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Research Note

A note on Saio's estimate of second-order effects of rotation on stellar oscillation frequencies

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Abstract. In many cases, oscillating stars rotate so rapidly that second-order effects must be taken into account in analyses of the oscillation frequencies. Such investigations have commonly been based on results provided by Saio (1981) for a polytropic model. Here we compare those with frequency changes for more realistic models; we point out that a simple correction to Saio's data allows them to be used in the observationally relevant case of comparing models at fixed luminosity and effective temperature.

Key words: stars: oscillations – stars: rotation – methods: data analysis

1. Introduction

The effects of stellar rotation must be taken into account in interpreting the spectrum of oscillations of many stars. In the case of a slow rotator, the effects of rotation are very simply interpreted: rotation removes the frequency degeneracy in the azimuthal order m of the modes, and in fact this allows the possibility of determining the angular velocity inside the star. The Sun is one such star: it has a rotation period of about one month, which is much longer than its global dynamical timescale of about one hour. (The dynamical timescale is essentially $(G\bar{\rho})^{-1/2}$, where $\bar{\rho}$ is the mean density and G is the gravitational constant.) Many pulsating stars rotate much more rapidly, however; delta Scuti stars, for example, are observed to rotate with surface velocities of the order 100–200 $\mathrm{km\,s^{-1}}$, corresponding to a rotation period of about one day. As well as the first-order effect of raising the m-degeneracy of the modes, rotation also makes the star oblate and perturbs the internal stratification. Such perturbations will result in a systematic change of all frequencies, if compared with a nonrotating star of similar mass and age. In this way even the frequencies of radial and axisymmetric modes are affected by the rotation of the star. These effects must evidently be taken into account, in order to identify the mode parameters and to use the observed frequencies to make inferences about the stars' internal structure and rotation.

An influential paper on the study of oscillations in rotating stars is that by Saio (1981). Saio's numerical results are still in use for interpreting observations. However, Saio's calculations were for simple polytropic models. More realistic models are required for the detailed mode interpretation and seismology of stars that modern observations make a possibility. For this reason, Kjeldsen et al. (1998) calculated stellar models with more realistic physics. This work raised a question about the proper interpretation of Saio's tabulated results, which is addressed in this note.

The full calculation of the normal modes of oscillation of the model of a rotating star is quite an undertaking; the usual approach to estimating the frequencies of such a star is therefore to evaluate the frequency changes induced by the effects of rotation, treating them as a perturbation about a nonrotating model. But to make use of such calculations to compare with the observed frequencies of stars, it is necessary to consider carefully what properties of the star are held fixed when making the perturbation.

Saio compared the frequencies of rotating and nonrotating polytropes. In that comparison, he kept the central density and central pressure (and hence polytropic constant) fixed. An observationally more relevant comparison, though, would be to keep fixed the observed global properties of the star. This is considered here.

2. Results

Kjeldsen et al. (1998) compared the frequencies of a rotating model that matched the observed stellar luminosity (L) and effective temperature ($T_{\rm eff}$) with a nonrotating model with the same luminosity and temperature. For fixed stellar mass and age, rotation increases the radius of the star and decreases its central temperature and hence luminosity. Thus, as pointed out in that paper, a rotating star must be younger and slightly more massive than a nonrotating star at the same L and $T_{\rm eff}$. However, the adjustment in mass is very small, at least for the stars con-

Send offprint requests to: J. Christensen-Dalsgaard sidered by Kjeldsen et al., since the luminosity has a sensitive dependence on mass. Moreover, the radius is fixed since the luminosity and effective temperature both are. This means in fact that the mean density $\bar{\rho}$ is almost the same in the rotating and nonrotating models that have the same luminosity and effective temperature.

Before we go on, we note that there is a natural scaling of the frequency ω of a star with mass M and radius R of the star through the dynamical timescale $(GM/R^3)^{-1/2}$. Thus it is often convenient to think in terms of the dimensionless frequency

$$\sigma = (GM/R^3)^{-1/2}\omega \tag{1}$$

(this is essentially a scaling by the root of the mean density $\bar{\rho}$). The second-order effects of rotation have a natural scaling with $\Omega^2/(GM/R^3)$ where Ω is the (uniform) rotation rate, so this is also often conveniently factored out. It may also be noted that, following the common modern notation, we use ω for the dimensional frequency and σ for the scaled dimensionless frequency; whereas Saio uses σ for the dimensional frequency and ω for the dimensionless frequency scaled by properties of the nonrotating model.

Fig. 1 (adapted from Kjeldsen et al. 1998) compares scaled relative differences in dimensionless frequency σ between our rotating and nonrotating models (for radial modes). Three models of masses $1.648M_\odot$, $1.980M_\odot$ and $2.045M_\odot$ are included, corresponding approximately to three delta Scuti stars in the Praesepe cluster, observed by Arentoft et al. (1998); further details on the models are provided in the figure caption. Thus what is plotted (lines without symbols) is $GM/(\Omega^2R^3)$ times

$$\frac{\omega/\bar{\rho}^{1/2} - \omega_0/\bar{\rho}_0^{1/2}}{\omega_0/\bar{\rho}_0^{1/2}} \; . \tag{2}$$

To make a proper comparison with the results of Saio (1981) we need to evaluate this scaled difference from his tables.

Saio's formulae estimate the relative change in frequency scaled by the mean density of the nonrotating model. Thus the quantities provided by Saio essentially determine

$$\frac{\omega/\bar{\rho}_0^{1/2} - \omega_0/\bar{\rho}_0^{1/2}}{\omega_0/\bar{\rho}_0^{1/2}} \; ; \tag{3}$$

this is plotted in Fig. 1 with stars. (Actually Saio's Table 1 only provides results for the first three $l = 0$ modes: we have been able to extrapolate to higher orders by treating Saio's quantity Z as a function of frequency independent of degree – indeed it is almost a linear function of the squared frequency over the range of interest.) Now ω in Eq. (3) is evaluated in Saio's rotating polytropic model, but the mean density has changed relative to the nonrotating model. Thus to obtain a quantity directly comparable to our results, where the mean density is essentially constant, we must add to what we get from Saio's table a correction

$$\left(\frac{\bar{\rho}_0}{\bar{\rho}}\right)^{1/2} - 1 \;\equiv\; -\frac{1}{2}\frac{\delta\bar{\rho}}{\bar{\rho}} \; , \tag{4}$$

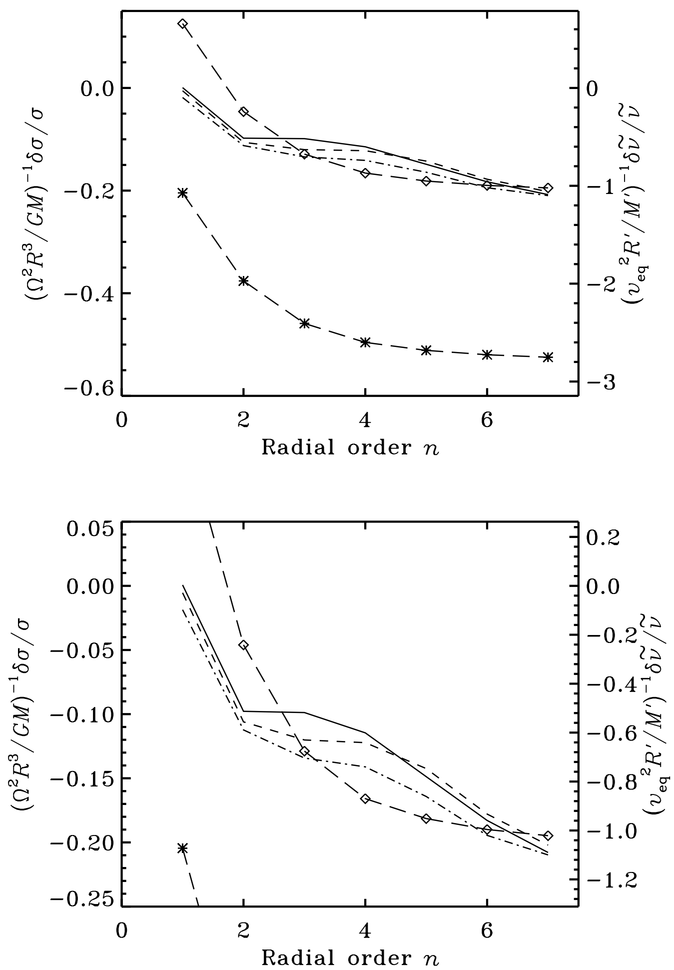

Fig. 1. Scaled relative changes in the dimensionless frequencies of radial modes. The right axis shows these changes in terms of observational quantities: $R' = R/R_\odot$ and $M' = M/M_\odot$ are radius and mass in solar units and $v_{\rm eq}$ (the equatorial rotational velocity) is in $\rm Mm\,s^{-1}$ ($= 10^3\,\rm km\,s^{-1}$); also $\tilde{\nu} = \sigma/2\pi$ is dimensionless cyclic frequency. Line with stars are the original results of Saio (1981) for a polytrope of index 3 (see text) whereas the line with diamonds is the result of shifting these results by 0.33 (in the units of the left-hand ordinate). The remaining lines are for realistic models, of masses $1.648M_\odot,\ v_{\rm eq} = 0.102\,\rm Mm\,s^{-1}$ (solid), $1.980M_\odot,\ v_{\rm eq} = 0.135\,\rm Mm\,s^{-1}$ (dashed), $2.045M_\odot,\ v_{\rm eq} = 0.222\,\rm Mm\,s^{-1}$ (dot-dashed). The lower panel shows the same results, on an expanded scale. [Adapted from Fig. 6 of Kjeldsen et al. (1998).]

where $\delta\bar{\rho}$ is the difference in mean density between Saio's rotating and nonrotating models. As already noted, Saio keeps central density $\rho_{\rm c}$ and central pressure $p_{\rm c}$ (and hence polytropic constant) fixed between the nonrotating and rotating models; the corresponding relative change in mean density is obtained as (Chandrasekhar 1933; Eq. 44)

$$\frac{\delta\bar{\rho}}{\bar{\rho}} = \frac{\Omega^2}{2\pi G\rho_{\rm c}}\frac{\frac{1}{3}\xi_1^2 - \psi_0'(\xi_1)\xi_1 - 3\psi_0(\xi_1)}{\xi_1|\theta_1'|} \; , \tag{5}$$

where ξ is the polytropic radius variable, ξ_1 its value at the surface, θ_1' is the gradient of the Lane-Emden function at the surface and the function ψ_0 is given in Table VI of Chandrasekhar

(1933). Noting that the central density is $\bar{\rho}\xi_1/(3|\theta_1'|)$, this expression computes for a polytrope of index 3 to

$$0.0123\frac{\Omega^2 R^3}{GM} \times (-53.7) \ = \ -0.66\frac{\Omega^2 R^3}{GM} \ . \qquad (6)$$

After multiplying by $-1/2$, and taking account of the $\Omega^2 R^3/(GM)$ scaling, we get that the immediate result from Saio's table must be increased by 0.33. This is in close accord (see the figure) with what is required to bring Saio's and Kjeldsen et al.'s results into agreement at the high ($n = 7$) end of the range of comparison. The results agree less well at low n, but here the frequency changes no doubt depend on the detailed structure of the model. Indeed, one might expect that the response for high-order modes is essentially homologous, and hence less sensitive to the details of the structure.

3. Conclusion

Saio (1981) evaluated frequency changes caused by rotation of a polytropic model, keeping the central pressure and density constant between the nonrotating and rotating model. Here we have evaluated the correction required to obtain the observationally more relevant change at fixed luminosity and effective temperature and shown that the result is quite similar to what is obtained for more realistic stellar models. We have shown that at least for higher-order modes the relative perturbation in dimensionless frequency σ, with the natural scaling $\Omega^2/(GM/R^3)$ factored out, can be obtained from Saio's results simply by adding 0.33. Thus, in particular, the second-order perturbation to the frequencies of the radial modes due to rotation, at fixed effective temperature and luminosity, is well-approximated from Saio's tabulated results as

$$\left(\frac{GM}{\Omega^2 R^3}\right)\frac{\delta\sigma}{\sigma} \ = \ \left(\frac{X_1 + Z}{\sigma^2}\right) \ + \ 0.33 \ , \qquad (7)$$

where σ is in our notation and X_1 and Z are as defined by Saio (with $X_1 = 4/3$ for $l = 0$ and Z is interpolated in frequency from Saio's Table 1). Although we have considered only radial modes, precisely the same correction must be applied in the case of nonradial modes.

Finally, we note that our analysis, as well as Saio's, was restricted to effects linear and quadratic in the angular velocity Ω. It was pointed out by Soufi, Goupil & Dziembowski (1998) that terms of order Ω^3 must also be taken into account at the angular velocities found in some pulsating stars. Additional complications in the analysis of data from rapidly rotating stars arise from the fact that rotation affects the relation between observed intensities and colours and the true luminosity and effective temperature of the star (e.g. Maeder & Peytremann 1970; Collins & Smith 1985). Thus the proper interpretation of the frequency spectrum of rapidly rotating stars is evidently nontrivial; however, their prevalence dictates that this is an important problem in asteroseismology.

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