self-gravitating accretion disk is highly attractive especially because some *global* regulation aspects that are emerging as interesting and relevant from various lines of thought (see also Coppi 1980) should find a natural manifestation in systems where the long-range forces are bound to generate an inherently global behavior.

The basic framework for the construction of models of accretion disks is traditionally "asymmetric". While the momentum transport equations are generally replaced by a physically based prescription that bypasses our ignorance on the detailed mechanisms that are involved (Shakura & Sunyaev 1973; see also the recent modification proposed by Heyvaerts et al. 1996 and Bardou et al. 1998), the energy transport equations are usually kept in their ideal form, either following a detailed inclusion of the radiation transport across the disk (e.g., see Shakura & Sunyaev 1973; Bardou et al. 1998) or by invoking some ideal "equation of state" and by arguing that the energy dissipated by viscosity is partly redistributed in the disk (see Narayan & Yi 1994 and many following articles). In a situation where we lack strong empirical constraints (such as those that might be available in the laboratory) on the detailed mechanisms at the basis of the various transport processes involved, one might try to explore models where self-gravity, much like in galaxy disks, plays a major role at all the three levels mentioned above, taking

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the view that *both* momentum and energy transport equations should be handled heuristically. This is indeed the step taken in an earlier exploratory paper (Bertin 1997), where the viscosity prescription suggested by Shakura & Sunyaev (1973) is retained, and, on a similar footing, the energy transport equations are replaced by a physical condition of self-regulation, related to marginal Jeans stability, as suggested by the dynamics of galaxy disks (see Sect. 3).

In this paper we develop this latter point of view by describing the entire parameter space available for *self-regulated accretion disks* (Sects. 2 and 4), and by providing two non-trivial extensions of the model. The first extension is that of a "bimodal" disk, where full matching with an inner "standard" Keplerian accretion disk is obtained (see Sect. 5.1). The second extension is the construction of self-regulated accretion disks embedded in a spherical diffuse "halo", which may find application to the extended disk associated with AGN's. The analysis is further strengthened by a careful discussion of the vertical structure of the disk, in such a way that the transition from self-gravity dominated to non-gravitating disks is covered uniformly (Appendix A).

2. Self-regulated accretion disks

2.1. Basic equations and parameter space

We consider a steady, axisymmetric, geometrically thin accretion disk, with constant mass and angular momentum accretion rates, which we call $\dot{M}$ and $\dot{J}$, respectively. The conservation laws for mass and angular momentum, in cylindrical coordinates, are:

$$\dot{M} = -2\pi r \sigma u, \tag{1}$$

$$\dot{J} = \dot{M} r^2 \Omega + 2\pi \nu \sigma r^3 \frac{\mathrm{d}\Omega}{\mathrm{d}r}, \tag{2}$$

where σ is the surface density of the disk, u is the radial velocity, Ω is the angular velocity, and ν is the viscosity coefficient. Note that we take $\dot{M} > 0$ in the case of inflow. Usually on the right hand side of Eq. (2) the two terms have opposite signs; the first term describes convection of angular momentum, while the second (negative in the common situation where $d\Omega/dr < 0$) accounts for the angular momentum transport due to viscous torques. Thus $\dot{J} > 0$ corresponds to a net angular momentum influx.

For a cool, slowly accreting disk, the radial balance of forces requires:

$$\Omega^2 \sim \frac{1}{r}\frac{\mathrm{d}\Phi_\sigma}{\mathrm{d}r} + \frac{GM_\star}{r^3}, \tag{3}$$

where $M_\star$ is the mass of the central object and Φ_σ is the disk contribution to the gravitational potential. The radial gravitational field generated by the disk can be written as:

$$\frac{\partial \Phi_\sigma}{\partial r}(r,z) = \frac{G}{r}\int_0^\infty \left[K(k) - \frac{1}{4}\left(\frac{k^2}{1-k^2}\right)\times \left(\frac{r'}{r} - \frac{r}{r'} + \frac{z^2}{rr'}\right)E(k)\right]\sqrt{\frac{r'}{r}}k\sigma(r')dr', \tag{4}$$

where $E(k)$ and $K(k)$ are complete elliptic integrals of the first kind, and $k^2 = 4rr'/[(r+r')^2 + z^2]$ (see Gradshteyn & Ryzhik 1980). The field $\mathrm{d}\Phi_\sigma/\mathrm{d}r$ in the equatorial plane is obtained by taking the limit $z \to 0$. Here we have preferred to refer directly to the field (and not to the potential, as was done by Bertin 1997) and to the formulae applicable to the case $z \neq 0$; this representation is more convenient in view of the numerical investigations that we have in mind (see Sects. 4 and 5).

For the viscosity we follow the standard prescription (Shakura & Sunyaev 1973):

$$\nu = \alpha c h, \tag{5}$$

where c is the effective thermal velocity and h the half-thickness of the disk. Here α is a dimensionless parameter, with $0 < \alpha \lesssim 1$; in the following we take it to be a constant.

In the case of dominant disk self-gravity, one may adopt the requirement of hydrostatic equilibrium in the z direction for a self-gravitating slab model, which gives

$$h = \frac{c^2}{\pi G \sigma}. \tag{6}$$

A more refined analysis of the vertical equilibrium is provided in Appendix A. Substituting Eq. (6) into Eq. (5) we obtain $\nu\sigma = (\alpha/\pi G)c^3$, which inserted in Eq. (2) yields:

$$G\dot{J} = G\dot{M} r^2 \Omega + 2\alpha c^3 r^3 \frac{\mathrm{d}\Omega}{\mathrm{d}r}. \tag{7}$$

In order to close the set of equations (Eq. (3), Eq. (4), and Eq. (7)), we need an additional relation. In standard studies this is provided by an energy transport equation (see, for example, Pringle 1981; Narayan & Popham 1993; Narayan & Yi 1994). Here we consider an alternative scenario (Bertin 1997) where the energy equation is *replaced* by the requirement of marginal Jeans stability:

$$\frac{c\kappa}{\pi G \sigma} = \bar{Q} \approx 1, \tag{8}$$

where κ is the epicyclic frequency. This assumes the presence of a suitable *self-regulation mechanism* (see Sect. 3 below). For a given value of α and $\bar{Q}$, we now have a complete set of equations, depending on the three parameters $M_\star$, $\dot{M}$, and $\dot{J}$. We recall that the parameter $c\kappa/\pi G\sigma$ is the fluid analogue of the axisymmetric stability parameter introduced and described by Toomre (1964) in his analysis of the stability of a stellar disk.

2.2. The self-similar pure disk solution

In the special case of $\dot{J} = 0$ and $M_\star = 0$, the problem is solved by the self-similar solution with flat rotation curve (Bertin 1997), characterized by:

$$2\pi G\sigma r = r^2\Omega^2 = V^2 = const. \tag{9}$$

The other properties of the disk are specified by: $c = (G\dot{M}/2\alpha)^{1/3}$, $V = (2\sqrt{2}/\bar{Q})c$, $u = -(\alpha\bar{Q}^2/4)c$, $h/r = \bar{Q}^2/4$. Note that Eq. (9) describes the self-similar disks found by Mestel (1963) in the non-accreting case.

The same solution is also valid asymptotically at large radii even when $M_\star$ and $\dot{J}$ do not vanish. In fact, the effect of a non-zero $M_\star$ should be unimportant for $r \gg r_s = 2GM_\star(\bar{Q}/4)^2(G\dot{M}/2\alpha)^{-2/3}$ while the effects of a non-zero $\dot{J}$ should be unimportant for $r \gg r_J = \sqrt{2}(|\dot{J}|/\dot{M})(\bar{Q}/4)(G\dot{M}/2\alpha)^{-1/3}$. Here the definitions of r_s and r_J are slightly different from those adopted earlier (Bertin 1997).

2.3. The iteration scheme for the general case

For a given value of α and $\bar{Q}$ and for a specified accretion rate $\dot{M}$, the general case is a problem with a well-defined physical lengthscale r_s and one dimensionless parameter $\xi = sgn(\dot{J})r_J/r_s$. In the following we wish to explore the properties of the mathematical solutions in the entire parameter space available (in particular, for both positive and negative values of ξ). The knowledge of these solutions is a prerequisite for a discussion of any specific astrophysical application. At a later stage, the physical problem under investigation is expected to restrict the relevant parameter space.

In order to calculate such one-parameter family of solutions, we start by writing the relevant equations in dimensionless form, in such a way that the self-similar solution described above is easily recognized. We thus introduce three *deviation functions* ρ, ϕ, and χ, which are related to the dimensional disk density σ, rotation curve V, and effective thermal speed c by the following definitions:

$$2\pi G\sigma = \frac{V_0^2}{r}(1+\rho), \tag{10}$$

$$V^2 = V_0^2\phi^2, \tag{11}$$

$$c = V_0\left(\frac{\bar{Q}}{2\sqrt{2}}\right)\chi, \tag{12}$$

with $V_0^2 = (8/\bar{Q}^2)(G\dot{M}/2\alpha)^{2/3}$. Clearly the functions $(1+\rho)$, ϕ^2, and χ are all positive definite; the self-similar solution corresponds to $\rho = 0$, $\phi = 1$, $\chi = 1$.

In terms of the dimensionless radial coordinate $x = r/r_s$, the basic set of equations (corresponding to Eqs. (3), (4), (7), and (8)) thus become:

$$\phi^2 = 1 + \hat{\phi}^2[\rho/x] + \frac{1}{x} \tag{13}$$

$$\begin{aligned}&\hat{\phi}^2[\rho/x] = \frac{1}{2\pi}\int_0^\infty dx' \frac{k}{\sqrt{xx'}}\times\\ &\left[K(k) - \frac{1}{4}\frac{k^2}{1-k^2}\left(\frac{x'}{x} - \frac{x}{x'} + \frac{\delta^2}{xx'}\right)E(k)\right]\rho(x'),\end{aligned} \tag{14}$$

$$\chi(x) = \left(1 - \frac{\xi}{x\phi}\right)^{1/3}\left(1 - \frac{\mathrm{d}\ln\phi}{\mathrm{d}\ln x}\right)^{-1/3}, \tag{15}$$

$$\rho(x) = \chi\phi\sqrt{1 + \frac{\mathrm{d}\ln\phi}{\mathrm{d}\ln x}} - 1, \tag{16}$$

with $k^2 = 4xx'/[(x+x')^2 + \delta^2]$, in the limit $\delta \to 0$.

In the case of outward angular momentum flux ($\xi < 0$), the above equations can be readily solved by iteration in the following way. An initial seed solution $\phi_0 = \sqrt{1+1/x}$ is inserted in Eqs. (15) and (16), thus leading to a first approximation to the density deviation $\rho_0(x)$. This is inserted in Eq. (14), then producing (via Eq. (13)) a new expression for the rotation curve deviation $\phi_1(x)$. Typically three or four iterations are sufficient to reach a satisfactory convergence.

For the case when the net angular momentum flux is inwards ($\xi > 0$), Eq. (15) shows that we may run into a difficulty at small radii, where the quantity $(1 - \xi/x\phi)$ changes sign. Here we may proceed by analogy with standard studies (e.g., see Pringle 1981), by restricting our analysis to the outer disk defined by $x > x_{in}$, with the condition $x_{in}\phi(x_{in}) = \xi$, i.e. $\dot{J} = \dot{M}r_{in}^2\Omega(r_{in})$. This is taken to occur at a point where the angular velocity $\Omega(r)$ reaches a maximum (so that $\mathrm{d}\ln\phi/\mathrm{d}\ln x = 1$ and χ may remain finite); such maximum is identified as a location, in the vicinity of the surface of the central object, to which the accretion disk is imagined to be connected by a relatively narrow boundary layer. Of course, in the boundary layer the physical processes will be different from those described in our model (for example, there will be effects due to pressure gradients and the disk will be thick). Thus we will not be able to follow the associated (inwards) decline of the angular velocity away from the Keplerian profile. As a result, in our calculation we let the effective thermal speed vanish at r_{in}; this unphysical behavior would be removed when one describes the boundary layer with more realistic physical conditions (see Popham & Narayan 1991).

3. Self-regulation in self-gravitating accretion disks

Before proceeding further, we should make here a digression and try to better explain the physical justification at the basis of the regulation prescription set by Eq. (8), which is the distinctive property of the class of models addressed in this paper. This discussion expands and clarifies a brief description provided earlier (Bertin 1997).

3.1. The mechanism

We start by noting that, when self-gravity dominates, processes associated with the Jeans instability basically determine the relevant scales of inhomogeneity of the system under consideration. On the other hand, it is well known (Toomre 1964) that in the plane of a thin, self-gravitating, rotating disk the tendency toward collapse driven by self-gravity, which is naturally expected to be balanced locally only at small wavelengths by pressure forces, can be fully healed by rotation also at small wavenumbers. As a result, if the disk is sufficiently warm, it can be locally stable against all axisymmetric disturbances. The condition of marginal stability thus ensured is usually cast in the form $Q = 1$, where Q is proportional to the effective thermal speed of the disk.

Therefore, in galactic dynamics it has long been recognized that an initially cold disk is subject to rapid evolution, on the dynamical timescale. For a stellar disk, the Jeans instability induces fast overall heating of the disk up to levels of the effective thermal speed for which the instability is removed, basically following the above criterion ($Q \approx 1$). The heating rate is thus very sensitive to the instantaneous value of Q. Therefore, even in the absence of dissipation, for relatively low values of Q the collective Jeans instability provides a mechanism to stir the system and to heat it. However, a collisionless non-dissipative disk would be unable to evolve in the opposite direction, i.e. to cool if initially hot ($Q > 1$), and, because of other residual processes (such as tidal interactions), may still be subject to some perennial heating even when Jeans instability is ineffective. In turn, if efficient dissipation is available (such as that associated with the interstellar medium in galaxy disks), a competing mechanism can cool an initially hot disk down toward conditions of marginal Jeans instability.

These important dynamical ingredients have been studied from various points of view, giving rise to interesting scenarios, also by means of numerical experiments (see Miller et al. 1970, Quirk 1971, Quirk & Tinsley 1973, Sellwoood & Carlberg 1984, Ostriker 1985), so that eventually the possibility of an actual *self-regulation* in self-gravitating disks has been formulated and explored. Accordingly, the two competitive mechanisms, of dynamical heating and dissipative cooling, can set up a kind of dynamical thermostat. In particular, it has been shown that self-regulation is very important in determining the conditions for the establishment of spiral structure in galaxies (Bertin and Lin 1996; and references therein). The general concept has also found successful application to the dynamics of the interstellar medium, even when dynamically decoupled from the stellar component, in relation to the interpretation of the observed star formation rates (Quirk 1972; Kennicutt 1989). Note that the most delicate aspect of the regulation process is the cooling mechanism; in galaxy disks, an important contribution to cooling is thus provided by cloud-cloud inelastic collisions that dissipate energy on a fast timescale.

A final remark is in order. As shown by the case of galaxy disks, the dynamics involved may be extremely complex, so that there is little hope of providing a simple "ideal" energy equation for the description of a coupled disk of stars (of different populations) and gas (in different forms and phases). This point has an additional important consequence. If we try to describe such an inherently complex system by means of an idealized one-component model, we should be ready to introduce the use of some *effective* quantities (in particular, of an effective thermal speed; see also the use of the term "effective" after Eq. (5) and at the beginning of this subsection).

3.2. Viability of the mechanism for some accretion disks

It should be emphasized that the main strength of the self-regulation scenario described above is rooted in semi-empirical arguments. The actual data from many galaxy disks (among which the Milky Way Galaxy) and from planetary rings show that conditions of marginal stability ($Q \approx 1$) often occur and can indeed be established in systems subject to complex dynamical processes. For accretion disks, there may already be some empirical indications in this direction, to the extent that the application of standard models to some observed systems points to very cold disks, with Q well below unity (e.g., see Kumar 1999, as briefly mentioned in Sect. 6). Additional clues in the same direction also derive from numerical experiments; in the context of protostellar disks, numerical simulations show that disks formed under conditions where self-gravity is important include wide regions characterized by a constant Q-profile (Pickett et al. 1997).

An accretion disk may be subject to efficient cooling by a variety of mechanisms, depending on the physical conditions that characterize the specific astrophysical system under consideration. From this point of view, the cooling necessary for the establishment of self-regulation may occur efficiently already via the radiative processes included in "standard" models (see the general discussion of the outer disk by Bardou et al. 1998, which we will summarize in Sect. 5.1). In reality, the dynamics of matter slowly accreting in a disk can be significantly more complex. Cold systems, such as a protogalactic disk, a protostellar disk, or the outer parts of the disk in an AGN, may have a composite and complex structure. They may include dust, gas clouds, and other particulate objects with a whole variety of sizes and "temperatures". Much like for the HI component of the interstellar medium, the main contribution to the effective temperature of the disk might be from the turbulent speed of an otherwise cold medium. On the one hand, for these systems it may be hard or even impossible to write out a simple "ideal" energy transport equation. On the other hand, such a complex environment is likely to possess all the desired cooling and heating mechanisms that cooperate in self-regulation. In this respect, one is thus encouraged to bypass the problem of defining a representative set of equations for energy transport, and to use instead the semiempirical prescription of Eq. (8). Somewhat in a similar way, our inability to derive from first principles a satisfactory set of equations for momentum transport is often taken to justify the adoption of the α-prescription of Eq. (5). These phenomenological prescriptions have several limitations, but may still work as a useful guide to our efforts and provide interesting models to be compared with the observations.

To be sure, some types of accretion disk, or some regions inside accretion disks (for example, very close to the center; see Sect. 5.1), may lack the physical ingredients invoked above. In fact, there is no reason to claim that self-gravity must *always* be important. Therefore, we will study the structure of self-regulated accretion disks, as a viable class of astrophysical systems, while we do recognize that warmer, non-regulated disks may exist and are likely to be basically free from the effects associated with the self-gravity of the disk.

3.3. The impact of Q on momentum and energy transport

The self-regulation mechanism has been demonstrated by considering a simplified set of equations (Bertin 1991) where efficient cooling is included and the role of self-gravity is modeled by means of a heating term with an analytic expression (in-

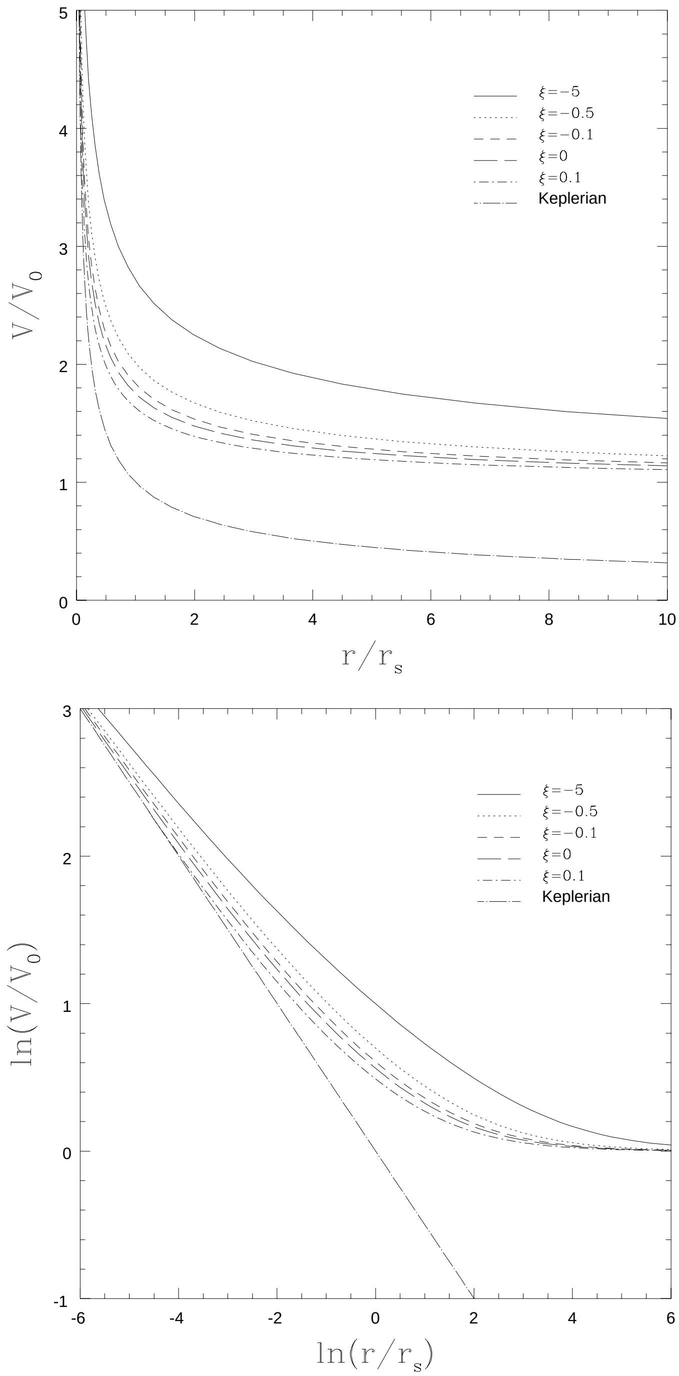

Fig. 1. *Top panel:* Rotation curve of the disk in the presence of a central point mass for different values of the angular momentum flux parameter. *Bottom panel:* Same as top panel, but in logarithmic scale.

versely proportional to a high power of Q) meant to incorporate the results of dynamical studies that show that heating is indeed very sensitive to the value of Q. The main features of this formula, with its threshold at $Q \approx 1$, aimed at representing the "thermal evolution" of the disk, are somewhat analogous to the heuristic characterization of the viscosity dependence on Q adopted by Lin & Pringle (1990) in the parallel problem of constructing the momentum transport equations when self-gravity is important.

In our discussion of self-regulated accretion disks, we actually have no doubt that self-gravity is likely to have an important impact on viscosity, and this is still tacitly incorporated in the α prescription. This impact is even more obvious if one recalls that a self-gravitating disk can be subject to non-axisymmetric instabilities, which are bound to contribute significantly to angular momentum transport. Our class of axisymmetric, steady-state accretion models represents only one approximate idealization of the actual system that we are addressing. Given the indications of several dynamical studies (in addition to those of Lin & Pringle, see, for example, Laughlin & Bodenheimer 1994, Laughlin & Różyczka 1996), a more complete analysis should thus include one further relation between α and Q. In general, this might practically require that the phenomenological prescription (5) be used with a parameter α varying with radius. In reality, we believe that the proper way to include the relevant physical effects, especially those associated with non-axisymmetric instabilities, would be through some *global* constraint. Until such global description remains not available, the assumption of a free, constant α may provide a first approximation, best applicable when Q is self-regulated. This choice can be physically consistent *a posteriori*, at least for the self-similar solution of Sect. 2.2.

4. Properties of models with a central point mass

After this digression, we can now proceed with the analysis of the problem as formulated in Sect. 2. In the presence of a central point mass ($M_\star \neq 0$) we expect the lengthscale r_s to mark the transition from a Keplerian disk to a fully self-gravitating disk with flat rotation curve. Surprisingly, in our class of self-regulated accretion disks the role of the disk self-gravity turns out to be significant all the way down to the center.

4.1. Rotation curves

In Fig. 1 we illustrate the behavior of the rotation curves in our class of models for several values of the angular momentum flux parameter ξ. For comparison, we also show the Keplerian curve V_K that is obtained by setting $1+\rho=0$. We note that the difference from the Keplerian decline is significant even at $r = O(r_s)$. For example, for the $\xi = 0$ case we find $(V - V_K)/V_K \approx 100\%$ at $r = 2r_s$.

4.2. Effective thermal speed

Disks where the angular momentum is transported outwards ($\xi < 0$) tend to develop a warmer core, while the opposite trend occurs for disks where the angular momentum is carried inwards ($\xi > 0$). This is shown in Fig. 2 (Fig. 2 and the following Fig. 3, Fig. 4, and Fig. 5 are shown in logarithmic scale to better bring out the behavior in the inner parts of the disk). Note that in the case of inward angular momentum flux ($\xi > 0$) the effective thermal speed need not vanish at the inner edge of the disk (as

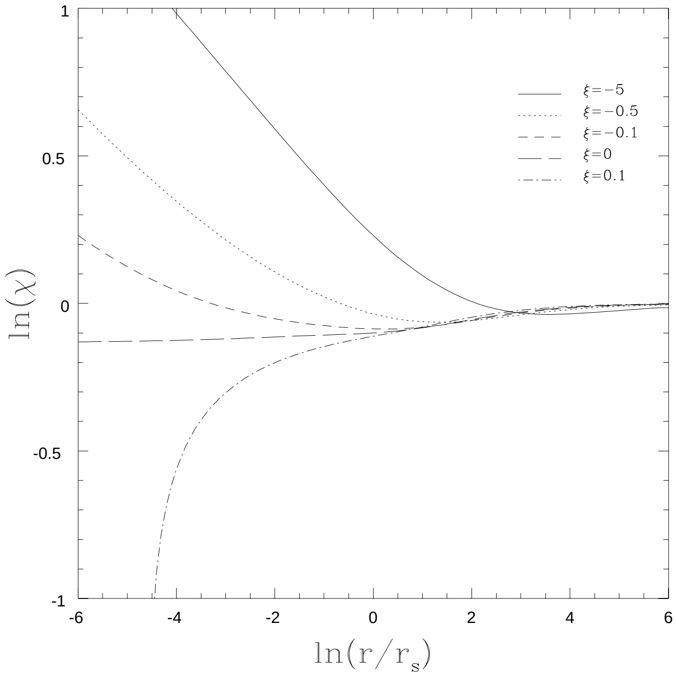

Fig. 2. Equivalent thermal speed of the disk. The two cases ($\xi > 0$ and $\xi < 0$) show opposite behavior in the inner disk.

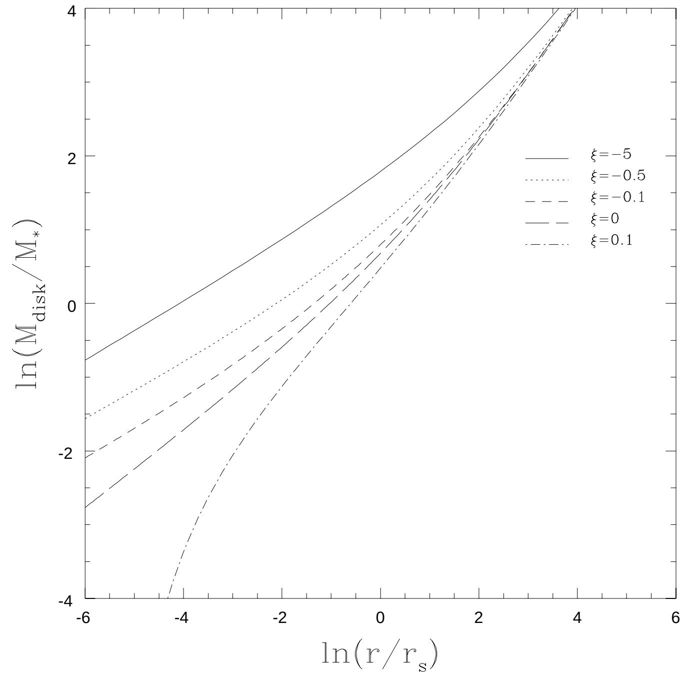

Fig. 3. Cumulative mass of the disk relative to that of the central object.

it does in our models) where a boundary layer is expected to be generated (see discussion at the end of Sect. 2.3).

4.3. Role of the disk self-gravity close to the central point mass

There are several quantities directly related to the disk density distribution in the disk that allow us to characterize the role of the disk self-gravity.

The most natural quantity to consider is the ratio of the mass of the disk to that of the central object. Obviously, for our non-truncated models this quantity is meaningful only when referred to a given radius. Fig. 3 shows how rapidly in radius the system becomes dominated by the mass of the disk. Note that, in any case, $M_{disk}(r)/M_\star \to 0$ for $r \to 0$.

In galactic dynamics the local disk self-gravity is usually measured in terms of the parameter $\epsilon = \pi G\sigma/r\kappa^2$. The fully self-gravitating self-similar disk (with flat rotation curve) is characterized by $\epsilon = 1/4$. The profile of this parameter (see Fig. 4) confirms that indeed, close to the center, the influence of the central mass becomes stronger and stronger.

Given the behavior of the profiles $M_{disk}(r)/M_\star$ and $\epsilon(r)$, one might conclude that the innermost disk should be treated as a standard Keplerian accretion disk. This conclusion is contradicted by the following argument. In setting up the equations of our models, we have taken the vertical equilibrium to be dominated by the disk self-gravity (see Eq. (6)). If the innermost disk were fully Keplerian, at small radii the vertical scaleheight $h_\star = cr/V_K$ should become much smaller than the thickness h associated with our models. Instead, a plot of the ratio $h_\star/h$ only shows that the two scales become *comparable* to each other (see

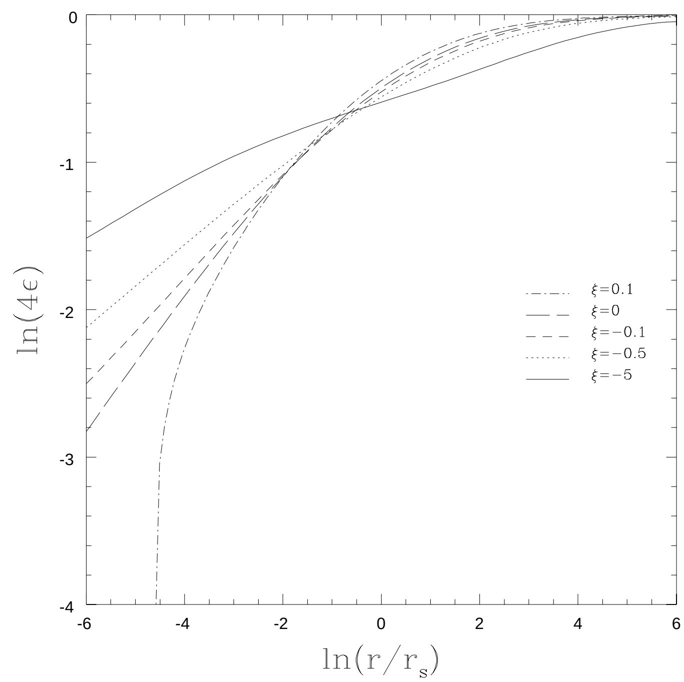

Fig. 4. Local self-gravity of the disk, as measured by $\epsilon = \pi G\sigma/r\kappa^2$, for different values of the angular momentum flux parameter.

Fig. 5), thus demonstrating that the influence of the disk self-gravity is significant all the way to the center. This, of course, reflects our choice of imposing the self-regulation prescription (Eq. (8)) at all radii; in the next section we will show that the disk can indeed make a true transition to a Keplerian disk if the self-regulation prescription is suitably relaxed.

In view of the above considerations, in order to check that no major consequences arise from the use of a vertical equilib-

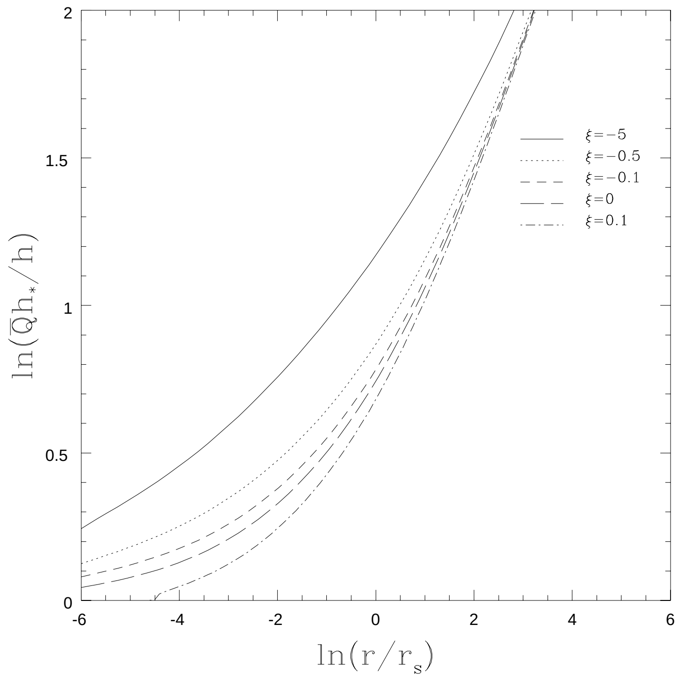

Fig. 5. Ratio of the vertical scaleheight $h_\star = cr/V_K$ to the thickness of the disk for various models.

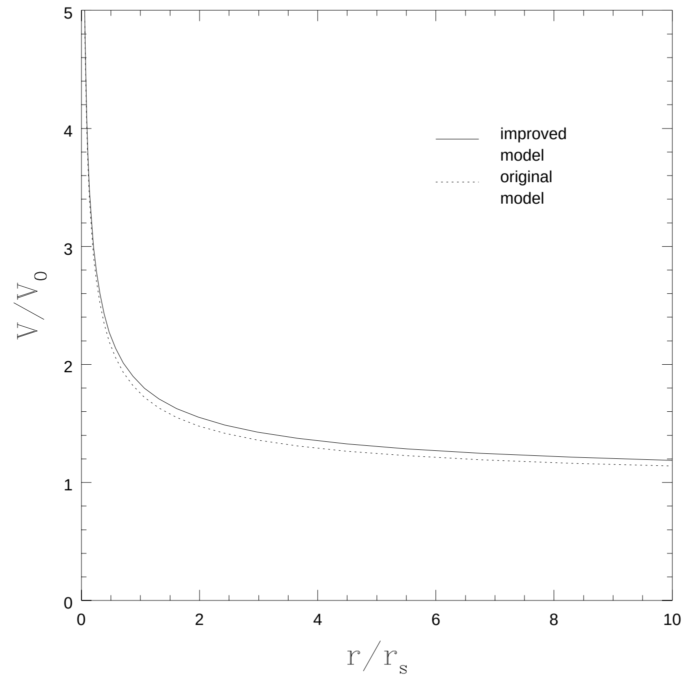

Fig. 6. Rotation curves for two different treatments of the vertical structure for the case $\xi = 0$, $\bar{Q} = 1$.The solid line represents the improved model, based on Eq. (17); the dotted line is the original model, based on Eq. (6).

rium prescription only partly justified for $r < r_s$, we have also considered models based on the improved prescription for the disk thickness (see Appendix A):

$$h = \frac{c^2}{\pi G\sigma}\frac{\pi}{4Q^2(2\Omega^2/\kappa^2-1)}\times \left[\sqrt{1+\frac{8}{\pi}Q^2\left(\frac{2\Omega^2}{\kappa^2}-1\right)}-1\right]. \quad (17)$$

Note that in the limit $2\Omega^2/\kappa^2 \to 1$, applicable to the self-similar disk, this prescription reduces to Eq. (6). In the innermost region $\Omega \to V_K/r$, $\Omega/\kappa \to 1$, so that $h_\star/h \to (4\bar{Q}/\pi)/[\sqrt{1+8\bar{Q}^2/\pi}-1]$.

In Fig. 6 we show the rotation curve of the improved model, compared to that of the original one, for the $\xi = 0$ case. This plot (along with similar results for the other relevant physical quantities of the disk) shows that, qualitatively, the more refined vertical analysis leaves the models basically unchanged.

5. Extensions

5.1. Matching with an inner Keplerian, non-self-regulated accretion disk

If we take an astrophysical situation (such as an AGN or a protostar) with specific physical conditions, it is likely that the arguments that support the adoption of the self-regulation constraint, followed in this paper, fail outside a well-defined radial range, either at small or at large radii. For example, we may recall that in the context of the dynamics of spiral galaxies the relevant Q profile is argued to be flat in the outer disk, but is thought to increase inwards inside a circle of radius r_Q often identified as the radial scale of influence of the bulge; even in the absence of a bulge, the central parts of the disk are thought to be generally hotter (e.g., see Bertin & Lin, 1996). In our context, we may then imagine an accretion disk where at small radii self-regulation fails, Q becomes large, and, correspondingly, the disk self-gravity ceases to be important.

This discussion suggests that it should be interesting to explore the possibility where Eq. (8) is replaced by a condition of the form $c\kappa/\pi G\sigma = Q(r)$, with a Q profile that decreases monotonically with radius and reaches the self-regulation value $\bar{Q}$ only beyond a certain radial scale r_Q. What would be the impact of such a choice on the structure of the accretion disk? What might be the physical arguments leading to the justification of such a profile? In particular, what would set the scale r_Q for given values of the parameters α, $M_\star$, $\dot{M}$, and $\dot{J}$? Note again that we are reversing the standard point of view, whereby one may ask what is the Q profile for an accretion disk, based on the structure calculated from a given choice of the energy equations.

Imposing, as we are going to do, a given profile $Q(r)$ may, at first sight, appear to be arbitrary. In reality, the freedom in the choice of the profile allows us to test quantitatively how the dynamical characteristics $V(r)$, $M_{disk}(r)$, $\epsilon(r)$, and $h(r)$ of the disk change when the self-regulation constraint is partially relaxed, in the inner disk, in a variety of ways. It is up to us to test the different possibilities (which may correspond to completely different sets of energy balance equations in the inner disk) that might be considered. For our purposes, since we wish to study the deviations from the standard "Keplerian" case, we only need to take into account that the relevant physical processes match so that in the outer disk self-regulation is enforced. Note that in the transition region where matching between the inner and the outer disk occurs, for reasons expressed in Sect. 3, it may be practically impossible to define, from first principles, a satisfactory set of energy equations able to include all the desired radiation processes and the non-linear effects associated with Jeans instability.

This procedure draws considerable support from a recent analysis of "standard" disks (Bardou et al. 1998) aimed at detecting evidence for the importance of disk self-gravity in the outer disk. Based on an extension of the "standard" α-disks (characterized by Kramers' opacity and by neglect of radiation pressure), the effects of the disk self-gravity have been here incorporated by means of an improved thickness prescription (somewhat in the spirit of our Appendix A) and of a modified viscosity prescription, but the (Keplerian) Ω profile is left unaltered. Therefore, this study is ideally suited to describe the conditions of our inner disk, as we intend to partially relax the self-regulation requirement. A very important result of the analysis by Bardou et al. (1998) is that their "standard" description breaks down beyond a radius r_Q, well inside which the local stability parameter behaves approximately as $Q \sim r^{-9/8}$; as it might have been anticipated, the location where the standard model breaks down coincides with the location where Q becomes approximately equal to unity. In conclusion, the analysis by Bardou et al. (1998) encourages us to consider the following choice of Q profile

$$\frac{c\kappa}{\pi G\sigma} = \bar{Q}\{1+(r/r_Q)^{-9/8}\exp[-(r/r_Q)]\} \tag{18}$$

to be used instead of Eq. (8). (The formula is meant to be used as a semi-empirical tool; one should keep in mind that the exact form of the Q profile will be determined by the detailed energy processes occurring in the inner disk and by the progressively important role of collective instabilities.) If we express the scale r_Q found in that study in terms of our scale r_s, we find

$$\frac{r_Q}{r_s} \simeq 0.8\cdot 10^{-4}\frac{\alpha^{-2/45}}{\bar{Q}^2}\left(\frac{M_\star}{10^8 M_\odot}\right)^{-2/3}\left(\frac{\dot{M}}{10 M_\odot/yr}\right)^{8/45}. \tag{19}$$

Note that the ratio r_Q/r_s *decreases* while $M_\star$ increases, and that its value is only weakly dependent on α and on $\dot{M}$. For parameters typical of an AGN, self-regulation may thus be ensured very far in, while an extrapolation of the above recipe to parameters typical of a disk surrounding a T Tauri star suggests that, for these latter objects, r_Q should become $O(r_s)$.

Some examples of *partially self-regulated models* computed on the basis of Eq. (18) (see Fig. 7) show how an outer disk dominated by self-gravity can match in detail with an inner standard Keplerian accretion disk.

5.2. *The effect of a diffuse "halo"*

So far we have considered the case where the mass is all distributed in a disk (either at the center, as a point mass, or in diffuse form). In view of possible applications to AGN configurations or to the general galactic context, it is important to consider an extension of the models to the case where part of the gravitational field is determined by a diffuse spherical component, which we will call *halo* (even if it may just correspond to the central region of an elliptical galaxy). This will lead to rotation curves otherwise not accessible by our models. On the other hand, it is easily recognized that this natural extension is going to leave the slow density decline of the disk unaltered. Therefore, if we are interested in producing models with finite total mass, we should be ready to impose an outer truncation radius or, which might effectively be equivalent, to relax the self-regulation prescription in the outermost disk.

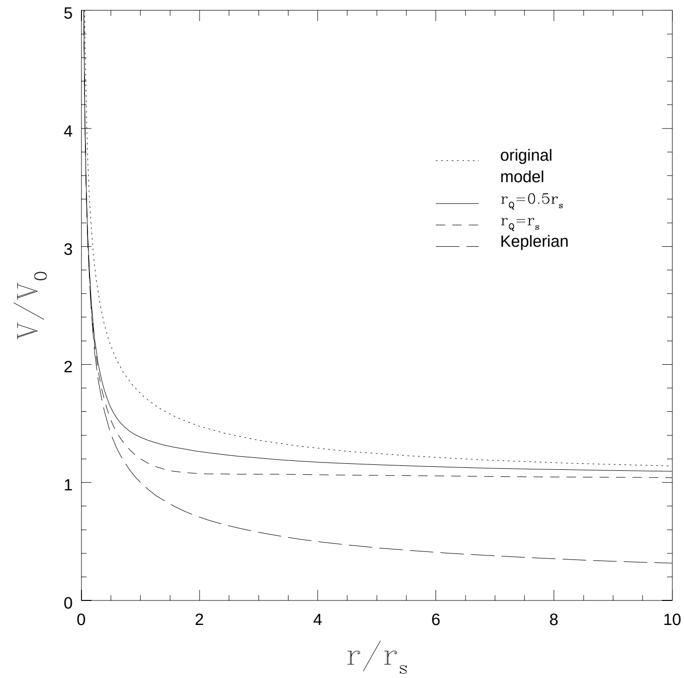

Fig. 7. Rotation curves of partially self-regulated models with the Q-profile given in Eq. (18), for different values of r_Q. The thickness prescription used is that of Eq. (17).

We have thus considered a set of models where the field external to the disk is produced by the joint contribution of a central point mass and of a halo (which, for simplicity, we take to be spherical). In view of the case of a disk embedded in an elliptical galaxy, we have modeled the halo as approximately isothermal, with a finite core radius. In this case the dimensionless equation giving the rotation curve (Eq. (13)) is modified as follows:

$$\phi^2 = 1 + \frac{1}{x} + \hat{\phi}^2[\rho/x] + \frac{f^2x^2}{x_0^2+x^2}. \tag{20}$$

We see that now the equations depend on two additional parameters: f^2, giving the relative strength of the external field, and x_0, which measures the size of the core radius. In this case it is easy to demonstrate that at large radii the density deviation ρ approaches f^2 if $f \ll 1$, and f if $f \gg 1$.

In Fig. 8 we show examples of the rotation curve of models with a diffuse halo, for the case $\xi = 0$, $x_0 = 1$. For the vertical

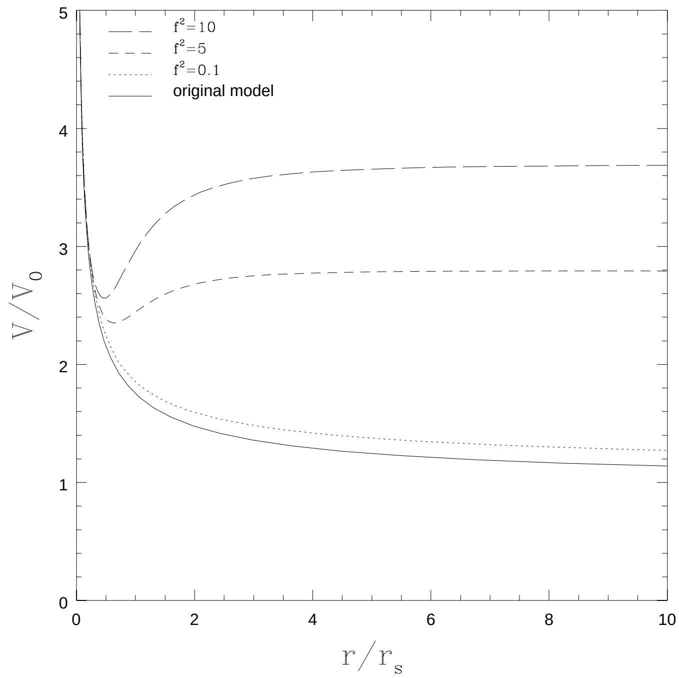

Fig. 8. Rotation curves of models with a diffuse halo, with $\xi = 0$, and core radius equal to r_s ($x_0 = 1$), for different values of f^2.

structure, we have referred to the improved vertical prescription of Eq. (A9), with $\Omega_{ext}^2 \propto f^2/(x_0^2 + x^2)$.

5.3. Disks with an outer truncation radius

Self-regulated accretion disks with finite mass can be easily constructed by imposing the existence of an outer truncation radius. Either the study of the collapse of a gas cloud with finite mass or the consideration of the physical conditions in the outer parts of some astrophysical objects will naturally bring us to address such models.

6. Discussion

In this paper we have developed a framework for the construction of a class of self-gravitating accretion disks, independently of the specific conditions characterizing the astrophysical systems where the accretion disk paradigm applies. In principle, we might even speculate that the above framework could be the starting point to describe some stages of the formation of protogalactic disks (e.g. see Fall & Efstathiou 1980) or the establishment of some regular extended HI disks in elliptical galaxies (see Morganti et al. 1997). In practice, our class of models appears to be best applicable to two categories of astrophysical objects, i.e. to Active Galactic Nuclei and to protostellar disks. Concrete applications, which will require a detailed consideration of the available observational constraints, are beyond the scope of the present paper. Still we would like to make a few comments that should illustrate why the models appear to be promising for the above categories of objects.

Recently, from radio maser emission, it has been possible to obtain accurate measurements of the rotation of the disk in the central parts of a few Active Galactic Nuclei; these include NGC 4258, NGC 1068, Circinus, NGC 4945, NGC 3079, and NGC 1386 (see Moran, 1998). These measurements have shown that, although in some cases (as for NGC 4258, see Miyoshi et al. 1995) the rotation curve is Keplerian to a high degree of approximation, there may be significant deviations from the $r^{-1/2}$ profile. For example, in NGC 1068 the rotation velocity at distances ≈ 1 pc from the "center" turns out to decline as $r^{-0.35}$ (see Greenhill et al. 1996; Kumar 1999). The general conclusion is that rather extended disks, which may contain large amounts of mass, are present; indeed one possible reason for the deviation from the Keplerian decline has been identified in the influence of the gravitational field contributed by the accretion disk itself. At the same time, the application of "standard" accretion disk models to some of these objects has led to finding values of the Q parameter well below unity (for NGC 1068, see Kumar 1999). All these are clues that show that a model where the role of the disk self-gravity is fully incorporated is called for. Note that if we adopt the numbers suggested by the data for NGC 1068 we would find $r_s \approx 25$ pc and $r_Q \approx 1.6\ 10^{-4} r_s$; then, it is curious to find that a self-regulated disk with these characteristics would have, for $\xi = -5$, a gradient of the rotation curve at $r \approx 1$ pc compatible with $V \sim r^{-0.37}$.

On a completely different mass scale, if we refer to the case of protostellar disks, typically quoted parameters are $\dot{M} \approx 10^{-8} M_\odot/yr$ and $M_\star \approx 0.5 M_\odot$ (Hartmann et al. 1998). Under these circumstances we find $r_s \approx 1000$ AU and $r_Q \approx 0.8 r_s$. Interestingly, protostellar disks have been observed to extend out to a radius from ≈ 100 AU to ≈ 1000 AU (Dutrey et al. 1996; McCaughrean & O'Dell 1996). As for the case of AGN's, a check on the values of the temperatures anticipated on the basis of the effective thermal speeds predicted by the self-regulated models shows that the numbers fall reasonably within the range suggested by the observations.

Finally, it is interesting to note that studies of the *spherical, inviscid* collapse of a molecular cloud, imagined to eventually generate a system composed of a protostar and a circumstellar disk, lead to mass accretion rates $\dot{M} \propto c^3$ (Hunter 1977, Shu 1977), curiously analogous to those of our completely different accretion scenario. This coincidence may offer interesting clues for modeling the entire process where a cloud, starting out in an initially spherical collapse, ends up in a protostellar accretion disk.

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Appendix A: modeling the vertical density profile

The vertical structure of the disk is generally derived by imposing hydrostatic equilibrium in the z-direction under the assumption of isothermality. The relevant Poisson equation and hydrostatic equilibrium condition for an axisymmetric configuration can be written as (here the symbol ρ denotes volume mass density and should not be confused with the density deviation function of the main text):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{\partial^2\Phi}{\partial z^2}=4\pi G\rho+4\pi G\rho_{ext}, \quad \text{(A1)}$$

$$\frac{c^2}{\rho}\frac{\partial \rho}{\partial z}=-\frac{\partial\Phi}{\partial z}, \quad \text{(A2)}$$

where we have taken polar cylindrical coordinates. For simplicity, we assume that the external field associated with ρ_{ext} is spherical. [In the class of models studied in Sect. 2 and Sect. 4 we have taken the case where $\rho_{ext}=M_\star\delta(\mathbf{x})$.] From the definitions of Ω and of κ, it is readily shown that, close to the equatorial plane, Eq. (A1) can be rewritten as:

$$\frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho+4\pi G\rho_{ext}+2\Omega^2-\kappa^2 \quad \text{(A3)}$$

$$= 4\pi G\rho+(2\Omega^2-\kappa^2)_\sigma+\Omega^2_{ext}. \quad \text{(A4)}$$

The latter expression follows from the exact relation $4\pi G\rho_{ext}=(\kappa^2-\Omega^2)_{ext}$ applicable to a spherical density distribution. The expression $(2\Omega^2-\kappa^2)_\sigma$ indicates that the quantity in parentheses should be calculated based on the radial field given by Eq. (4). For a relatively thin disk the quantity $(2\Omega^2-\kappa^2)_\sigma+\Omega^2_{ext}$ may be taken to be nearly independent of z, so that the gravitational field obtained by integrating Eq. (A4) includes a term that is approximately linear in z. The result can be inserted in the right hand side of Eq. (A2), which becomes:

$$\frac{\mathrm{d}^2\sigma_z}{\mathrm{d}z^2}=-\frac{\mathrm{d}\sigma_z}{\mathrm{d}z}\left\{\frac{2\pi G}{c^2}\sigma_z+\left[\frac{(2\Omega^2-\kappa^2)_\sigma}{c^2}+\frac{\Omega^2_{ext}}{c^2}\right]z\right\}, \quad \text{(A5)}$$

where we have introduced the integrated density $\sigma_z=2\int_0^z\rho(z')\mathrm{d}z'$. Note that the term $\Omega^2_{ext}z$ is just the vertical component of the spherical external field, as might have been anticipated. If $\rho_0=\rho(z=0)$ is the density on the equatorial plane, we can define two scales, $\Sigma=\sqrt{2\rho_0/\pi G}c$ and $H=c/\sqrt{2\pi G\rho_0}$, and the natural dimensionless parameter $A=[(2\Omega^2-\kappa^2)_\sigma+\Omega^2_{ext}]/4\pi G\rho_0$, so that Eq. (A5) becomes:

$$\frac{\mathrm{d}^2y}{\mathrm{d}\zeta^2}=-2\frac{\mathrm{d}y}{\mathrm{d}\zeta}\left[y+A\zeta\right], \quad \text{(A6)}$$

to be solved under the conditions $y(0)=0$, $y'(0)=1$; here we have used the dimensionless variables $y=\sigma_z/\Sigma$ and $\zeta=z/H$. For each value of A one can compute the desired density profile and the associated surface density $\sigma=\sigma_z(z=\infty)$. One can then introduce a scaleheight h such that $\sigma=2\rho_0h$, and the value of h can be computed directly from the relation $h=Hy(z=\infty)$.

The standard non self-gravitating model, where the contribution of the gas density ρ to the vertical gravitational field is negligible, corresponds to the limit equation $y''=-2A\zeta y'$. Usually the analysis is carried out with the additional approximation $A\approx\Omega^2_{ext}/4\pi G\rho_0$, but this is not needed; by retaining the contribution $(2\Omega^2-\kappa^2)_\sigma$ one can thus keep track of the effects associated with the rest of the disk mass distribution, which may be significant even where the local disk density ρ_0 is small. In any case, such limit equation yields the well known Gaussian profile $\rho=\rho_0\exp(-z^2/2h_\star^2)$, with $h_\star^2=H^2/2A=c^2/[(2\Omega^2-\kappa^2)_\sigma+\Omega^2_{ext}]$, here improved with respect to the standard expression $h_\star\approx c/\Omega_K$ used in the so-called Keplerian limit. Note that $h_\star$ is slightly different from the scaleheight h defined above, since $h=\sqrt{\pi/2}h_\star$.

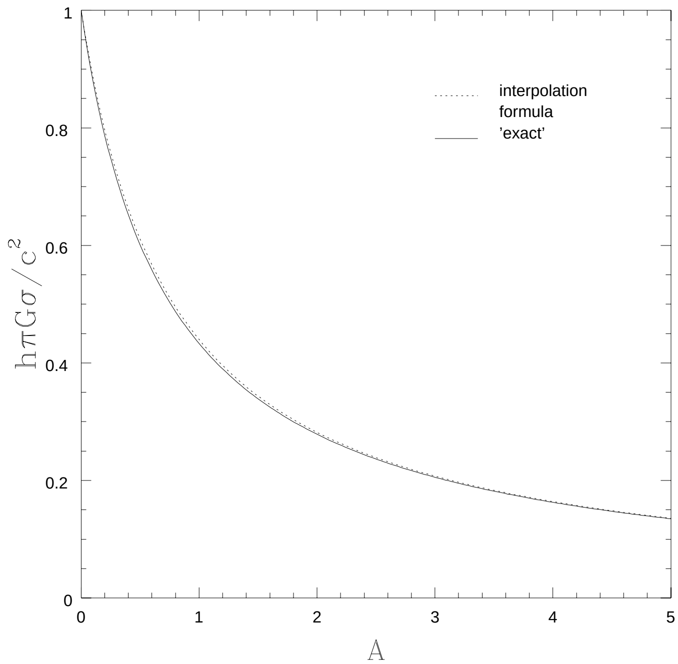

Fig. A1. Comparison between the exact (numerically computed based on Eq. (A6)) and approximate (Eq. A8) expression for the disk thickness.

The limit of the homogeneous fully self-gravitating slab corresponds to the equation $y''=-2yy'$, so that the vertical density profile is given by (Spitzer 1942) $\rho=\rho_0\,\mathrm{sech}^2(z/h)$, with $h=c^2/\pi G\sigma$; here the scale h is the same as that defined by the relation $\sigma=2\rho_0h$. Note that even when no external (spherical) field is present, the solution for an axisymmetric disk obtained from Eq. (A6) is not exactly the one derived in the homogeneous, self-gravitating slab, unless the rotation curve is flat.

The density profile associated with Eq. (A6) is neither Gaussian nor sech^2. For practical purposes it may be convenient to use a simple interpolation formula for the vertical scale, which is justified by the following description "biased" towards the Gaussian limit. If we start from Eq. (A2), naively expand the vertical field as $(\partial\Phi/\partial z)(z)\sim(\partial^2\Phi/\partial z^2)_{z=0}z$, and then use Eq. (A1) to estimate the factor $(\partial^2\Phi/\partial z^2)_{z=0}$, we find an equation leading to an unrealistic Gaussian density profile with scaleheight h given by:

$$\frac{\pi}{2h^2}=\frac{4\pi G\rho_0}{c^2}[1+A]. \quad \text{(A7)}$$

The quantity ρ_0 contains a dependence on h (for given σ). This suggests the use of the following interpolation formula:

$$h=\left(\frac{c^2}{\pi G\sigma}\right)\frac{1}{1+4A/\pi}. \quad \text{(A8)}$$

Note that the exact calculation, from $h = Hy(z = \infty)$ gives a relation $h = (c^2/\pi G\sigma)f(A)$, with $f(A) = [y(z = \infty)]^2$. The choice of the factor $(4/\pi)$ in Eq. (A8) guarantees the proper limits for $A = 0$ and for $A \to \infty$. The accuracy of the interpolation formula is illustrated in Fig. (A1). In turn, since A depends on h and on σ via the quantity ρ_0, for the purposes of the present paper it may be convenient to reexpress Eq. (A8) as:

$$h = \frac{c^2}{\pi G\sigma}\frac{\pi}{4a}\left(\sqrt{1+8a/\pi}-1\right), \qquad \text{(A9)}$$

with $a = Q^2[(2\Omega^2 - \kappa^2)_\sigma + \Omega^2_{ext}]/\kappa^2$. This leads to Eq. (17) of the main text, applicable when the external field is produced by a central point mass. This interpolation formula improves on earlier analyses (Sakimoto & Coroniti 1981, Bardou et al. 1998), in several respects, allowing us to better describe the transition between a Keplerian and a self-gravity dominated disk and to extend the treatment to the case where the external field is not just that of a simple point mass at the center.

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