Astron. Astrophys. 350, 1051–1059 (1999)

ASTRONOMY AND ASTROPHYSICS

An iterative method for the reconstruction of the solar coronal magnetic field

I. Method for regular solutions

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Received 17 February 1999 / Accepted 4 June 1999

Abstract. We present a method for reconstructing the coronal magnetic field, assumed to be in a non-linear force-free state, from its values given in the photosphere by vector magnetograph measurements. In this paper, that is the first of a series, we propose a method that solves the boundary value problem set in the functional space of regular solutions (i.e., that do not contain current sheets). This is an iterative method introduced by Grad and Rubin. It is associated with a well-posed boundary-value problem. We present some results obtained with this method on two exact solutions of the magnetostatic equations, used as theoretical magnetograms. Unlike some other extrapolations methods, that are associated with ill-posed boundary value problems, our method allows extrapolation to arbitrarily large heights, with no blowing up due to the presence in these methods of an intrinsic instability that makes errors growing up exponentially.

Key words: rodynamics (MHD) – Sun: corona – Sun: magnetic fields

1. Introduction

The magnetic field dominates most of the corona, and it is probably the origin of a large variety of structures and phenomena, such as flares, Coronal Mass Ejections, prominences and coronal heating (Priest 1982). Unfortunately the magnetic field is not yet observationally accessible in the tenuous and hot plasma that fills the corona (see Sakurai 1989, Amari & Demoulin 1992, and references therein). One possible familiar approach consists in solving the equations of a model (defined by some reasonable assumptions about the physical state of the corona) as a Boundary Value Problem (BVP), the boundary conditions being taken to be the measured values of the magnetic field in the denser and cooler photosphere: this is the so called *Reconstruction* problem of the coronal magnetic field. Many problems have been encountered since the early attempts of Schmidt (1964), as the observational problems to get rid of the ambiguity that remains in the transverse component of the photospheric magnetic field (Amari & Demoulin 1992, McClymont et al. 1997, and references therein), or the problems related to the choice of boundary conditions that make a well set BVP (Aly 1989)

In the simplest approximation the coronal magnetic field is current-free. This only requires the longitudinal component of the photospheric field as a boundary condition (Schmidt, 1964), and the solution can be computed using either a Green's function method or Laplace solver methods for the magnetic field or the vector potential. The mathematics of the various related BVPs (e.g., their well-(or ill-) posedness properties), are also known (Aly 1987, Amari et al. 1998).

In many active regions, where the magnetic configuration is known to have stored free energy, the current free assumption is not relevant. One can then introduced the so-called constant-α force-free hypothesis, which allows for the presence of electric currents in the corona. The magnetic field is computed, for a given value of α, from its longitudinal component by using either Fourier transform (Nakagawa et al. 1973, Alissandrakis 1981) or Green's function (Chiu & Hilton 1977, Semel 1988) techniques. Other spectral methods have been recently proposed (Boulmezaoud et al. 1998). It is also possible to solve it by regularizing an ill-posed BVP (in which the three components of the magnetic field are used) (Amari et al. 1998). However, the non-regularization or partial regularization of the so called Vertical Integration Method (VIM), leads in general to an amplification of errors (Wu et al. 1990 and referncess therein, Cuperman et al. 1990a-b, Cuperman et al. 1991, Demoulin et al. 1992). In addition the total energy of the linear force-free field in an unbounded domain such as the exterior of a star shaped domain is infinite, and is in general infinite in the case of the upper half space (except for some particular periodic solution satisfying some special conditions, Alissandrakis 1981, Aly 1992). Moreover the electric currents are uniformly distributed, while observations clearly show strong localized shear along the neutral line of many active region magnetic configurations (Hagyard 1988, Hofmann & Kalman 1991).

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Modeling such strong localized electric currents needs to assume that the coronal magnetic configuration is in a non-linear, force-free state. In this case one can distinguish two types of methods associated to different classes of BVPs, Extrapolation Methods and Reconstruction Methods. In the first class of methods the three components of the magnetic field are used as boundary conditions. The equations are thus vertically integrated step by step, from the photosphere towards the corona, without incorporating any type of asymptotic boundary conditions. This give rise to the VIM (Wu et al. 1990 and references therein, Cuperman et al. 1990a-b, 1991, Demoulin et al. 1992). This method, associated to an ill-posed boundary value problem, has not yet been proved to be convincingly regularized, still ending with an exponential growing of the errors with height, prohibiting extrapolation up to reasonable heights. The second class of methods considers a BVP that only requires the normal component of the field on the boundary (B_n) and the normal component of the electric current say, where $B_n > 0$. Now the problem is considered in the whole domain and the solution is globally sought. It has been tackled by the use of iterative methods introduced by Grad and Rubin (Grad & Rubin 1958, Sakurai 1981, Sakurai et al. 1985) and by the Resistive MHD Relaxation Method (Mikic & McClymont 1994, Jiao et al. 1997). Roumeliotis (1997) presented a Relaxation Method in which the three components of the magnetic field are used at the photospheric level. Another method (see Amari & Demoulin 1992), is the Method of Weighted Residuals (Pridmore-Brown 1981). This method is based on the minimization of two residuals, one associated with the Laplace force that has to vanish for a force-free magnetic field, and the other one with the difference between the directions of the observed transverse photospheric magnetic field and of the computed one. However, some aspects, such as the choice of test functions to be used for scalar products, as well as some other points concerning the definite positiveness of one functional to be minimized, are not yet clear. Other computational schemes such as collocation or least square methods have also been proposed in Amari & Demoulin (1992), but they have not been tested so far.

Sakurai (1981, 1985) presented a Green's function approach of the Grad-Rubin formulation. Practically, the standard Green-Function formulation is however numerically expensive, since at each step of the iterative scheme one would need to compute an integral over the whole volume to get the value of the magnetic field at each point! An alternative approach proposed by Sakurai (1981, 1985) is to discretize the integral involving the Green's function by introducing "finite-element"-like dicretization for the electric currents. The process thus consists in starting from an initial current-free field line, putting current on it, and then retracing the correct perturbed field line carrying the electric current just put on. In this method, the field lines are discretized into a finite number of nodes (which define the degrees of freedom of the problem) and the nodes locations then become the unknowns of the problem for tracing the field lines. The latter are determined by solving a system of nonlinear algebraic equations, whose convergence is related in some sense to the absolute value of α, and has not been proved to hold for large values of α.

In this paper we consider another class of Grad-Rubin Methods that used the vector potential representation of the magnetic field. The paper is organized as follows. In Sect. 2 we present the general problem that is solved. In Sect. 3, we present the class of Grad-Rubin-like computational methods for solving the non-linear force-free case. We introduce in particular a new Vector Potential formulation in Sect. 4. We then present some results obtained with our method when applied to some particular known solutions in Sect. 5. Sect. 6 gathers concluding remarks.

It should be noted that a portion of the present study has been published in the proceeding of a conference (Amari et al.1997).

2. The problem

The set of equations that describe the equilibrium of the coronal magnetic field in the half-space $\Omega = \{z > 0\}$, when plasma pressure and gravitational forces are neglected, are the well known force-free equations (Parker 1979):

$$\nabla \times \mathbf{B} = \alpha(\mathbf{r})\mathbf{B} \quad \text{in} \quad \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{in} \quad \Omega, \tag{2}$$

in which $\alpha(\mathbf{r})$ as well as $\mathbf{B}$ are unknowns.

The analysis of set of characteristics curves of this system, which is in general nonlinear (Grad and Rubin 1958, Parker 1995), shows that this system has a mixed elliptic-hyperbolic type structure. This complex structure of the problem (already known in fluid mechanics as the Beltrami field equations) makes this problem a formidable task to solve, and still makes it an open field of research in applied mathematics (Laurence & Avellaneda 1993), even in bounded domains. Moreover the astrophysical constraints, as seeking a solution in a domain that may be unbounded as $\Omega = \{z > 0\}$ add another non-trivial difficulty.

This mixed nature implies the requirement of two types of boundary conditions:

- First of all, the elliptic part, resulting from the assumption that the RHS of Eq. (1) is given (the electric current), is rather well known, since it is nothing else than the Biot and Savart law, and just requires the value of B_n on $\partial\Omega$ to compute $\mathbf{B}$ in the whole domain, as expected for any elliptic problem:

$$B_n|_{\partial\Omega} = b_0, \tag{3}$$

  where b_0 is a given regular function.
- Then from Eqs. (1)-(2) one gets a hyperbolic equation for α (for $\mathbf{B}$ given):

$$\mathbf{B}.\nabla\alpha(\mathbf{r}) = 0\,, \tag{4}$$

  and therefore one may give the value of α in the part $\partial\Omega^+$ of $\partial\Omega$ where $B_n > 0$, say:

$$\alpha|_{\partial\Omega^+} = \alpha_0\,, \tag{5}$$

where α_0 is a given regular function. Note that this type of boundary condition is sufficient if one reasonably assumes that every field line of the coronal magnetic field has its two footpoints connected to the boundary $\partial\Omega$. Configurations having non-connected field lines (magnetic islands) would otherwise lead to the impossibility of transporting information from the boundary $\partial\Omega$ (Aly 1988).

Because Ω is unbounded, one may also require the asymptotic boundary condition:

$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{B}| = 0. \tag{6}$$

3. Grad-Rubin approach

Let us follow the approach that was proposed by Grad and Rubin (1958). The previous underlying mixed elliptic-hyperbolic structure of the system of equations is exploited by introducing the following sequences of hyperbolic and elliptic linear BVPs:

$$\mathbf{B}^{(n)}.\nabla\alpha^{(n)} = 0 \quad \text{in} \quad \Omega, \tag{7}$$

$$\alpha^{(n)}|_{\partial\Omega^+} = \alpha_0\,, \tag{8}$$

and

$$\nabla\times\mathbf{B}^{(n+1)} = \alpha^{(n)}\mathbf{B}^{(n)} \quad \text{in} \quad \Omega, \tag{9}$$

$$\nabla\cdot\mathbf{B}^{(n+1)} = 0 \quad \text{in} \quad \Omega, \tag{10}$$

$$B_z^{(n+1)}|_{\partial\Omega} = b_0\,, \tag{11}$$

$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{B}^{(n+1)}| = 0\,. \tag{12}$$

with $\mathbf{B}^0$ the unique solution of:

$$\nabla\times\mathbf{B}^0 = 0 \quad \text{in} \quad \Omega, \tag{13}$$

$$B_z^0|_{\partial\Omega} = b_0\,, \tag{14}$$

$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{B}^0| = 0\,, \tag{15}$$

that is given by (Aly 1989):

$$\mathbf{B}^0 = \nabla\phi^0\ , \tag{16}$$

$$\phi^0(\mathbf{r}) = \frac{1}{2\pi}\int_{\partial\Omega} B_z(\mathbf{r}')\frac{dx'\,dy'}{|\,\mathbf{r}-\mathbf{r}'\,|}\ . \tag{17}$$

These sequences of problems must be proved to converge towards the solution of the original BVP defined by Eqs. (1) and (2) provided with the set of boundary conditions given by Eqs. (3),5). One can use them to address theoretical issues such as i) existence of solution ii) uniqueness iii) continuity of the solution with the respect to the boundary conditions. These three points define a well-posed BVP in the sense of Hadamard (1932) and has been discussed for other BVP associated to extrapolation methods (Low & Lou 1991, Amari et al. 1998). Note that the last point is important because of the presence of errors in the measurements of the photospheric magnetic field and of the possible non-force-free character of the field at the photospheric level, where pressure and dynamic forces can play a non-negligible role (Aly 1989, McClymont et al. 1997). Of course those three points depend on the functional space in which one seeks the solution, and in particular on the the regularity of the solution (Amari 1991).

Bineau (1972), considered this BVP in the Holder functional spaces (set of functions sufficiently regular and whose derivatives are also regular enough, Brezis 1983). The BVP is then proved to be well-posed when $\alpha < \alpha_c$. However, this proof rests on the following assumptions: (i) The domain Ω is bounded. (ii) The field $\mathbf{B}_0$ as well as $\mathbf{B}$ have a simple magnetic topology (then they must not vanish in Ω). It is however possible to show the existence of a solution for Ω bounded, in more general spaces (when $(\alpha,\mathbf{B}) \in L^\infty \times H^1(\Omega)$), that is in a functional space such that solution may admit separatrices surfaces, null points, and current sheets, (Boulmezaoud, Amari & Maday, in preparation). Uniqueness of the solution has not yet been proved.

4. A vector potential formulation

4.1. Gauge for B_n fixed on $\partial\Omega$

To ensure that $\mathbf{B}$ is divergence free (Eq. (2)) we use the vector potential representation for $\mathbf{B}$. Since in BVP (10)-(12) $B_n|_{\partial\Omega}$ is fixed, the vector potential $\mathbf{A}$ should be determined such that

$$\mathbf{B} = \nabla\times\mathbf{A} \quad \text{in} \quad \Omega, \tag{18}$$

$$B_n|_{\partial\Omega} = b_0 \tag{19}$$

This representation is not yet unique, since if $\mathbf{A}$ is a potential for $\mathbf{B}$ then:

$$\hat{\mathbf{A}} = \mathbf{A} + \nabla\phi \tag{20}$$

where ϕ is an arbitrary scalar function, is also a vector potential for $\mathbf{B}$. Uniqueness is obtained by the choice of a particular gauge. There are several possible choices (Dautray & Lions 1982), but these do not in general take into account Eq. (11) (one well known choice is for example $\nabla\cdot\mathbf{A} = 0$ and $A_n|_{\partial\Omega} = 0$).

Our gauge is fixed by imposing that $\mathbf{A}$ *is the unique vector potential such that*:

$$\mathbf{B} = \nabla\times\mathbf{A} \quad \text{in} \quad \Omega, \tag{21}$$

$$\nabla\cdot\mathbf{A} = 0 \quad \text{in} \quad \Omega, \tag{22}$$

$$\nabla_t\cdot\mathbf{A}_t = 0 \quad \text{on} \quad \partial\Omega \tag{23}$$

where the subscript t in ξ_t stands for the trace (when it exists) of the operator or the field ξ on the boundary (in particular in cartesian coordinates: $\nabla_t\cdot\mathbf{g} = \partial_i g_i + \partial_j g_j$ on the plane $\Sigma_k = \{\mathbf{r}\cdot\hat{n}_k = \mathit{constant}\}$ with $(i,j,k) := (x,y,z)$ and $\hat{n}_k$ standing for the unit vector normal to the current boundary plane Σ_k). Note also that one readily gets:

$$\partial_n\mathbf{A}_n = 0 \quad \text{on} \quad \partial\Omega, \tag{24}$$

where $\nabla_n(f) = \hat{n}\cdot\nabla f$. The proof that $\mathbf{A}$ is unique is straightforward since from Eqs. (20)-(23) one gets that ϕ is the unique solution of a Laplace equation:

$$\Delta\phi = 0 \quad \text{in} \quad \Omega, \tag{25}$$

$$\phi = \phi_0 \quad \text{on} \quad \partial\Omega \tag{26}$$

(note that ϕ_0 is also obtained once solving a Laplace equation on $\partial\Omega$, and is also unique once ϕ_0 is prescribed on the border Γ of the boundary $\partial\Omega$).

Then with this choice of Gauge, $\mathbf{A}$ is the unique vector potential that satisfies:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{in} \quad \Omega, \tag{27}$$
$$\nabla \cdot \mathbf{A} = 0 \quad \text{in} \quad \Omega, \tag{28}$$
$$\nabla_t \cdot \mathbf{A}_t = 0 \quad \text{on} \quad \partial\Omega \tag{29}$$
$$\mathbf{A}_t = \nabla^{\perp}\chi \quad \text{on} \quad \partial\Omega \tag{30}$$
$$\partial_n \mathbf{A}_n = 0 \quad \text{on} \quad \partial\Omega \tag{31}$$

where $\nabla^{\perp}$ is the operator defined on $\partial\Omega$ such that $\nabla^{\perp} \cdot \nabla = 0$ (i.e., $\nabla^{\perp}\chi = \nabla_t\chi \times \hat{n}$) and where χ is the unique solution of

$$-\Delta_s \chi = b_0 \quad \text{in} \quad \partial\Omega\,, \tag{32}$$
$$\chi = 0 \quad \text{or} \quad \partial_n\chi = 0 \quad \text{on} \quad \Gamma\,. \tag{33}$$

where Δ_s is the Laplacian operator on $\partial\Omega$ (i.e., $\Delta_s f := \nabla_t^2 f$)

4.2. BVP for $\mathbf{A}$

One can then rewrite BVP(10)-((12) in terms of the potential vector $\mathbf{A}$ that is then the unique solution of the following BVP (referred to hereafter as BVP-A):

$$\nabla \times \mathbf{A}^{(n)}.\nabla\alpha^{(n)} = 0 \quad \text{in} \quad \Omega, \tag{34}$$
$$\tag{35}$$
$$\alpha^{(n)}|_{\partial\Omega^+} = \alpha_0\,. \tag{36}$$

and

$$-\Delta\mathbf{A}^{(n+1)} = \alpha^{(n)}\nabla \times \mathbf{A}^{(n)} \quad \text{in} \quad \Omega\,, \tag{37}$$
$$\mathbf{A}_t^{(n+1)} = \nabla^{\perp}\chi \quad \text{on} \quad \partial\Omega\,, \tag{38}$$
$$\partial_n\mathbf{A}_n^{(n+1)} = 0 \quad \text{on} \quad \partial\Omega \tag{39}$$
$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{A}^{(n+1)}| = 0\,. \tag{40}$$

The solution $\mathbf{A}^{(n+1)}$ of the linear elliptic mixed Dirichlet-Neumann BVP is in general regular ($\mathbf{A}^{(n+1)} \in C^2(\Omega) \cup C^1(\partial\Omega)^3$).

One can then prove that:

$$\forall n \geq 1\;;,\mathbf{A}^{(n)} \quad \text{satisfies} \quad \nabla \cdot \mathbf{A} = 0 \quad \text{in} \quad \Omega,$$

Proof: Applying the operator $\nabla\cdot$ to both sides of Eq. (37) and using Eq. (2) for $\mathbf{B}^n$, one gets:

$$-\Delta(\nabla \cdot \mathbf{A}^{(n+1)}) = \nabla \cdot (\alpha^{(n)}\nabla \times \mathbf{A}^{(n)}) \tag{41}$$
$$= \nabla \times \mathbf{A}^{(n)}.\nabla\alpha^{(n)} = 0 \quad \text{in} \quad \Omega \tag{42}$$
$$\nabla \cdot \mathbf{A}^{(n+1)} = 0 \quad \text{on} \quad \partial\Omega\,, \tag{43}$$

Whence $\nabla \cdot \mathbf{A}^{(n+1)} = 0$ is the unique solution tending to zero at infinity for this BVP. Note that since the initial potential magnetic field $\mathbf{B}^{(0)}$ clearly satisfies $\nabla \cdot \mathbf{J}^{(0)} = 0$ (where $\mathbf{J}$ stands for the associated electric current), this property is preserved for all $n > 0$.

4.3. A two-level iteration procedure

Let us define the sequence $(\alpha_{0_p})_{1\leq p\leq P}$ and the mononotic increasing sequence $(u_p)_{1\leq p\leq P}$ such that:

$$\alpha_{0_p}(x) = u_p\alpha_0(x) \quad \text{for} \quad x \in \partial\Omega\,, \tag{44}$$
$$u_1 = \epsilon\,, \tag{45}$$
$$u_P = 1\,, \tag{46}$$

where ϵ is a ''small enough" real number, and P is a ''large enough" integer.

One can then generate a more general sequence of linear BVP for $(\mathbf{A}_p^{(n)}, \alpha_p^{(n)})_{n\geq 1, 1\leq p\leq P}$ given by:

$$\nabla \times \mathbf{A}_p^{(n)}.\nabla\alpha_p^{(n)} = 0 \quad \text{in} \quad \Omega, \tag{47}$$
$$\tag{48}$$
$$\alpha_p^{(n)}|_{\partial\Omega^+} = \alpha_{0p}\,. \tag{49}$$

and

$$-\Delta\mathbf{A}_p^{(n+1)} = \alpha^{(n)}\nabla \times \mathbf{A}_p^{(n)} \quad \text{in} \quad \Omega\,, \tag{50}$$
$$\mathbf{A}_p{}^{(n+1)}{}_t = \nabla^{\perp}\chi \quad \text{on} \quad \partial\Omega\,, \tag{51}$$
$$\partial_n\mathbf{A}_p{}^{(n+1)}{}_n = 0 \quad \text{on} \quad \partial\Omega \tag{52}$$
$$\lim_{|\mathbf{r}|\to\infty} |\mathbf{A}_p^{(n+1)}| = 0\,. \tag{53}$$

One may initialize the iteration procedure for $p = 0$, with the unique solution of BVP(13)-(15) (which would be equivalent to choose $u_0 = 0$). A possible choice for $(u_p)_{1\leq p\leq P}$ is for P given:

$$u_p = p\frac{1}{P}\,, \tag{54}$$
$$\tag{55}$$

One clearly notices that for every value of p one needs to solve a sequence of linear BVPs for all $n > 0$. This corresponds to a progressive injection of α at the boundary which turns out to improve convergence of the classical Grad-Rubin scheme.

4.4. Numerical implementation

We have developed a code called EXTRAPOL, based on the method described in the previous sections.

- i) The computational domain Ω is supposed to be the bounded cubic box $[0, Lx] \times [0, Ly] \times [0, Lz]$ (instead of the infinite upper half space), that we discretize as Ω_h using a non-uniform structured mesh for finite difference approximation. This staggered mesh used for the the various components of the vector potential $\mathbf{A}$, the magnetic field $\mathbf{B}$ and α, is the same as the one used in our MHD code METEOSOL used for three-dimensional dynamic evolution (Amari et al. 1996).
- ii) We use as a boundary condition for BVP (37)-(40) on the lateral and top boundaries of the box, $\mathbf{B}_n = 0$, which owing to our gauge choice (Eqs. (22)-23)) is equivalent to impose

on these boundaries:

$$\mathbf{A}_t = 0 \tag{56}$$

$$\partial_n \mathbf{A}_n = 0 \tag{57}$$

This type of boundary conditions, whose aim is to mimic the far field behaviour at infinity (as one would expect for the magnetic field in the actual infinite half-space), implies that the top and lateral boundaries of the box have to be chosen sufficiently far away from the main region of interest. This can be achieved at relatively low cost since our mesh is not uniform, and therefore large cells can be put in the far-field region.

- iii) The various differential operators (Eqs. (47)-(53)) are then dicretized on this mesh to second order accuracy. The Laplacian operator (in the Dirichlet-Neuman BVP (50)-(53)) leads to a 7 diagonal sparse positive definite matrix. The corresponding linear system is solved by use of an iterative method, in which the matrix is not stored but the matrix-vector product is generated explicitly by the operator (and only one more array is stored for building a preconditioner to accelerate the convergence of the method). This memory space saving allows the method to be implemented on a workstation with reasonable central memory size, and not only on supercomputers. We actually run the code on both machine types although the results presented here correspond to runs performed on a CRAY C90 machine.
- iv) The numerical solution of Eqs. (47)-(49) is performed by using a characteristics method approach, since those curves are the field lines. Let $(\mathbf{X}; s)$ be the characteristics, solution of

$$\mathbf{X}' = \mathbf{B}(\mathbf{X}), \mathbf{X}(0) = \mathbf{q} \tag{58}$$

for $\mathbf{q}$ given in Ω_h (the prime symbol standing for differentiation with respect to the parameter that runs along the characteristics). Then for any node $\mathbf{q}_h \in \Omega_h$ on which α is defined, one gets α_h as

$$\alpha(\mathbf{q}_h) = \alpha_0(\mathbf{X}_{\partial\Omega^+}(\mathbf{q}_h)) \tag{59}$$

where $\mathbf{X}_{\partial\Omega^+}(\mathbf{q}_h) = \mathbf{X}(\mathbf{q}_h, s_{\partial\Omega^+})$ is the intersection of $\{\mathbf{X}(\mathbf{q}; s) : s < 0\}$ with $\partial\Omega^+$. Since α_0 is known at the nodes that do not in general coincide with $\alpha_0(\mathbf{X}_{\partial\Omega^+}(\mathbf{q}_h))$, an interpolation from its four nearest neighbors eventually gives $\alpha(\mathbf{q}_h)$. We have then derived two methods: **a)** In the first one, once a step is chosen for field line integration, one goes backwards along the characteristics using a second order predictor-corrector scheme. Clearly one can save computation time by avoiding going back up to $\partial\Omega^+$. This is achieved by marking the nodes in the domain where α has already been computed, and then linearly interpolating α from its nearest neighbors as soon as the current node is surrounded by such marked nodes. **b)** In a second method (Pironneau 1988) one avoids fixing a step by using a slightly less accurate scheme that consists in going backwards along the characteristics following the faces of each cubic cell that is centered on an α-node, approximating the characteristic curve by a polygonal line made of the segments $[\mathbf{q}^k, \mathbf{q}^{k+1}]$ where $\mathbf{q}^0 = \mathbf{q}_h$ and $\mathbf{q}^{k+1}$ is the intersection of the line $\{\mathbf{q}^k - \mu\mathbf{B}(\mathbf{q}^k)\}_{\mu>0}$ with the boundary ∂C_m of the cubic α-cell C_m that contains $\mathbf{q}^k$ and $\mathbf{q}^k - \eta\mathbf{B}(\mathbf{q}^k)$ (with $\eta > 0$). This method is then faster than the previous one since there is no step size to be fixed a priori. Unlike for the first method, in a non-uniform mesh, each cell is crossed in 'one step' only, which makes this method faster in the big cells region. Despite this difference in the computational speed we have kept the two methods available because of their slight accuracy difference.

5. Application to some known exact force-free solutions

We now test the scheme presented in the previous section by running our code EXTRAPOL on some analytical and semi-numerical exact solutions of the non linear force-free equations Eq. (1)-(2). The boundary values of these exact solutions are used as simulated magnetograms. Hopefully, in these cases one knows the solution above in the domain too, and compare the reconstructed and the exact solutions (which is not the case for the actual corona!). There are only very few known exact solutions of the force-free equations. Let us presents the results obtained with our code on two cases that have been also used by other methods such as the VIM (Demoulin et al. 1992) for the first one and the Resistive Relaxation Method (Mikic & McClymont 1994) for the second one. Note that another class of related solutions that will not be tested here are those found by Cuperman & Ditkowski (1991).

5.1. The Low (1982) solution

Our first target is the well-known solution of Low (1982) for which the magnetic field $\mathbf{B}$ is given by:

$$B_x = -\frac{B_0}{r}\cos\phi(r), \tag{60}$$

$$B_y = \frac{B_0 x_1 y_1}{r\rho^2}\cos\phi(r) - \frac{B_0 z_1}{\rho^2}\sin\phi(r), \tag{61}$$

$$B_z = \frac{B_0 x_1 z_1}{r\rho^2}\cos\phi(r) + \frac{B_0 y_1}{\rho^2}\sin\phi(r), \tag{62}$$

where $x_1 = x - \frac{L_x}{2}$, $y_1 = y - \frac{L_y}{2}$, $z_1 = z + 1$, $\rho^2 = y_1^2 + z_1^2$, $r^2 = x_1^2 + \rho^2$. The generating function $\phi(r)$ is related to $\alpha(r)$ by:

$$\alpha(r) = -\phi'(r) \quad . \tag{63}$$

We choose for the function ϕ

$$\phi(r) = r_0\alpha_m \tanh(r/r_0), \tag{64}$$

which owing to Eq. (63) gives:

$$\alpha(r) = -\frac{\alpha_m}{\tanh(r/r_0)} \quad . \tag{65}$$

We fix hereafter $r_0 = 4$ and $\alpha_m = 0.2$

Our numerical box size corresponds to the choice $L_x = 48$, $L_z = 48$, $L_z = 48$. A non-uniform mesh with $51 \times 51 \times 51$ nodes was chosen as in Demoulin et al. (1992). The analytical solution is then computed on the mesh and in particular the values taken by B_z and α on the boundary provide boundary conditions for our force-free reconstruction procedure. We choose $P = 25$ in Eq. (53) (i.e., the parameter necessary to fix the outer iteration corresponding to the injection of α_0). Our method converges up to a Lorentz force of order 10^{-3} for a number of inner Grad-Rubin iterations $N_{iterations} = 6$. The numerical error is defined as in Amari et al. (1998). We also found that choosing $P = 15$ implies increasing $Ngradrub$ up to about 12 to reach a Lorentz force of the same order. Fig. 1 shows some field lines of the exact solution (top) and the the corresponding field lines obtained from our computation.

Some discrepancies (up to few percents) between the exact and the computed solution are found in the domain, and these can reach almost .2 for the field lines approaching the lateral boundaries of the box. These can be explained by our choice for the boundary condition ($B_n = 0$) on these boundaries for the computed solution, while the exact solution does not decrease fast enough and even more pathologically in the horizontal plane (see Amari et al. 1998). Note that because applying this boundary condition results in a difference between the computed and exact solution, but still allows to reach a force-free equilibrium. However this equilibrium shows a different behaviour than the exact solution near the boundary, but there is no intrinsic instability as in the VIM (Wu 1990 and references therein, Cuperman et al. 1990a-b, Demoulin et al.1992). It is worth noting that we have also performed some higher resolution run, with $Nx = 101$,$Ny = 101$,$Nz = 101$ which, unlike the VIM, gave even better results, allowing the boundaries to be pushed far away. Note that this ‘’robustness ‘’ property (good behaviour while increasing spatial resolution) as well as the convergence of the method even for this type of lateral and top boundary conditions results from the well-posed formulation we have adopted, unlike for the VIM which is associated to an ill-posed mathematical problem (see Low & Lou 1991 and Amari et al. 1998). In this latter method errors increase exponentially with height (Demoulin et al. 1992) and this is a property intrinsic to the method (and not the numerical scheme used for the extrapolation), which implies that the computed solution will eventually diverge, while our solution never diverges for an arbitrarily large box. Actually the bigger is our box, the bigger the region of agreement between our solution and the exact one is, a property that we checked with the higher resolution run, pushing the lateral boundaries to $L_x = 120$, $L_z = 120$, $L_z = 120$.

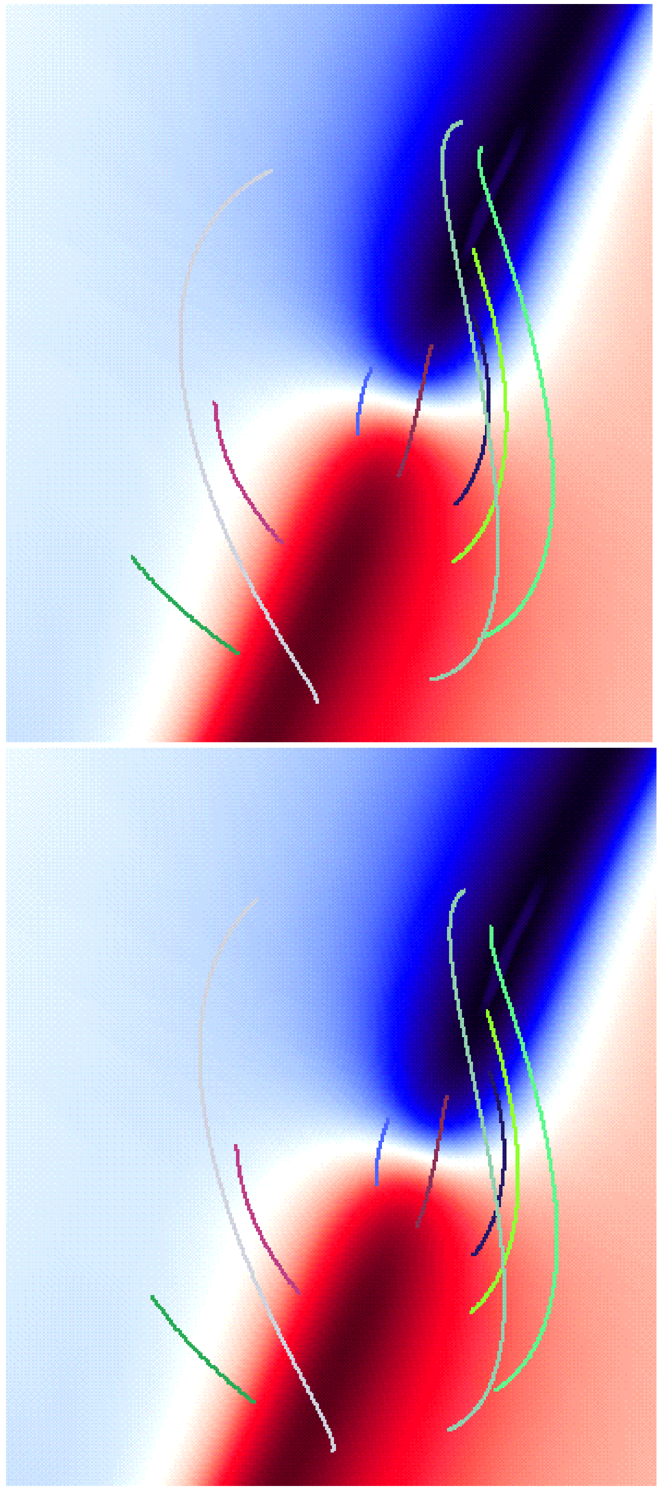

Fig. 1. Example of force-free reconstruction with the Vector Potential Grad Rubin Method compared to the exact analytical Low's (1982) solution. The computed solution (*bottom*), matches the exact solution (*top*) up to few percents in most of the interior of the computational box. Some discrepancies occur for field lines near the lateral and top boundaries because of the slow asymptotic decreasing behaviour of this particular solution, while the boundary condition $B_n = 0$ has been imposed on the boundaries in the computation.

5.2. *Low & Lou's (1991) solution*

We have also tried the particular case of the exact force-free solution presented in Low & Lou (1991). Unlike Low's (1982) solution, it requires some numerical calculations.

The solution is supposed to be axi-symmetric and writes in spherical coordinates:

$$B_r = \frac{1}{r^2 \sin\theta} \frac{\partial A}{\partial \theta} \ , \quad (66)$$

$$B_\theta = -\frac{1}{r \sin\theta} \frac{\partial A}{\partial r} \ , \quad (67)$$

$$B_\phi = \frac{1}{r \sin\theta} Q \ , \quad (68)$$

where $Q(A)$ is an a priori unknown function of $A(r,\theta)$, a solution of the nonlinear partial differential equation (see Low & Lou 1991). A family of solutions can be generated by choosing

$$A = \frac{P(\cos\theta)}{r^n} \ , \quad (69)$$

$$Q = a A^{1+\frac{1}{n}} \ , \quad (70)$$

for odd n, and a a real constant. P is then the solution of the following boundary-value problem:

$$(1-\cos^2\theta)\frac{d^2 P}{d(\cos\theta)^2} + n(n+1)P + a^2 \frac{1+n}{n} P^{1+\frac{2}{n}} = 0 \ , \quad (71)$$

$$P(-1) = P(1) = 0 \ . \quad (72)$$

We then solve numerically Eqs. (71)- (72). Usual transformations (Low & Lou 1991) then allow to get the solution in cartesian coordinates, in the upper half space.

Our numerical box is taken such that $L_x = 8$, $L_z = 8$, $L_z = 4$. A non-uniform mesh is generated with $N_x = 60$, $N_y = 60$, $N_z = 40$ with most of the cells concentrated in the inner stronger field region. Once BVP (71)-(72) is solved, one deduces the corresponding three components (B_x,B_y,B_z), the associated electric currents and $\alpha = \frac{dQ}{dA}$ on the same nodes (x_h, y_h, z_h) of the mesh used by our force-free reconstruction code EXTRAPOL, and then computes the solution. One then use $B_z(x_h, y_h, 0)$ and $\alpha(x_h, y_h, 0)$ (for the nodes in $\partial\Omega^+$ only) as boundary condition for the reconstruction procedure. We found that using $P = 20$ and 4 inner iterations($N_{iterations} = 4$) allows to decrease the Lorentz force down to values of order 10^{-3}.

Fig. 2 shows some field lines of the exact solution (top) and the corresponding field lines resulting from our reconstruction procedure. The errors, defined as for the previous case (Low's (1982) solution), are even less or of the order of 1% in the larger part of the domain, except again near the lateral and top boundaries where the imposed boundary condition $B_n = 0$ and the exact one disagree. Actually, those discrepancies are however smaller than those of the case of Low's (1982) model for the lateral boundaries because the magnetic field now decreases faster with distance. The case of the top boundary is different because of the existence, in the exact solution, of a pathological field-line in the center of the box that crosses almost vertically the top boundary while it has to match the applied boundary condition $B_n = 0$ in the calculation, which will be difficult to fulfill, even with a large box. Note that despite the much better asymptotic behaviour of this force-free solution for the magnetic field the electric currents are distributed on a scale that is still large, which results in a configuration that does not quickly approach

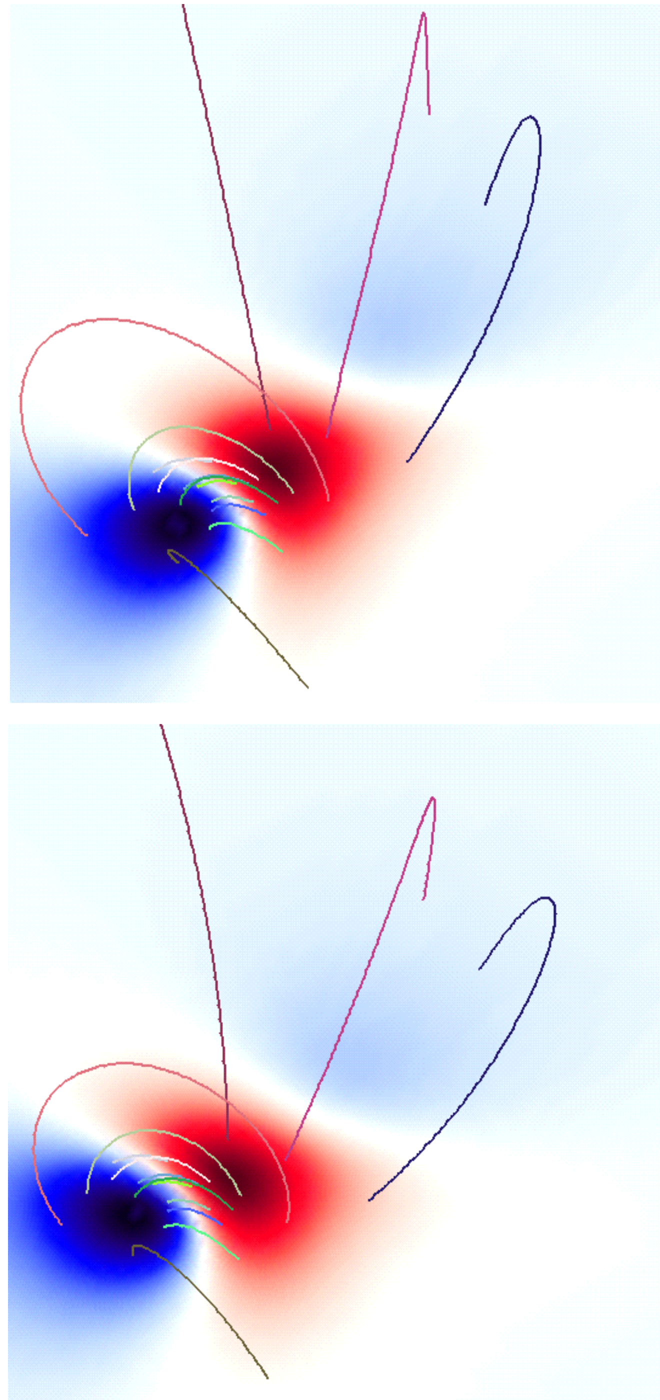

Fig. 2. Non linear force-free reconstruction (with the Vector Potential Grad Rubin Method) of the semi-numerical exact solution of Low & Lou (1990). The computed solution (*bottom*), and the exact solution (*top*) agree in most of the computed area. The existence of a pathological field line (in the exact solution) that crosses almost perpendicularly the top boundary, implies larger errors near this boundary since the computed solution corresponds to the boundary condition $B_n = 0$. The boundaries of the box are put far away enough from the inner stronger field area.

toward the potential field as it is often the case in the corona, outside regions of more localized electric currents.

6. Conclusions

In this paper, we have presented a numerical method for reconstructing the coronal magnetic field as a force-free magnetic field from its value given on the boundary of the domain. Let us summarize here the main points we have discussed and our main results:

(a) The boundary-value problem is formulated such that it corresponds to a well-posed mathematical problem: the normal component of the magnetic field is imposed on the boundary of the domain, and α only on that part of this boundary where $B_n > 0$. We impose $B_n = 0$ on the lateral boundaries so that α does not need to be specified on these boundaries, provided that these boundaries are put far enough to mimic the behaviour of the solution at infinity.

(b) We have derived a Grad Rubin Vector Potential formulation of this BVP to ensure div $\mathbf{B} = 0$ up to machine roundoff numerical errors. We have shown that this problem may be equivalent to solve a sequence of linear elliptic boundary-value problems for the Vector Potential, and hyperbolic ones for α. This current formulation is relevant for seeking regular enough solutions but not equilibria having current sheets. Weak formulations of these methods are however currently under study and would be reported in a next Paper of the series (Amari & Boulmezaoud 1999, in preparation). We have implemented this method in our computational code EXTRAPOL, in a relatively efficient numerical way. Other mathematical approaches allowing the existence of critical points in the configuration are also currently studied.

(c) We have successively applied our method to theoretical magnetograms obtained from two exact known solutions, the solutions of Low (1982) and Low & Lou (1991). The method converges up to a small residual Lorentz force, in a reasonable number of iterations. Some discrepancies between the exact solution and the reconstructed one occurred near the top or lateral boundaries of the computational box, and have been explained by the relatively bad asymptotic behaviour of Low's (1982) solution, or the existence of an almost vertical pathological field line in the solution of Low & Lou (1991), which makes difficult to match our applied boundary conditions on these boundaries ($B_n = 0$). Other approaches involving the assumption of potential field near the boundaries, or approximation of the Green formulae that can explicitly give the normal component at those boundaries are currently under development. Another approach could be to map the infinite upper half-plane onto the bounded square box by using a class of mappings that represent the generalization of conformal mappings used in two dimensions.

(d) Our formulation is better than the (VIM) (Wu 1985, Cuperman et al. 1990a-b, Demoulin et al.1992) since it corresponds to a mathematically well-posed boundary value problem. Although it may exhibit some residual discrepancies with the exact known solution, errors never increase exponentially up to blowing up as in the VIM. Moreover as it was shown by Bineau (1972) another consequence is that the solution is expected to be continuous respect to boundary conditions, at least for α not to large (Amari et al. 1998).

(e) Our method is different from Sakurai's (1981) approach in which, instead of solving an elliptic problem for $\mathbf{B}$, he uses a more local approach where the location of the nodes that discretize a given field line are computed once some electric currents (α) are injected in this field line, as the solution of non-linear system of equations that does not take into account the contribution of the whole computational domain (as one would expect in an elliptic problem). This approach allows a fast enough computation, which might be useful for some very concentrated (almost thin isolated) tube-like configurations, but it is not yet clear how this truncation procedure (by solving a single problem for each field line) may be involved in the numerical instabilities encountered in solving the nonlinear system for cases corresponding to large values of α. Indeed Sakurai's (1981) approach might be considered as a Lagrangian discretization method while we have presented a Eulerian type discretization that would be more suited to highly sheared magnetic configurations. The two methods should be worth to be kept and used for different types of data and configurations. The results presented in this Paper seem to be optimistic as regards the application of the method to simulated magnetograms. The next step currently under development is the application of this method to various sets of data provided by vector magnetographs. However there are several important points that need to be emphasized, and that make actual data much more difficult to handle than exact force-free solutions:

i) First of all data are much more noisy, because of the errors on the transverse magnetic field measurements that are larger than on the longitudinal one (Amari & Demoulin 1992, Klimchuk & Canfield 1993, McClymont et al. 1997). Other errors may also arise after the resolution of the 180^o ambiguity that exists on the transverse component. These errors depend on the method that is used (Mikic & Amari 1999, in preparation). Eventually the non-force-free character of the photosphere (Aly 1989) may be taken into account. Actually from point (b) above, the well-posedness of our formulation (for at least α not too large), would make the solution not very sensitive to errors expected on the photospheric measurements. We are currently working on the project of simulating the error effects (Amari et al. 1999, in preparation) of these instrumental errors on the transverse field components, by introducing some random noise in the simulated data obtained from some highly sheared force-free solutions obtained by a relaxation code (Yang et al. 1986, Klimchuk & Sturrock 1992), and then reconstructing them with our method.

ii) Related to these errors, one may also find that, unlike theoretical magnetograms, actual data are far from smooth. This implies that any reconstruction method should be either robust or one will have to smooth the data prior to reconstruction, which may introduce possible added deviations from the sought solution, since there is no unique way of smoothing.

iii) One non negligible difficulty that has to be taken into account is the needs for computing α from the photospheric normal

components of the magnetic field and of the electric current. Weak field regions cannot be ruled out in a straightforward way since high shear can be localized near the neutral line (Hagyard 1988).
iv) One final point is that unlike for theoretical magnetograms, one never knows a priori the solution in the corona in order to check the reconstructed one. However an alternative can be the use of YOHKOH or SOHO/EIT data (for different heights). These data would have to be used a posteriori to check if the computed structures has such loops or ‘’footpoints” that match the coronal observed ones, but not use these data set to fix a remaining free parameter such as α in linear force-free constant-α extrapolations!

Acknowledgements. The numerical computations presented in this paper have been performed on the CRAY supercomputers of the Commissariat à l’Energie Atomique and the Institut I.D.R.I.S of the Centre National de la Recherche Scientifique. This work benefits from a NATO Collaborative Research Grant support.

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