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LETTER

Letter to the Editor

Sensitivity of a pyramidic Wave Front sensor in closed loop Adaptive Optics

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Abstract. The Pyramidic wavefront has a very interesting property: its SNR has a significant increase once the loop is closed, similarly to the Curvature Sensor and contrasting with the Shack-Hartmann one. This result in a gain comparing the Pyramidic and the Shack-Hartmann sensors. It is shown by a combination of descriptive and numerical arguments that this gain is of the order of $(D/r_0)^2$ for low order modes and drift linearly down to the Shack–Hartmann case at the highest spatial frequency sampled. A discussion of the overall gain (including a first estimation of the magnitude gain) and procedures to force a true system to work in closed loop fashion are sketched.

Key words: telescopes – instrumentation: adaptive optics

1. Introduction

Recently a novel concept of wavefront sensor (WFS) based upon an oscillating pyramidic optical component (Ragazzoni, 1996) has been introduced into the field of Adaptive Optics (AO). Such a WFS retains a variable dynamic range and sampling, allowing it to be equivalent to a Shack–Hartmann (SH) WFS with the proper choice of the oscillation amplitude of the pyramid. The variable gain can be effectively used in closed loop operation, in a similar fashion to curvature WFS where defocus of the observed pupil is driven by a vibrating membrane whose amplitude is proportional to the gain (Roddier, Northcott & Graves, 1991). We recall a similar concept proposed by Pugh et al., 1995 which is however affected by a fixed gain and a limited dynamic range. The physical oscillation of the pyramid can be avoided by tilting a mirror in the pupil–plane, as suggested by Riccardi, 1996 and it has been successfully realized in a prototype laboratory version giving excellent results (Riccardi et al., 1998; Esposito et al., 1999). It has also been implemented onto the AO system of a 4m–class telescope (Ragazzoni et al., 1998) whose first–light is expected soon.

In literature, the pyramidic WFS has been considered identical to the SH, with different and attractive *practical* features. We want to highlight a remarkable property of this WFS that has not yet been investigated as far as we know. This property is the much higher sensitivity with respect to a SH, especially for low–order modes measurements.

2. A descriptive approach

Let us consider an AO system tuned to obtain full diffraction limited performances by a pupil sampling on scales of the order of r_0. Furthermore, we'll assume that the loop has been already successfully closed. In this configuration the wavefront approaching the WFS will be nearly flat. While the SH behavior doesn't change anything during the loop closure, the situation is expected to be different for the pyramidic one. In fact the pyramid is oscillating with an amplitude corresponding to the seeing size at the focal plane before the closure loop operations and, as much as the loop becomes properly closed, the vibration lowers and tends to zero. Let us compare a SH vs. a pyramidic WFS in this final configuration (equivalent to the one described by Pugh et al., 1995). In this Letter we assume that the inversion of the WF from the data coming from this WFS is accomplished correctly. Although both Pugh et al., 1995 and Horwitz, 1994 demonstrate that, under this Foucault limit, the data provided by this type of WFS are essentially the derivative of the incoming WF, some corrections due to diffraction effects might be introduced in the reconstruction process. We also assume that the lenslet size of the SH lenslet array and the sampling of the pupils in the pyramidic WFS are equal to each other and to r_0. Let us say that $N^2 \approx (D/r_0)^2$ is the number of subapertures (where D is the telescope diameter) and n^* is the number of photons effectively collected by the detector in a single sub–aperture for a single integration time. Each SH spot, characterized by an angular size of λ/r_0 (where λ is the effective wavelength of the WFS detector) will be characterized by a centering error due to photon shot noise given, in angular units, by $\sigma^2 = (\lambda/r_0)^2/n^*$.

Similarly, in the pyramidic WFS case, the light will be split equally in four parts and will reach the detector. Due to diffraction on the edges of the pyramid, these pupils might no longer reflect the true illumination of the entrance pupil. However, for symmetry reasons, although these pupils will be eventually smeared out, the light will be split exactly in four symmetrical and identical parts; we'll return on this point later. Let us

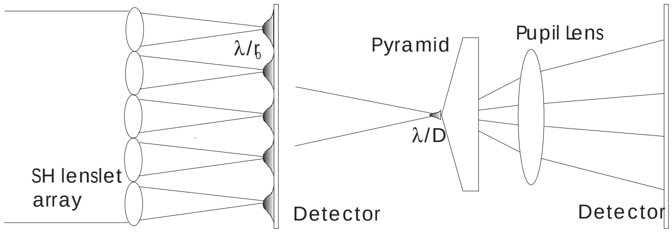

Fig. 1. The SH (left) and the pyramidic (right) WFS compared; both the WFS are shown in the perfectly close loop situation. When a tilt of the order of λ/D is introduced the efficiency of the pyramidic WFS is much larger because it acts *after* the recombination of the whole pupil light. In fact the movement of the spot is comparable to its size, while in the SH case all the spots will move of a fraction of their size.

consider a tilt of the wavefront of λ/D. Such a perturbation is big enough to destroy high Strehl ratios on the scientific focal plane of the telescope. The coherent displacement of the SH spots will be roughly equal to $r_0/D \approx 1/N$ of their size. On the other hand, in the pyramidic case the spot will move almost to one side of the pyramidic facets; assuming that the displacement moves in the same direction of one of the pyramidic edges, two pupils will be essentially unilluminated while the other two will be mostly completed illuminated (see also Fig. 1).

The estimation of the tilt from the averaging of all the SH spots, or by averaging the illumination of the four pupils in the pyramidic WFS will be affected by very different error. In the first case, in fact, the N^2 independent estimates will produce a final error on the tilt estimation given by $\sigma_{\rm SH-tilt} = (\lambda/Nr_0)^2/n^*$.

Recalling the meaning of N one can easily work out:

$$\sigma^2_{\rm SH-tilt} = \left(\frac{r_0}{D}\right)^2 \left(\frac{\lambda}{r_0}\right)^2 \frac{1}{n^*} = \left(\frac{\lambda}{D}\right)^2 \frac{1}{n^*} \qquad (1)$$

In the Pyramidic case the behavior will be almost identical to the one of a quadrant sensor located on the focal plane of the diffraction–limited whole aperture of the telescope, collecting the whole light of the telescope in a single diffraction limited spot as $\sigma^2_{\rm P-tilt} = (\lambda/ND)^2/n^*$.

The last result can be expressed in terms of Eq. (1):

$$\sigma^2_{\rm P-tilt} = \left(\frac{r_0}{D}\right)^2 \left(\frac{\lambda}{D}\right)^2 \frac{1}{n^*} = \sigma^2_{\rm SH-tilt} \left(\frac{r_0}{D}\right)^2 \qquad (2)$$

obtaining in this way a direct comparison of the two residuals. It is clear from Eq. (2) that the pyramidic WFS has a very huge gain in sensitivity, especially for large telescopes.

How does this extend to any other Zernike polynomial?

It has to be recalled that diffraction limit (resolution power of $\approx \lambda/D$) is a direct consequence of Heisenberg uncertainty principle; in fact the uncertainty on the measurement of the momentum of the photon along the focal plane is directly linked to the uncertainty on its entrance pupil position; the product of the latter with the uncertainty on the position of the same photon on the focal plane is limited.

In other words any measurement aimed to identify the location on the pupil of a photon approaching the focal plane in such a situation will destroy to some extent the λ/D resolving power capability.

For instance, when full diffraction limit is obtained, there is no way to get information on how the light coming onto the telescope is distributed on the pupil. In this case in fact the images on the four pupils are completely smeared out. When a tiny departure from this condition is accomplished, there should be a direct relationship between the size of the zone on the incoming pupil where the aberration measurement is taking place and the corresponding spot size. This means that in Eq. (1) one should replace D with ω, the latter being the typical scale of the sensed mode. For tilt, just $\omega = D$. Generally speaking, a Zernike polynomial of radial degree q will exhibits a scale length of the order of $\omega = D/q$. A reasonable guess is to estimate the error of the pyramidic WFS for a Zernike polynomial of q–th radial order, as given by:

$$\sigma^2_{\rm P-Z(q)} \approx \left(\frac{qr_0}{D}\right)^2 \sigma^2_{\rm SH-Z(q)} \qquad (3)$$

that is equivalent to assume that the sensitivity of the sensor is inversely proportional to the second power of the focal spot linear dimension. A statement already pointed out, concerning only the tilt term, by Sandler et al., 1994. It is remarkable that the gain at the largest modes is unchanged with respect to the SH case. However the final error budget will still benefit from the compensation of low order modes (which are sensed in the SH case with the same sensitivity of the highest one) and it is reasonable to expect that a substantially fainter limiting magnitude is obtained assuming a fixed WF residual. The shape of Kolmogorov spectra is, in this case, helpful because it will weight more the lowest modes, the ones which more benefit from this WFS.

3. A numerical simulation

In order to investigate the effect of pupil delocalisation under realistic circumstances and to confirm the reasoning that leads to Eq. (3) we developed code to simulate the pyramidic WFS including diffraction effects. We generate a wavefront $W(\rho, \theta)$ defined onto a normalized pupil as a complex function whose modulus is unit in a region limited by a circular aperture with a central obscuration characterized by a linear obstruction ratio $\varepsilon = 0.3$. Mapping this pupil onto a small subset of a larger matrix and by Fourier transforming this one obtains, for instance, the electric field intensity at the focal plane position. At this stage, taking the squared modulus of the complex amplitude of the electric fields, one could obtain the related Point Spread Function. The ratio between the size of the large matrix and the size of the pupil embedded in it, gives the resolution of the results, in terms of Airy size. We used, for instance, a 1024×1024 matrix where a 128×128 pupil is embedded. In this way a λ/D is mapped in roughly 8 pixels, allowing for a proper sampling of the edge of the pyramidic facets, being their resolution nearly one order of magnitude better than the diffraction limit. In the (complex) electric field space we masked all but one quadrant of the matrix for each of the four quadrants. By inverse Fourier

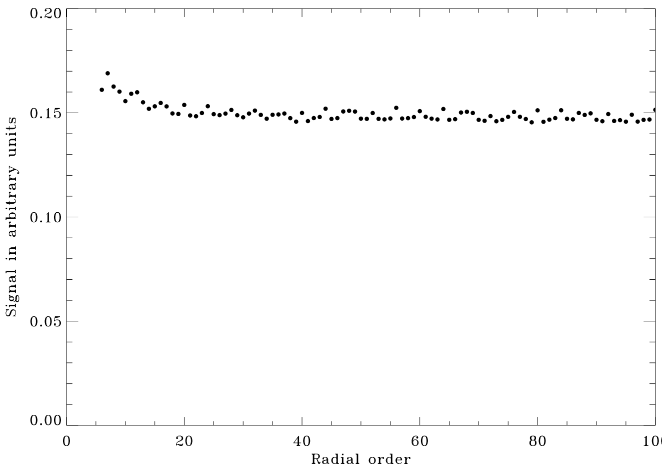

Fig. 2. In this plot the returning normalized Peak to Valley signal is given with respect to a Zernike polynomial wavefront of $\lambda/20$ Peak to Valley. Sampling of the pupil is made with a 128×128 array.

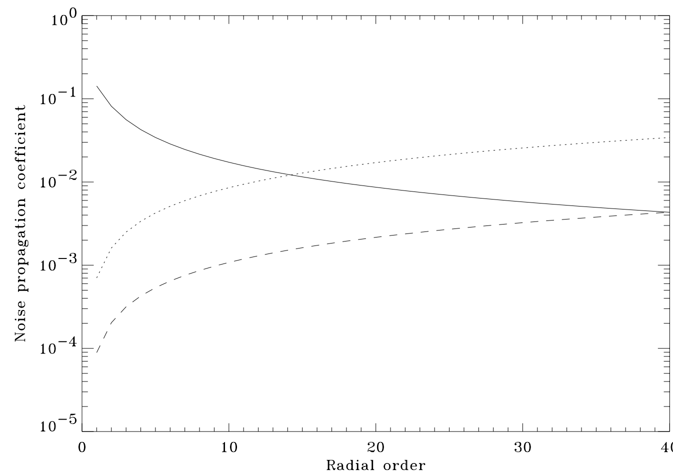

Fig. 3. Solid line: p_i vs. the radial order q for the SH case; dotted line: the same for the pyramidic case; dashed line: the same scaled in order to have the same integral area for the WF residual error.

transformation and again taking the squared modulus of the obtained electric field, the illuminations on the four quadrants of the pyramidic WFS are obtained. Combining these in the usual way (Ragazzoni, 1996) as for the SH–WFS, the estimate of the wavefront derivative is obtained. The latter can be compared to the derivatives of the original wavefront used in the simulation. In Fig. 2 we plot the normalized Peak to Valley signal vs. a specific Zernike polynomial for a given radial order. We taken these as representative of a specific rank of pupil sampling. The signal $\Delta I/I$ is expected to be proportional to the derivative of the wavefront assuming the geometrical rays approximation holds. The derivative of the incoming WF, however, is proportional to q and the constancy of the signal in Fig. 2 is to be interpreted as the proportionality mentioned above is attenuated by a factor nearly proportional to q^{-1}. This is also a direct verification of Eq. (3).

4. Limiting magnitude gain estimate

The arguments presented in the previous sections give the evidence that a gain in terms of limiting magnitude can be achieved using the pyramidic WFS. In the following we quantify such a gain compared with a SH based WFS. We assume that the same reconstructor is applied to both the WFSs, assuming they are measuring the first derivative of the wavefront, as in the geometrical approximation. Hence, no speculations are made a better way to comply with the different behaviors of the two WFSs (and, in addition, a reconstructor based upon the SH is assumed). The noise propagation coefficients p_i, in the SH case, for each i–th Zernike mode Z_i characterized by an azimuthal order n and a radial order m, is given by (Rigaut & Gendron, 1992):

$$
\begin{aligned}
p_i &= 0.295(q_i+1)^{-2.05} \quad (n=m) \\
p_i &= 0.174(q_i+1)^{-2} \qquad (n \neq m)
\end{aligned}
\tag{4}
$$

The total variance of the WF estimation is given by:

$$
\sigma^2_{\rm SH} = \sigma^2_{\rm ph} \sum_{i=1}^{M} p_i \tag{5}
$$

where $\sigma^2_{\rm ph}$ is the photon noise error in the WFS, proportional to n^{*-1} and M is the rank of the highest Zernike polynomial considered. Relationship (9) can be explicitly written in terms of the radial orders q ranging from 1 to the maximum radial order $Q = D/r_0$ under the assumptions given in this Letter. Recalling that, for each radial order q, there are 2 modes corresponding to $n = m$ and $(q-1)$ corresponding to $n \neq m$, one can rewrite Eq. (5) as:

$$
\begin{aligned}
\sigma^2_{\rm SH} = \sigma^2_{\rm ph} \textstyle\sum_{q=1}^{Q} \big[\; & 0.590(q+1)^{-2.05} + \\
& + 0.174(q-1)(q+1)^{-2} \big]
\end{aligned}
\tag{6}
$$

On the other hand the last relationship is to be corrected for the pyramidic case multiplying it by a term $(q/Q)^2$ (accordingly to Eq. (3)) leading in this way to:

$$
\begin{aligned}
\sigma^2_{\rm P} = \sigma^2_{\rm ph} \textstyle\sum_{q=1}^{Q} \left(\frac{q}{Q}\right)^2 \big[\; & 0.590(q+1)^{-2.05} + \\
& + 0.174(q-1)(q+1)^{-2} \big]
\end{aligned}
\tag{7}
$$

In Fig. 3 the two relationships in the argument of the summations are plotted versus q for the case of $Q = 40$ corresponding, for instance, to a somewhat realistic case of $D = 8$m and $r_0 = 0.2$m.

Eq. (6) and (7) can be numerically evaluated for several values of Q. In order to have the same residual WF variance, hence the same Strehl ratio in the compensated image, the two summations should have identical values. This can be accomplished by a *degradation* of $\sigma^2_{\rm ph}$, corresponding to a weaker requirement in terms of collected photons. For a matter of comparison, in Fig. 3 it is also plotted the relationship for the pyramidic WFS scaled in a way to have the same integral value. It is to be pointed out that the spectra of the residuals are however substantially dif-

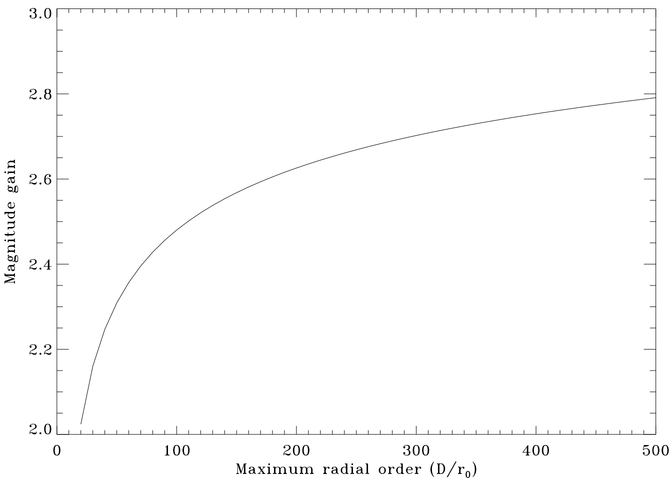

Fig. 4. The gain in limiting magnitude for different $Q = D/r_0$ cases.

ferent. The ratio of Eqs. (6) and (7) leads directly to an estimate of the gain Δm in the limiting magnitude:

$$\Delta m = -2.5 \log \left(\frac{\sigma^2_{\rm P}}{\sigma^2_{\rm SH}} \right) \quad (8)$$

In Fig. 4 a plot of $\Delta m(Q)$ is given for Q ranging from 20 to 500. The first figure can be considered as characteristic of a $D = 4$m class telescope under a median seeing of $0.5''$. The latter one represent the case of a giant $D = 100$m telescope under the same seeing conditions.

5. Conclusion

In the case where $N \gg 1$ the pyramidic WFS appears to be much more sensitive than the SH one. This makes this WFS very attractive for high order Adaptive Optics system and especially for extremely large apertures (Gilmozzi et al., 1998; Mountain, 1997). It is also noticeable that a substantial increase of limiting magnitude can extend reliability of multiple NGS concepts (Ragazzoni, 1999). A key problem, however, is represented by the ability of such a system to effectively bootstrap (Farinato, Marchetti & Ragazzoni, 1996) or, in other words, to close the loop under poor SNR conditions experienced during open loop (although one time the loop would be successfully closed the system could easily retain such a status). A possible solution could be to close the loop using at the beginning a poor sampling of the pupil (this can be easily done in the pyramidic WFS, allowing also to minimize effects of detector read out noise). This will translate into a low–order correction that will shrink a bit the used reference. This could allow an increase in the pupil sampling and, by successive iterations, to effectively close the loop up to the highest possible mode. Finally, we also point out that Laser Guide Stars references will not benefit from such a gain, unless they are fired using the whole telescope aperture through the AO system, that is unlikely for several reasons. Of course the pyramidic WFS, in the latter case, will retain the other known practical advantages.

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