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Analytical solution of the radiative transfer equation for polarized light

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Abstract. A new formalism is introduced for the transfer of polarized radiation. Stokes parameters are shown to be four–vectors in a Minkowski-like space and, most strikingly, the radiative transfer equation (RTE) turns out to be an infinitesimal transformation under the Poincaré (plus dilatations) group. A solution to the transfer equation as a finite element of this group is proposed.

Key words: radiative transfer – polarization

1. Introduction

Since the pioneering paper of Unno (1956) in which for the first time a transfer equation was derived for polarized light in the presence of a magnetic field, a number of attempts have been made to obtain a solution for it. Rachkowksy (1967), after completing the equation by adding anomalous dispersion effects, gave the first analytical solution for the case of Milne-Eddington atmospheres. While numerical solutions were more and more successful (Wittmann, 1974; Landi Degl'Innocenti, 1976; Rees et al., 1989; Bellot Rubio et al., 1998; López Ariste & Semel, 1999) analytical trials struggled to overcome the problem of non-commuting absorption matrices. This problem, already noted in a paper of Landi Degl'Innocenti & Landi Degl'Innocenti (1985), is pointed out clearly in the work of Semel & López Ariste (1999) (hereafter referred as Paper I) as the origin of all previous limitations. In this paper, deep insights are given on how to benefit from the physical significance of the various terms in the transfer equation to attain an analytical solution as general as possible. Several transformations used here to simplify the RTE seem to indicate that the physics of the Stokes parameters is best described in geometrical terms. Indeed, we shall show in this paper that the Poincaré (plus dilatations) group is at the origin of these geometrical aspects, and we shall use its algebraic properties to solve the problem even for non–commuting absorption matrices and give a general solution for this equation of transfer.

The fundamental problem addressed in the present paper is the existence of an analytical solution to the RTE. At present, analytical solutions exist for only very limited cases and the known formal solutions (Landi Degl'Innocenti & Landi Degl'Innocenti, 1985) do not offer, in the general case, any advantage from the computational point of view. Apart from the interest of a computable analytical solution by itself, it would be of greater importance for testing numerical codes which nowadays are considered acceptable only by observing their *convergence* upon increasing number of layers. This is not a very satisfactory situation. We think that the solution presented in this paper is the first step towards a general solution suitable for testing computations.

Why an analytical solution could not be found so far is our first question. It is our belief that group theory is the key to the solution and that explains why it had not been reached up to now (this could already be understood from the conclusions in Paper I which stressed the importance of the non-commutativity of matrices in this problem, but it is still clearer here). Group theory is not common in astronomical literature. However the RTE of polarized light has become a more and more important problem in astrophysics, related, for instance, to the measurement of magnetic fields via the Zeeman or Hanle effects. Hence group theory should be accessible to the concerned astrophysical community, and with its help we can give a method to find the solution, and explicitly give the full expression of the analytical solution for the most general case. Any other particular or general solution will benefit from the use of advanced linear algebra, and so avoid wasted effort. Last but not least, it may deepen our understanding of Stokes polarimetry.

We begin our research by disclosing the mathematical nature of the Stokes vector, starting with its physical definition and extending to the appropriate mathematics to treat our problem. Extensive literature has already been devoted to the existing relations between polarization and the Lorentz group, mainly from the optical point of view (see for instance Cloude 1986; Givens & Kostinski 1993; Sridhar & Simon 1994 and references therein). Usually these works take off from the Jones formalism for polarized light (Jones, 1941) and develop these relations. Here a similar path is followed to show that the Stokes vector is a 4–vector in a Minkowski-like space. The demonstration is based on the comparison between the usual definition for the Stokes parameters (see for example Shurcliff, 1962, or Jefferies, Lites & Skumanich, 1989) and the well–known relations between the

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definition of a spinor and its different representations (see for instance Landau & Lifshitz, 1971). In the same line of thought, we will propose in Sect. 3 that the RTE is just a representation of an infinitesimal Poincaré (plus dilatations) transformation in the Minkowski-like space where the 4–vectors are best described. This new interpretation of the equation of transfer suggests that any solution to this equation must be a finite Poincaré transformation. We calculate it in Sects. 4 and 5. We note that nothing differentiates the transfer equation as written elsewhere in the literature from a Poincaré transformation, although at present a demonstration is not available, apart from this similarity. We anticipate that a finite Poincaré transformation may be a solution for the RTE.

The concepts of 4–vector, Minkowsky space or Lorentz and Poincaré transformations must be understood throughout this paper in their purest mathematical sense, beyond their historical meaning. These concepts originated in the framework of the special theory of relativity, however mathematics did abstraction of these tools and incorporated them into more general frames of geometry and group theory. Therefore we define 4–vectors as sets of 4 numbers characterized by their Minkowsky norm and described in a hyperbolic 4–dimensional space called the Minkowsky space, which, in this paper, we will refer to as the *Minkowsky–like space*, in order to stress the difference with the usual Minkowsky space used in relativity. Lorentz and Poincaré transformations describe movements in this space: generalized translations and rotations. In the relativistic formalism these movements are interpreted as a change of reference system. In this paper they are not given that meaning, but are seen as changes in the polarization state. The manipulation of these concepts is identical here and in the relativistic formalism, if one takes care in substituting the four Stokes parameters for space and time, and for the speed its analogues as shown in Sect. 3.

2. Stokes parameters as a 4–vector

The 4 Stokes parameters are usually represented by I, Q, U, V, where I stands for the total intensity of light, Q and U for the linear polarized light in two axes rotated by $45°$ one from the other, and V for the circularly polarized light. These four quantities are not completely free but must satisfy an energy condition: there cannot be more polarized light than total light. This condition suggests an interpretation of the Stokes parameters as a 4–vector in a Minkowski-like space. The norm of a vector in such a space is defined as

$$\|\boldsymbol{I}\|^2 = I^2 - Q^2 - U^2 - V^2. \tag{1}$$

Hence the above condition is naturally satisfied by vectors with a positive norm, in parallelism to *time-like* vectors in special relativity. Continuing this parallelism, a *light-cone* can be defined as the surface which obeys the condition

$$\|\boldsymbol{I}\| = 0, \tag{2}$$

that is

$$I^2 = Q^2 + U^2 + V^2. \tag{3}$$

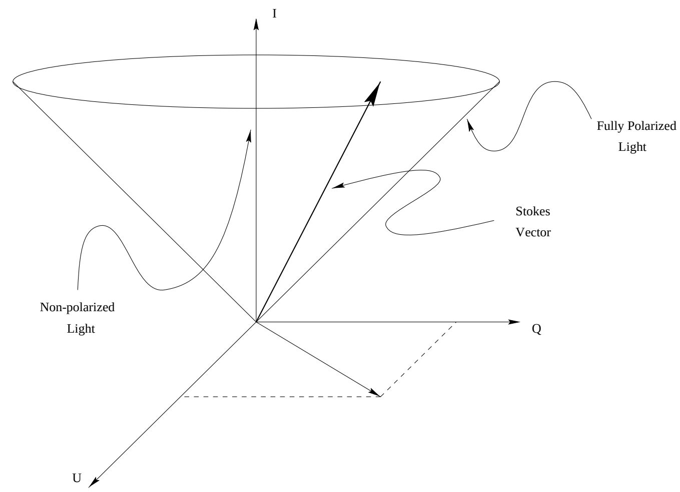

Fig. 1. 3-dimensional projection of the light cone for the Stokes vector

In our context, this surface contains all the different possibilities for fully polarized light. The Stokes vector must be inside or, in the limit, on this *light-cone* (see Fig. 1) to obey condition (1). An exception to this parallelism with special relativity: the *backward light cone* does not have an equivalent with the Stokes vectors.

To consolidate and extend this interpretation of the Stokes parameters we begin with the definition of the Stokes parameters in terms of the electric field.

A transversal monochromatic light wave is completely described by E_x and E_y, the components of the electrical field in a plane perpendicular to the direction of the propagation of light, z. Following definitions in Landau & Lifshitz (1971), these two components can be arranged in a 2–dimensional vector. Since it transforms linearly under the proper Lorentz group (ibid.) this vector can be called a *spinor of rank one*. As an illustration of this kind of transformation, we profit from the fact that any element of this group in its 2–dimensional representation can be written as a linear combination of the Pauli matrices (plus the 2×2 identity matrix):

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{5}$$

and transform the electric field vector by these matrices:

$$\sigma_0 \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad \sigma_1 \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_x \\ -E_y \end{pmatrix}$$
$$\sigma_2 \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} E_y \\ E_x \end{pmatrix}, \quad \sigma_3 \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} -iE_y \\ iE_x \end{pmatrix}. \tag{6}$$

The result is always a new 2-dimensional electric field vector which describes a different state of polarization.

A spinor of rank two can be easily constructed by multiplying conveniently two spinors of rank one. For our particular spinor we obtain

$$\mathsf{J}' = \begin{pmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{pmatrix},$$

where the $*$ symbol stands for complex conjugated. This 2nd rank spinor is to be compared with the *coherency matrix* (see Born & Wolf 1980). In fact the last is defined for any given light beam, and a mean over frequencies or time is necessary in the above expression of J' to fulfill the definition. The average of J' is a linear combination of matrices of the form referred. Since spinors of rank two form a linear vector space, any linear combination of spinors will be a spinor as well. Hence the coherency matrix, defined usually as

$$\mathsf{J} = \begin{pmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{pmatrix},$$

is still a spinor of rank two.

Since a spinor built in this way has 4 independent components (the four entries of the matrix), there must exist a relation between it and a 4-vector, which also has 4 independent components. Technically speaking, both must be different *realizations of the same irreducible representation of the Lorentz group* (Landau & Lifshitz 1971, page 55). The components (I, Q, U, V) of this 4–vector are indeed related to the components of J as

$$\begin{aligned} I &= \frac{1}{2}(\mathsf{J}_{11} + \mathsf{J}_{22}), \\ Q &= \frac{1}{2}(\mathsf{J}_{11} - \mathsf{J}_{22}), \\ U &= \frac{1}{2}(\mathsf{J}_{12} + \mathsf{J}_{21}), \\ V &= \frac{1}{2} i (\mathsf{J}_{12} - \mathsf{J}_{21}). \end{aligned} \qquad (7)$$

These components are identical to the definition of the Stokes parameters as given by Jefferies *et al.* (1989), or Born & Wolf (1980) for instance. The conclusion is evident: the resulting 4–vector, derived from the coherency matrix, a spinor of rank 2, is the Stokes vector. This formalism can be given in an alternative way: an usual basis for spinors is the set of Pauli matrices plus the 2×2 identity matrix. The coherency matrix expressed in this basis has for coefficients the Stokes parameters

$$\mathsf{J} = I\sigma_0 + Q\sigma_1 + U\sigma_2 + V\sigma_3 \qquad (8)$$

Using this relation we can write in a more compact form relations (7) as:

$$\boldsymbol{I} = \frac{1}{2} Tr \left[\mathsf{J}\sigma \right], \qquad (9)$$

where the vector σ has four components, the 2×2 identity matrix and the 3 Pauli matrices, and *Tr* denotes the trace operation on matrices.

These relations stress further the interpretation of the Stokes parameters as a 4–vector in a Minkowski-like space. We stress that when talking about a Minkowski–*like* space we mean that the coordinates in our 4–dimensional space are no longer space and time, but the Stokes parameters, contrary to the usual Minkowski space used in relativistic formalism. On the other hand the underlying *hyperbolic* geometry is exactly the same in both cases, mathematically speaking they are the same space. All the usual properties of the Stokes parameters are recovered in this space. For instance the sum of two 4–vectors is a new 4–vector, a well-known property of the Stokes parameters. Contrary to the usual Minkowsky space, the absence of the *backward light cone* implies not only that negative intensities are meaningless, but also that the negative of a Stokes vector does not exist. Hence subtraction of Stokes vectors is naturally forbidden in this space.

3. Radiation transfer equation as a Poincaré (plus dilatations) infinitesimal transformation

Since the Stokes vector is a 4–vector in a Minkowski-like space, one may wonder what would be the meaning of a Lorentz transformation over the Stokes parameters.

Homogeneous Lorentz transformations form a 6–parameter Lie group: 6 generators suffice to describe all possible infinitesimal transformations. These generators are (see for example Greiner 1990, or any textbook in special relativity or group theory)[1]:

– The three 4×4 matrices for 3-dimensional spatial rotations

$$\mathsf{S}_Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \mathsf{S}_U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathsf{S}_V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

– The three 4×4 matrices for hyperbolic rotations (or Lorentz boosts in relativistic terms)

$$\mathsf{K}_Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathsf{K}_U = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{K}_V = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

An infinitesimal Lorentz transformation over the Stokes 4–vector $\boldsymbol{I}$ can be expressed as a sum of generators multiplied by their respective infinitesimal parameters:

$$\boldsymbol{I}' = \boldsymbol{I} + \sum_{i=Q,U,V} \beta_i \mathsf{S}_i \boldsymbol{I} + \sum_{i=Q,U,V} \gamma_i \mathsf{K}_i \boldsymbol{I}. \qquad (10)$$

For convenience, we can re-write all these parameters(β_i, γ_i) in terms of a common parameter $d\tau$, expressed in differential form (we are dealing with an infinitesimal transformation)

$$\boldsymbol{\beta} = (\beta_Q, \beta_U, \beta_V) = -(\rho_Q, \rho_U, \rho_V) \cdot d\tau, \qquad (11)$$

$$\boldsymbol{\gamma} = (\gamma_Q, \gamma_U, \gamma_V) = -(\eta_Q, \eta_U, \eta_V) \cdot d\tau. \qquad (12)$$

[1] Everywhere in this paper we use $g = diag(1, -1, -1, -1)$ as the metric for the Minkowski space

And the infinitesimal transformation reads now:

$$\boldsymbol{I}' = \boldsymbol{I} - \left(\sum_{i=Q,U,V} \rho_i \mathsf{S}_i + \sum_{i=Q,U,V} \eta_i \mathsf{K}_i \right) \boldsymbol{I} d\tau. \tag{13}$$

Given the infinitesimal character of the transformation, expressed explicitly by putting the common parameter $d\tau$, it easily leads to a differential equation for $\boldsymbol{I}$ in the variable τ:

$$\frac{d}{d\tau}\boldsymbol{I} = -\mathsf{K}'\boldsymbol{I}, \tag{14}$$

where K' is a 4×4 matrix given by

$$\mathsf{K}' = \sum_{i=Q,U,V} \rho_i \mathsf{S}_i + \sum_{i=Q,U,V} \eta_i \mathsf{K}_i =$$

$$= \begin{pmatrix} 0 & \eta_Q & \eta_U & \eta_V \\ \eta_Q & 0 & \rho_V & -\rho_U \\ \eta_U & -\rho_V & 0 & \rho_Q \\ \eta_V & \rho_U & -\rho_Q & 0 \end{pmatrix}.$$

Matrix K' looks like the well-known absorption matrix (Landi Degl'Innocenti, 1992). At this point the possibility that the RTE could be written as an infinitesimal transformation involving the Lorentz group seems to be at hand. But K' still differs from the general form of the absorption matrix. In particular the inclusion of a diagonal term is necessary if we want to take into account the scalar absorption represented by η_I in the usual matrix. To include it we just add to the usual 6 generators of the Lorentz group the one for the dilatation transformation[2]. While dealing with the homogeneous group, we can represent the generator of dilatations by the 4×4 identity matrix. We will repeat all the previous steps calling the new parameter for this transformation η_I:

$$\boldsymbol{I}' = \boldsymbol{I} - \left(\eta_I \mathbb{1} + \sum_{i=Q,U,V} \rho_i \mathsf{S}_i + \sum_{i=Q,U,V} \eta_i \mathsf{K}_i \right) \boldsymbol{I} d\tau,$$

to finally obtain:

$$\frac{d}{d\tau}\boldsymbol{I} = -\mathsf{K}\boldsymbol{I}, \tag{15}$$

where K is given by

$$\mathsf{K} = \eta_I \mathbb{1} + \sum_{i=Q,U,V} \rho_i \mathsf{S}_i + \sum_{i=Q,U,V} \eta_i \mathsf{K}_i =$$

$$= \begin{pmatrix} \eta_I & \eta_Q & \eta_U & \eta_V \\ \eta_Q & \eta_I & \rho_V & -\rho_U \\ \eta_U & -\rho_V & \eta_I & \rho_Q \\ \eta_V & \rho_U & -\rho_Q & \eta_I \end{pmatrix}.$$

[2] The dilatation transformation does not belong to the usual Lorentz group. A usual definition of the transformations belonging to this group is that they do not change the Lorentz norm of any 4–vector. By definition, a dilatation transformation does change this norm. Fortunately the new set of 7 generators is still a group.

Matrix K can now be compared to the so-called absorption matrix which appears in RTE (Landi Degl'Innocenti, 1992; Jefferies et al., 1989). Coefficient η_I gives the scalar absorption, independent of polarization. This absorption has its equivalent in a contraction (that is a negative dilatation) of the Stokes 4–vector. The 3–vector $\boldsymbol{\eta}$ is responsible for the creation and absorption of polarization, which is understood here to be a hyperbolic rotation (or Lorentz boost) of the Stokes 4-vector. Finally the 3–vector $\boldsymbol{\rho}$ gives the so-called Faraday rotation in Zeeman effect and, as its name seems to indicate, it rotates the Stokes 4–vector inside the 3-dimensional space of polarized states (also called *Poincaré sphere*). Note that from the previous paragraph it cannot be stated that any infinitesimal Lorentz (plus dilatations) transformation is a transfer equation. We state that the absorption matrix as it is usually written elsewhere in the literature, cannot be differentiated from an infinitesimal Lorentz transformation (indeed we have decomposed the absorption matrix in the RTE in terms of the infinitesimal generators of the Lorentz transformations). In fact, it is well known that the actual absorption matrix for Zeeman effect must still obey some further constraints. In this sense a general Lorentz (plus dilatations) transformation is too general: absorption matrices standing for a real physical process would constitute only a subset of all possible Lorentz (plus dilatations) transformations. While it is evident that a further study of this relation is necessary, in this paper we will only make use of the *mathematical advantage* and postpone the rest for a forthcoming paper.

Eq. (15) is not yet the complete transfer equation. An inhomogeneous term, the emission vector, is still needed. For this purpose there are appropriate movements in the Minkowski space: the homogeneous Lorentz group can be extended to the inhomogeneous Poincaré group. This 10-parameters Lie group shares 6 infinitesimal generators with the Lorentz group and adds 4 more generators ($\mathsf{P}_I, \mathsf{P}_Q, \mathsf{P}_U$ and P_V) to take into account translations along I, Q, U and V (what in relativistic formalism would be translations in time and space). These generators operate on a generic Stokes vector as follows:

$$\mathsf{P}_I\boldsymbol{I} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathsf{P}_Q\boldsymbol{I} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathsf{P}_U\boldsymbol{I} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathsf{P}_V\boldsymbol{I} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{16}$$

After including dilatations, the infinitesimal inhomogeneous transformation is given by

$$\boldsymbol{I}' = \boldsymbol{I} - \mathsf{K}\boldsymbol{I} d\tau + d\tau \sum_{i=I,Q,U,V} j_i \mathsf{P}_i \boldsymbol{I} = \boldsymbol{I} - \mathsf{K}\boldsymbol{I} d\tau + \boldsymbol{J} d\tau \tag{17}$$

where $\boldsymbol{J}$ is the emission vector. Some algebraic manipulations equivalent to those used for the homogeneous group will lead to the complete radiative transfer equation.

It is interesting to note the fact that these generators allow one to write an inhomogeneous term (the source function) in a pseudo-homogeneous way:

$$\boldsymbol{I}' = \boldsymbol{I} + \left(-\mathsf{K} + \sum_{i=I,Q,U,V} j_i \mathsf{P}_i \right) \boldsymbol{I} d\tau.$$

A useful representation to understand this apparent paradox is the one using differential operators. Let us call $\boldsymbol{I}_I = I, \boldsymbol{I}_Q = Q$ and so on, and write

$$\mathsf{P}_i = \frac{\partial}{\partial \boldsymbol{I}_i}$$

where $i = I, Q, U, V$. It is evident that with such an operator, relations (16) hold. All the other generators can be rewritten in this representation. For instance, the dilatation generator can be written

$$\mathsf{D} = \sum_{i=I,Q,U,V} \boldsymbol{I}_i \frac{\partial}{\partial \boldsymbol{I}_i}$$

and the generator of rotations in the plane UV, for instance, is

$$\mathsf{S}_Q = -\boldsymbol{I}_U \frac{\partial}{\partial \boldsymbol{I}_V} + \boldsymbol{I}_V \frac{\partial}{\partial \boldsymbol{I}_U},$$

The interested reader will find good discussions on the representations of the Poincaré group and the dilatation transformation in Greiner (1990), Gourdin (1982) or Jones (1996) for example.

4. Finite transformations as a solution

In view of the results obtained in the last section, the solution to the RTE appears to be quite straightforwardly a finite Poincaré (plus dilatations) transformation. The important fact now is that we already know how to write such a finite transformation: if we denote the 11 generators by t_i, a finite element of this group can always be written as

$$\exp\left(\sum_i \xi_i \mathsf{t}_i\right),$$

where ξ_i are the parameters of the transformation for each movement, the equivalent of angles for usual rotations. The next problem is how to calculate those finite parameters ξ_i from their infinitesimal counterparts (η_I, η_Q and so on). Unfortunately this is not an easy task. The problem resides in the non–commutativity of the generators. The next paragraph proposes a solution to this problem, but first we consider it useful to clarify why other approaches will not work. Magnus (1954) (a brief introduction to this paper can be found in the Appendix of the paper by Semel & López Ariste, 1999) has already written a finite transformation in terms of such a unique exponential. A quick inspection of the expression given there illustrates why we consider that this calculation is not an easy task. Nevertheless some simple cases can be proposed in which the relation between the ξ_i and the respective infinitesimal parameters is plain. For instance in the case of 3–dimensional rotations it is always possible to transform (by means of the Euler angles) our initial reference system into another one for which the rotation axis is parallel to one of the new reference axes. In the case of a fixed axis, just one generator suffices to describe the movement. If we call this generator t, and the infinitesimal parameter $d\xi$, the finite transformation results in

$$\exp\left(\xi \mathsf{t}\right) = \exp\left(\mathsf{t} \int_C d\xi\right).$$

Apparently, our problem is solved if we deal with a unique generator per exponential. Hence, we can propose for instance a solution in the form of a product of several exponentials, one for each generator:

$$\prod_{i=1,11} \exp\left(\xi_i \mathsf{t}_i\right).$$

Now the derivative of each exponential in the product can be easily calculated in a very compact form, and so the product of all of them, and consequently the new finite parameters ξ_i (as in the above example). But a new question arises: which order should be chosen for the exponentials? Again due to the non-commutativity of the infinitesimal generators, different orders produce different results,

$$\mathrm{e}^{\xi_i \mathsf{t}_i} \mathrm{e}^{\xi_j \mathsf{t}_j} \neq \mathrm{e}^{\xi_j \mathsf{t}_j} \mathrm{e}^{\xi_i \mathsf{t}_i}, \tag{18}$$

and in general no particular order will be the solution. An answer to this kind of problem has been given by Wei & Norman (1963). In what follows, we sketch the solution there proposed and apply it to our particular problem. We start with the homogeneous equation (just the Lorentz group plus dilatations) and in the next section we will incorporate the inhomogeneous part and handle the full Poincaré group.

The ordering problem can be recast as follows: we may say that by introducing an order in the exponentials we introduce an error. In spite of this error, let us choose a particular order for the exponentials and then substitute the ξ_i (which could be calculated straightforwardly as the integrals over the path of the corresponding infinitesimal parameters) for some unspecified scalar functions g_i:

$$\prod_{i=1,7} \exp\left(g_i \mathsf{t}_i\right). \tag{19}$$

The new functions g_i have to, in a certain sense, take into account the effect of the ξ_i and correct the error introduced by the chosen order. Existence for those g_i functions can only be ensured after introducing the proposed solution into the RTE. The following consistency equation for the g_i's is obtained as a necessary condition for Eq. (19) to be a solution:

$$\begin{aligned} -\mathsf{K}(\tau) \ = \ & \sum_{i=1,7} \dot{g}_i(\tau) \left[\prod_{j=1}^{i-1} \exp(g_j \mathsf{t}_j) \right] \\ & \times \mathsf{t}_i \left[\prod_{j=i-1}^{1} \exp(-g_j \mathsf{t}_j) \right], \end{aligned} \tag{20}$$

where the dot denotes derivative over the integration variable τ. This equation was the aim of this section so far. A less heuristic but more direct way of introducing it is to look for a solution of the form (19) and introduce it into the transfer equation. It is straightforward to fall upon Eq. (20) as the condition for (19) to be a solution.

In what follows in this and the next sections we shall solve Eq. (20) for the g_i's. The details being quite technical, the reader may wish to skip this and go directly to Eq. (28).

Eq. (20) is quite an involved equation. It requires calculation of a non negligible number of expressions of the form

$$\exp(g_j \mathsf{t}_j)\mathsf{t}_i \exp(-g_j \mathsf{t}_j).$$

This is to be done by means of the Baker-Hausdorff formula, which states that

$$\begin{aligned} \mathrm{e}^X Y \mathrm{e}^{-X} &= Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] \\ &+ \frac{1}{3!}[X,[X,[X,Y]]] + \ldots \end{aligned}$$

In the form they have been written in the last section, the generators of the Lorentz group obey the following Lie algebra (with ε_{ijk} the totally antisymmetric index):

$$\begin{aligned} [\mathsf{K}_i, \mathsf{K}_j] &= -\varepsilon_{ijk}\mathsf{S}_k \\ [\mathsf{S}_i, \mathsf{S}_j] &= \varepsilon_{ijk}\mathsf{S}_k \\ [\mathsf{S}_i, \mathsf{K}_j] &= \varepsilon_{ijk}\mathsf{K}_k \\ [\mathsf{S}_i, \mathbb{1}] &= 0 \\ [\mathsf{K}_i, \mathbb{1}] &= 0, \end{aligned}$$

which does not facilitate calculations at all. A suitable combination of generators will yield a new base of generators with a gentler (from our point of view) Lie algebra. For instance, the following set,

$$\begin{aligned} \mathsf{H}_1 &= \mathsf{S}_Q + i\mathsf{K}_Q, \\ \mathsf{H}_2 &= (\mathsf{K}_U - \mathsf{S}_V) - i(\mathsf{K}_V + \mathsf{S}_U), \\ \mathsf{H}_3 &= (\mathsf{K}_U + \mathsf{S}_V) + i(\mathsf{K}_V - \mathsf{S}_U), \\ \mathsf{L}_i &= \mathsf{H}_i^* \mathrm{for} i = 1,2,3, \end{aligned}$$

to which we add the identity, $\mathbb{1}$, for completion, obeys the following commutation rules

$$\begin{aligned} [\mathsf{H}_1, \mathsf{H}_2] &= 2i\mathsf{H}_2, \\ [\mathsf{H}_1, \mathsf{H}_3] &= -2i\mathsf{H}_3, \\ [\mathsf{H}_2, \mathsf{H}_3] &= -4i\mathsf{H}_1, \\ [\mathsf{L}_1, \mathsf{L}_2] &= -2i\mathsf{L}_2, \\ [\mathsf{L}_1, \mathsf{L}_3] &= 2i\mathsf{L}_3, \\ [\mathsf{L}_2, \mathsf{L}_3] &= 4i\mathsf{L}_1, \\ [\mathsf{H}_i, \mathsf{L}_j] &= 0, \forall i,j. \end{aligned}$$

The initial algebra of 6 generators has been decomposed into two sub-algebras of 3 generators each, with the particularity that each generator of one sub-algebra commutes with every generator of the other one. The dilatations generator, which must be added to them, commutes with every other generator (remember that, while constrained to the homogeneous group, the dilatation generator can be represented by the identity matrix) and therefore there is no ordering problem associated with it. The initial problem of ordering 7 exponentials is reduced to ordering a subset of 3 exponentials; the order of each one of the H's with respect to the L's generators or the identity being immaterial.

We can rewrite solution (19) using the new set of infinitesimal generators in explicit form as

$$\begin{aligned} \boldsymbol{I}(\tau) &= \mathrm{e}^{g_3(\tau)\mathsf{H}_3}\mathrm{e}^{g_2(\tau)\mathsf{H}_2}\mathrm{e}^{g_1(\tau)\mathsf{H}_1}\mathrm{e}^{g_6(\tau)\mathsf{L}_3}\mathrm{e}^{g_5(\tau)\mathsf{L}_2} \\ &\times \mathrm{e}^{g_4(\tau)\mathsf{L}_1}\mathrm{e}^{g_7(\tau)\mathbb{1}}\boldsymbol{I}(\tau_0), \end{aligned} \tag{21}$$

where a very special order has already been chosen. Different orders, while yielding equivalent solutions, can make calculations affordable or desperate. With this problem in mind we have chosen a particular order.

We now express K in the new basis:

$$\mathsf{K} = \sum_i a_i \mathsf{H}_i + \sum_i b_i \mathsf{L}_i + \eta_I \mathbb{1}$$

where

$$\begin{aligned} a_1 &= -\frac{1}{2}(\rho_Q + i\eta_Q) \\ a_2 &= \frac{1}{4}[(\eta_U + \rho_V) + i(\eta_V - \rho_U)] \\ a_3 &= \frac{1}{4}[(\eta_U - \rho_V) - i(\eta_V + \rho_U)] \\ b_i &= a_i^*, \forall i = 1,2,3. \end{aligned}$$

Calculation of the Baker–Hausdorff series for the new generators simplifies a lot, and Eq. (20) reads now:

$$\begin{aligned} &-\sum_i a_i \mathsf{H}_i - \sum_i b_i \mathsf{L}_i - \eta_I \mathbb{1} \qquad (22) \\ &= \dot{g}_7 \mathbb{1} + \dot{g}_3 \mathsf{H}_3 + \dot{g}_2 \left[\mathsf{H}_2 + 4ig_3\mathsf{H}_1 - 4g_3^2\mathsf{H}_3\right] \\ &+ \dot{g}_1 \left[-2ig_2\mathsf{H}_2 + (1 + 8g_2g_3)\mathsf{H}_1 + \left(2ig_3 + 8ig_3^2g_2\right)\mathsf{H}_3\right] \\ &+ \dot{g}_6 \mathsf{L}_3 + \dot{g}_5 \left[\mathsf{L}_2 - 4ig_6\mathsf{L}_1 - 4g_6^2\mathsf{L}_3\right] \\ &+ \dot{g}_4 \left[2ig_5\mathsf{L}_2 + (1 + 8g_5g_6)\mathsf{L}_1 - \left(2ig_6 + 8ig_6^2g_5\right)\mathsf{L}_3\right]. \end{aligned}$$

Comparing coefficients on both sides, and after proper rearrangement the following set of differential equations is obtained:

$$\begin{aligned} \dot{g}_1 &= -a_1 + 4ig_3a_2 \\ \dot{g}_2 &= -2ig_2a_1 - (1 + 8g_2g_3)a_2 \\ \dot{g}_3 &= 2ig_3a_1 + 4g_3^2a_2 - a_3 \\ \dot{g}_4 &= -b_1 - 4ig_6b_2 \\ \dot{g}_5 &= 2ig_5b_1 - (1 + 8g_5g_6)b_2 \\ \dot{g}_6 &= -2ig_6b_1 + 4g_6^2b_2 - b_3 \\ \dot{g}_7 &= -\eta_I \end{aligned}$$

All of them share the same boundary condition, namely $g_i(\tau_0) = 0$, to satisfy the boundary condition of the RTE.

The function g_7 can be integrated at once to give

$$g_7(\tau) = -\int_{\tau_0}^{\tau} \eta_I(\tau')d\tau'.$$

As expected, because of the particularities of the Lie algebra, the set of equations for $g_{1,2,3}$ is separated from the one for $g_{4,5,6}$, and each set is the complex conjugated of the other, so that the solution to $g_{4,5,6}$ is straightforward once the one for $g_{1,2,3}$ is given. Furthermore each set can be solved by quadrature, as equations for g_1 and g_2 depend only on g_3, whose equation is disentangled from the others:

$$\dot{g}_3 = \alpha g_3^2 + \beta g_3 + \gamma, \tag{23}$$

where we have defined $\alpha = 4a_2$, $\beta = 2ia_1$ and $\gamma = -a_3$. This is a Riccati equation, and for its solution the explicit dependences of $a_{1,2,3}$ on the integration variable τ are required. For a constant K matrix the solution is straightforward, and from it those of g_2 and g_1. More complex dependences must be carefully managed (see for example Cariñena & Ramos (1998) and references therein for the integrability conditions of the Riccati equation).

Before passing to the next section, where we will generalize the method to the full Poincaré group, we go back to Eq. (21). Once we have integrated the Riccati equation and obtained all the g_i's, we still need to calculate the exponentials. To this end we profit from a remarkable property of matrices H_i and L_i:

$$\mathsf{H}_1^2 = \mathsf{L}_1^2 = -\mathbb{1}$$
$$\mathsf{H}_2^2 = \mathsf{H}_3^2 = \mathsf{L}_2^2 = \mathsf{L}_3^2 = 0,$$

by means of which:

$$\begin{aligned}
\mathrm{e}^{g_1 \mathsf{H}_1} &= \cos g_1 \mathbb{1} + \sin g_1 \mathsf{H}_1 \\
\mathrm{e}^{g_2 \mathsf{H}_2} &= \mathbb{1} + g_2 \mathsf{H}_2 \\
\mathrm{e}^{g_3 \mathsf{H}_3} &= \mathbb{1} + g_3 \mathsf{H}_3 \\
\mathrm{e}^{g_4 \mathsf{L}_1} &= \cos g_4 \mathbb{1} + \sin g_4 \mathsf{L}_1 \\
\mathrm{e}^{g_5 \mathsf{L}_2} &= \mathbb{1} + g_5 \mathsf{L}_2 \\
\mathrm{e}^{g_6 \mathsf{L}_3} &= \mathbb{1} + g_6 \mathsf{L}_3.
\end{aligned}$$

The final complete solution for the homogeneous part results in

$$\begin{aligned}
\boldsymbol{I}(\tau) = &\ [\mathbb{1} + g_3(\tau)\mathsf{H}_3] \cdot [\mathbb{1} + g_2(\tau)\mathsf{H}_2] \\
&[\cos g_1(\tau)\mathbb{1} + \sin g_1(\tau)\mathsf{H}_1] \\
&\cdot [\mathbb{1} + g_6(\tau)\mathsf{L}_3] \cdot [\mathbb{1} + g_5(\tau)\mathsf{L}_2] \cdot \\
&[\cos g_4(\tau)\mathbb{1} + \sin g_4(\tau)\mathsf{L}_1] \cdot \exp[g_7(\tau)\mathbb{1}]\boldsymbol{I}(\tau_0).
\end{aligned} \tag{24}$$

The validity of this solution is almost evident: Its derivative results in the RTE just by making use of the differential equations satisfied by the functions g_i.

This is a solution to the homogeneous equation

$$\frac{d}{d\tau}\boldsymbol{I}_L = -\mathsf{K}\boldsymbol{I}_L.$$

Let us write this solution as

$$\boldsymbol{I}_L(\tau) = \mathsf{O}(\tau, \tau_0)\boldsymbol{I}_L(\tau_0),$$

where the explicit form of $\mathsf{O}(\tau, \tau_0)$ can be found by comparing this expression with the complete one in Eq. (24). This operator $\mathsf{O}(\tau, \tau_0)$ is often referred to as the *evolution operator* (see mainly Landi Degl'Innocenti & Landi Degl'Innocenti 1985, who first introduced it). This operator trivially obeys the homogeneous equation

$$\frac{d}{d\tau}\mathsf{O}(\tau, \tau_0) = -\mathsf{K}\mathsf{O}(\tau, \tau_0), \tag{25}$$

with initial condition

$$\mathsf{O}(\tau_0, \tau_0) = \mathbb{1}.$$

Eq. (24) provides on its own a general analytical solution for the evolution operator.

This solution is a fully general expression for finite Lorentz transformations plus dilatations. But radiative transfer cases do not cover the full spectrum of Lorentz transformations. In this sense the obtained solution is too general, in agreement with Sect. 3. As an illustration, consider the case when the ρ's and η's are zero except for η_Q. The function g_1 becomes

$$g_1 = -\frac{i}{2}\int \eta_Q d\tau,$$

and a term of the form

$$\cos g_1 = \cosh \frac{1}{2}\int \eta_Q d\tau$$

appears in the final solution. The hyperbolic cosine grows monotonously with its argument, therefore the intensity of the out-coming light would grow also monotonously for a semi-infinite atmosphere. This is a completely nonsensical result. To recover physical sense one must impose some constraint on the allowed transformations. This constraint evidently imposes a relation between η_Q and η_I, whose explicit form is outside the scope of this paper, but which should be derived from the assumed physical processes. This example can be extrapolated to all the η's and ρ's. The relations thus obtained will constrain the Lorentz transformations to a subset of matrices for which, nevertheless, the above solution (24) will remain valid.

5. Solution for the complete inhomogeneous equation

To solve the inhomogeneous equation, one would need to repeat the calculations shown in the previous section, but this time for the whole Poincaré group. To recalculate everything with 4 more generators involves a lot of work. The paper by Wei & Norman (1963) provides us with a way to avoid some of this work. The Poincaré group can be decomposed into the direct sum of a semi-simple algebra L and a radical R (whose definitions can be found in that same paper for instance). In terms of the previously used generators of the Poincaré group, the semi-simple algebra is given by

$$L = \{\mathsf{H}_1, \mathsf{H}_2, \mathsf{H}_3, \mathsf{L}_1, \mathsf{L}_2, \mathsf{L}_3\},$$

the generators of the homogeneous part. The radical is given by

$$R = \{\mathbb{1}, \mathsf{P}_I, \mathsf{P}_Q, \mathsf{P}_U, \mathsf{P}_V\},$$

the inhomogeneous part plus the identity. We will include the dilatation transformation in the semi-simple algebra set for easiness. If we write the transfer equation as

$$\frac{d}{d\tau}\boldsymbol{I} = \mathsf{H}\boldsymbol{I},$$

then, H, an element of the Poincaré group plus dilatations, can be decomposed into

$$\mathsf{H} = -\mathsf{K} + \mathsf{P},$$

where K is the usual absorption matrix, an element of the homogeneous group, and P, which stands for the set of four translations introduced in Sect. 3, is an element of both the radical and the inhomogeneous part of the equation. In the last section we dealt with the homogeneous equation and found a solution for the evolution operator $\mathsf{O}(\tau, \tau_0)$ by using the Lorentz group plus dilatations. Now, it is easy to demonstrate that if we are able to solve the equation

$$\frac{d}{dt}\boldsymbol{I}_R = \left(\mathsf{O}^{-1}(t,\tau_0)\mathsf{P}(t)\mathsf{O}(t,\tau_0)\right)\boldsymbol{I}_R,$$

a solution for the complete transfer equation can be written in the form

$$\boldsymbol{I}(\tau) = \mathsf{O}(\tau,\tau_0)\boldsymbol{I}_R(\tau). \tag{26}$$

In fact, this result is exactly equivalent to the formal solution given by Landi Degl'Innocenti & Landi Degl'Innocenti(1985). To prove it, we note that $\mathsf{O}(t,\tau_0)\boldsymbol{I}_R$ will give, by properties of the evolution operator, a new $\boldsymbol{I}_R(t)$. The effect of P is however independent of the actual value of $\boldsymbol{I}_R(t)$, we will always obtain that

$$\mathsf{P}\boldsymbol{I}_R(t) = \begin{pmatrix} j_I \\ j_Q \\ j_U \\ j_V \end{pmatrix} = \boldsymbol{J},$$

where the j_i's are the infinitesimal parameters of the translation transformation: the emission vector in our particular case. The previous equation results therefore in

$$\frac{d}{dt}\boldsymbol{I}_R = \mathsf{O}^{-1}(t,\tau_0)\boldsymbol{J}(t),$$

which can be integrated at once:

$$\boldsymbol{I}_R(\tau) = \boldsymbol{I}_R(\tau_0) + \int_{\tau_0}^{\tau} \mathsf{O}^{-1}(t,\tau_0)\boldsymbol{J}(t)dt.$$

Combining it with the homogeneous solution, we obtain the final complete solution:

$$\boldsymbol{I}(\tau) = \mathsf{O}(\tau,\tau_0)\left(\boldsymbol{I}_R(\tau_0) + \int_{\tau_0}^{\tau} \mathsf{O}^{-1}(t,\tau_0)\boldsymbol{J}(t)dt\right).$$

And benefiting from the well known properties of the evolution operator, we can transform this expression into the formal solution given in the above referred paper:

$$\boldsymbol{I}(\tau) = \mathsf{O}(\tau,\tau_0)\boldsymbol{I}_R(\tau_0) + \int_{\tau_0}^{\tau} \mathsf{O}(\tau,t)\boldsymbol{J}(t)dt.$$

Hence, once the evolution operator is solved as shown in the previous section we can use this expression to obtain the complete solution. Instead of doing that, we shall proceed with the techniques provided by the group theory and obtain a completely equivalent but independent expression for $\boldsymbol{I}_R$.

P belongs to the radical which, by definition, is an ideal of the Poincaré group, so that in fact the term $\mathsf{O}^{-1}\mathsf{P}\mathsf{O}$ is just a linear combination of the infinitesimal generators of R:

$$\left(\mathsf{O}^{-1}\mathsf{P}\mathsf{O}\right) = \eta_I \mathbb{1} + D_0\mathsf{P}_I + D_1\mathsf{P}_Q + D_2\mathsf{P}_U + D_3\mathsf{P}_V.$$

Obtaining the coefficients D_i (with $i = 0, 1, 2, 3$) is quite long, the detailed calculation is to be found in the Appendix. This calculation constitutes by itself a demonstration of the first statement of this paragraph for our particular case, a long one, but which does not require further knowledge in group theory. The next step is to solve the equation

$$\frac{d}{d\tau}\boldsymbol{I}_R = \left(D_0\mathsf{P}_I + D_1\mathsf{P}_Q + D_2\mathsf{P}_U + D_3\mathsf{P}_V\right)\boldsymbol{I}_R.$$

This is in fact a very easy equation, as every P_i commutes with each other. The solution can be given at once as

$$\boldsymbol{I}_R(\tau) = \mathrm{e}^{\mathsf{P}_I \int_{\tau_0}^{\tau} D_0(t)dt} \cdot \mathrm{e}^{\mathsf{P}_Q \int_{\tau_0}^{\tau} D_1(t)dt} \cdot$$

$$\mathrm{e}^{\mathsf{P}_U \int_{\tau_0}^{\tau} D_2(t)dt} \cdot \mathrm{e}^{\mathsf{P}_V \int_{\tau_0}^{\tau} D_3(t)dt}\boldsymbol{I}_R(\tau_0).$$

Calculation of the exponentials is straightforward: The P_i are the infinitesimal generators of translations in the four axes I, Q, U, V, hence by exponentiation we recuperate the finite transformation:

$$\begin{aligned}\boldsymbol{I}_R(\tau) &= \boldsymbol{I}_R(\tau_0) + \begin{pmatrix} \int_{\tau_0}^{\tau} D_0(t)dt \\ \int_{\tau_0}^{\tau} D_1(t)dt \\ \int_{\tau_0}^{\tau} D_2(t)dt \\ \int_{\tau_0}^{\tau} D_3(t)dt \end{pmatrix} \\ &= \boldsymbol{I}_R(\tau_0) + \boldsymbol{D}(\tau,\tau_0).\end{aligned} \tag{27}$$

The last step is to put together the homogeneous and inhomogeneous solutions by using expression (26). We obtain

$$\begin{aligned}\boldsymbol{I}(\tau) &= \mathrm{e}^{-\int_{\tau_0}^{\tau}\eta_I(t)dt}\left[\mathbb{1} + g_3(\tau)\mathsf{H}_3\right]\cdot\left[\mathbb{1} + g_2(\tau)\mathsf{H}_2\right] \\ &\cdot\left[\cos g_1(\tau)\mathbb{1} + \sin g_1(\tau)\mathsf{H}_1\right]\cdot\left[\mathbb{1} + g_6(\tau)\mathsf{L}_3\right] \\ &\cdot\left[\mathbb{1} + g_5(\tau)\mathsf{L}_2\right]\cdot\left[\cos g_4(\tau)\mathbb{1} + \sin g_4(\tau)\mathsf{L}_1\right] \\ &\cdot\left(\boldsymbol{I}(\tau_0) + \boldsymbol{D}(\tau,\tau_0)\right).\end{aligned} \tag{28}$$

Note that this solution is general for all the radiative transfer problems known to date in polarization, provided the source function is given. As discussed in the previous section, it is even too general.

This solution is also independent of any model atmosphere. This is necessary to ensure its generality, but presents the problem of the integrability: have the integrals for the g_i's and the inhomogeneous vector $\boldsymbol{D}$ an analytical expression for all and every interesting case? The most likely answer is no. But whatever the answer, physical intuition indicates that there must always exist at least a numerical solution to them. However further work must be developed on the subject.

It is also important to note that we are proposing not just an expression as solution of the RTE, but a method: particular cases may ask for different orderings of the generators or even a different decomposition of the product of exponentials. We have seen that we can in any case give a solution in the form of seven exponentials, but, for instance, when solving constant matrix atmospheres it may be more interesting to consider only the product of 3 exponentials, one with the H_i generators a second one for the L_i's, and a last for the dilatations, or even a sole one, in which case the solution for the evolution operator can be written at once:

$$\mathsf{O}(\tau,\tau_0) = \exp -\mathsf{K}(\tau-\tau_0),$$

in accordance with Magnus' solution (Magnus, 1954) or with the scalar–like exponential solution (Semel & López Ariste, 1999). For any number of exponentials, the method will work, the sole problem being to solve the subsequent scalar linear equations and integrals. In all the cases, a *compact and finite* expression for the solution is obtained and the problem is reduced to the ability to integrate scalar expressions.

6. Discussion and conclusion

In this paper we have introduced a new formalism to handle Stokes parameters and radiative transfer equations for polarized light. In this formalism, the Stokes parameters appear as a 4–vector in a Minkowski–like 4–dimensional space, and its evolution in time looks mathematically as typical rotations, contractions and translations in this space. These movements are completely described by the transformations of the group of Poincaré plus dilatations, a $10+1$ dimension group, well-known from other areas of physics and mathematics. The RTE is shown to be an infinitesimal transformation of this group. We therefore propose that a solution to the RTE can be given in the form of a finite transformation of the Poincaré plus dilatations group. Obtaining of this solution from the variables present in the transfer equation raises some technical difficulties which have been overcome by the use of the Wei-Norman method (Wei & Norman, 1963). The final obstacle is reduced to a scalar Riccati equation.

The Riccati equation is a well studied first order differential equation, characterized by its quadratic term. This non-linearity can at worst prevent an explicit solution, and usually make it difficult to calculate. In any case the problem of giving a solution for the RTE will have been reduced from solving a 4–dimensional vector equation to solving a scalar Riccati one. Whenever this Riccati equation can be integrated, a complete solution is obtained for the RTE.

Until now only numerical integration methods (see for instance Rees et al., 1989, Bellot Rubio et al., 1998 or López Ariste & Semel, 1999) were capable of integrating non-constant K matrices. The only way to test the validity of the solution and the convergence rates was to compare them with previous methods, known to converge asymptotically. The solution presented in this paper may allow a comparison with an analytically exact solution. We anticipate that new numerical methods will be developed taking advantage of the analytical solution; perhaps faster and more precise than previous ones.

In order to obtain this solution we made use of a mathematical frame, group theory, rarely seen in the astrophysical literature. The advantages gained in the integration of the polarized RTE warranted the efforts. We anticipate that new results in the study of polarized light transfer in astrophysical problems will be achieved by the use of this and related techniques.

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Appendix A: solution for the radical

As explained in Sect. 5, we need to calculate, in order to obtain the final solution, a term of the form

$$\left(\mathsf{O}^{-1}\mathsf{P}\mathsf{O}\right), \tag{A1}$$

where O is the evolution operator, solution of the homogeneous equation, i.e.,

$$\mathsf{O}(\tau,\tau_0) \;=\; \mathrm{e}^{g_3(\tau)\mathsf{H}_3}\mathrm{e}^{g_2(\tau)\mathsf{H}_2}\mathrm{e}^{g_1(\tau)\mathsf{H}_1}\mathrm{e}^{g_6(\tau)\mathsf{L}_3}\mathrm{e}^{g_5(\tau)\mathsf{L}_2} \times\mathrm{e}^{g_4(\tau)\mathsf{L}_1}\mathrm{e}^{g_7(\tau)\mathbb{1}},$$

and where P is the inhomogeneous part of the transfer equation which can be written as

$$\mathsf{P} = j_I(\tau)\mathsf{P}_I + j_Q(\tau)\mathsf{P}_Q + j_U(\tau)\mathsf{P}_U + j_V(\tau)\mathsf{P}_V,$$

where the j_i are the components of the emission vector. In a further effort to simplify the calculations, instead of this linear combination we will use

$$\mathsf{P} = j_I\mathsf{P}_I + j_Q\mathsf{P}_Q + j_A\mathsf{P}_A + j_B\mathsf{P}_B,$$

where $\mathsf{P}_A = \mathsf{P}_U + i\mathsf{P}_V$, and P_B is its complex conjugate. Consequently, j_A and j_B are given by

$$j_A = \frac{1}{2}(j_U - ij_V)$$
$$j_B = \frac{1}{2}(j_U + ij_V).$$

Working out expression (A1) implies the use of commutators of H_i, L_i and the dilatations with P_i. Those commutators, which can be found in any textbook on group theory, are, for P_I

$$[\mathsf{P}_I,\mathsf{H}_1] \;=\; -i\mathsf{P}_Q$$
$$[\mathsf{P}_I,\mathsf{H}_2] \;=\; -\mathsf{P}_B$$
$$[\mathsf{P}_I,\mathsf{H}_3] \;=\; -\mathsf{P}_A.$$

For P_Q we have

$$[\mathsf{P}_Q,\mathsf{H}_1] \;=\; -i\mathsf{P}_I$$
$$[\mathsf{P}_Q,\mathsf{H}_2] \;=\; \mathsf{P}_A$$
$$[\mathsf{P}_Q,\mathsf{H}_3] \;=\; -\mathsf{P}_A.$$

For P_A

$$[\mathsf{P}_A,\mathsf{H}_1] \;=\; i\mathsf{P}_A$$
$$[\mathsf{P}_A,\mathsf{H}_2] \;=\; -2\mathsf{P}_I$$
$$[\mathsf{P}_A,\mathsf{H}_3] \;=\; 0.$$

And for P_B

$$[\mathsf{P}_B, \mathsf{H}_1] = -i\mathsf{P}_B$$
$$[\mathsf{P}_B, \mathsf{H}_2] = -2\mathsf{P}_Q$$
$$[\mathsf{P}_B, \mathsf{H}_3] = -2(\mathsf{P}_I - \mathsf{P}_Q).$$

Commutators for the L_i can be obtained as the complex conjugated of the corresponding ones for the H_i. Finally, for the dilatation operator D (expressed in matrix representation by the identity matrix), we have:

$$[\mathsf{P}_i, \mathsf{D}] = \mathsf{P}_i, \tag{A2}$$

with $i = I, Q, A, B$.

Once we have all the rules of the game we can begin to play with expression (A1) and calculate its first term:

$$R_1 = \mathsf{e}^{-g_3\mathsf{H}_3}\mathsf{P}\mathsf{e}^{g_3\mathsf{H}_3}$$

(in what follows and for the sake of clarity we leave out the dependences on τ of g_i and j_i to recuperate them in the final expressions). To this end we will need to calculate and add afterwards all the terms of the form

$$\mathsf{e}^{-g_3\mathsf{H}_3}\mathsf{P}_i\mathsf{e}^{g_3\mathsf{H}_3}.$$

Each one of which is to be calculated using an equivalent of the Baker–Hausdorff formula, which, for example for P_I, affirms that

$$\mathsf{e}^{-g_3\mathsf{H}_3}\mathsf{P}_I\mathsf{e}^{g_3\mathsf{H}_3} = \mathsf{P}_I + g_3[\mathsf{P}_I, \mathsf{H}_3] + \frac{1}{2!}g_3^2[[\mathsf{P}_I, \mathsf{H}_3], \mathsf{H}_3] + \ldots$$

The result of these calculations is

$$\mathsf{e}^{-g_3\mathsf{H}_3}\mathsf{P}_I\mathsf{e}^{g_3\mathsf{H}_3} = \mathsf{P}_I - g_3\mathsf{P}_A,$$
$$\mathsf{e}^{-g_3\mathsf{H}_3}\mathsf{P}_Q\mathsf{e}^{g_3\mathsf{H}_3} = \mathsf{P}_Q - g_3\mathsf{P}_A,$$
$$\mathsf{e}^{-g_3\mathsf{H}_3}\mathsf{P}_A\mathsf{e}^{g_3\mathsf{H}_3} = \mathsf{P}_A,$$
$$\mathsf{e}^{-g_3\mathsf{H}_3}\mathsf{P}_B\mathsf{e}^{g_3\mathsf{H}_3} = \mathsf{P}_B + 2g_3(\mathsf{P}_I - \mathsf{P}_Q).$$

So one obtains

$$\begin{aligned} R_1 &= (j_I + 2g_3j_B)\mathsf{P}_I + (j_Q - 2g_3j_B)\mathsf{P}_Q + \\ &\quad (-j_Ig_3 - j_Qg_3 + j_A)\mathsf{P}_A + j_B\mathsf{P}_B = \\ &= c_{10}\mathsf{P}_I + c_{11}\mathsf{P}_Q + c_{12}\mathsf{P}_A + c_{13}\mathsf{P}_B. \end{aligned} \tag{A3}$$

The meaning of the coefficients c_{1i} is self-evident. Next term is

$$R_2 = \mathsf{e}^{-g_2\mathsf{H}_2}R_1\mathsf{e}^{g_2\mathsf{H}_2}.$$

Partial results involved are

$$\mathsf{e}^{-g_2\mathsf{H}_2}\mathsf{P}_I\mathsf{e}^{g_2\mathsf{H}_2} = S_0\mathsf{P}_I + S_1\mathsf{P}_Q + S_2\mathsf{P}_A + S_3\mathsf{P}_B$$
$$\mathsf{e}^{-g_2\mathsf{H}_2}\mathsf{P}_Q\mathsf{e}^{g_2\mathsf{H}_2} = S_0\mathsf{P}_Q - S_1\mathsf{P}_I - S_2\mathsf{P}_A + S_3\mathsf{P}_B$$
$$\mathsf{e}^{-g_2\mathsf{H}_2}\mathsf{P}_A\mathsf{e}^{g_2\mathsf{H}_2} = S_0\mathsf{P}_A + S_1\mathsf{P}_B + 2S_2\mathsf{P}_I - 2S_3\mathsf{P}_Q$$
$$\mathsf{e}^{-g_2\mathsf{H}_2}\mathsf{P}_B\mathsf{e}^{g_2\mathsf{H}_2} = S_0\mathsf{P}_B + S_1\mathsf{P}_A + 2S_2\mathsf{P}_Q - 2S_3\mathsf{P}_I,$$

where the S_0, S_1, S_2, S_3 are shortcuts for

$$S_0 = \cosh g_2 \cdot \cos g_2, \tag{A4}$$
$$S_1 = \sinh g_2 \cdot \sin g_2, \tag{A5}$$
$$S_2 = -\frac{1}{2}(\cosh g_2 \cdot \sin g_2 + \cos g_2 \cdot \sinh g_2), \tag{A6}$$
$$S_3 = \frac{1}{2}(\cosh g_2 \cdot \sin g_2 - \cos g_2 \cdot \sinh g_2). \tag{A7}$$

The result for R_2 is

$$\begin{aligned} R_2 &= (c_{10}S_0 - c_{11}S_1 + 2c_{12}S_2 - 2c_{13}S_3)\mathsf{P}_I + \\ &\quad (c_{10}S_1 + c_{11}S_0 - 2c_{12}S_3 + 2c_{13}S_2)\mathsf{P}_Q \\ &\quad +(c_{10}S_2 + c_{11}S_3 + c_{12}S_0 + c_{13}S_1)\mathsf{P}_A + \\ &\quad (c_{10}S_3 - c_{11}S_2 + c_{12}S_1 + c_{13}S_0)\mathsf{P}_B \\ &= c_{20}\mathsf{P}_I + c_{21}\mathsf{P}_Q + c_{22}\mathsf{P}_A + c_{23}\mathsf{P}_B. \end{aligned} \tag{A8}$$

Next term is

$$R_3 = \mathsf{e}^{-g_1\mathsf{H}_1}R_2\mathsf{e}^{g_1\mathsf{H}_1}$$

and, by means of the following partial results:

$$\mathsf{e}^{-g_1\mathsf{H}_1}\mathsf{P}_I\mathsf{e}^{g_1\mathsf{H}_1} = \cos g_1\mathsf{P}_I - i\sin g_1\mathsf{P}_Q$$
$$\mathsf{e}^{-g_1\mathsf{H}_1}\mathsf{P}_Q\mathsf{e}^{g_1\mathsf{H}_1} = \cos g_1\mathsf{P}_Q - i\sin g_1\mathsf{P}_I$$
$$\mathsf{e}^{-g_1\mathsf{H}_1}\mathsf{P}_A\mathsf{e}^{g_1\mathsf{H}_1} = \mathsf{e}^{ig_1}\mathsf{P}_A$$
$$\mathsf{e}^{-g_1\mathsf{H}_1}\mathsf{P}_B\mathsf{e}^{g_1\mathsf{H}_1} = \mathsf{e}^{-ig_1}\mathsf{P}_B$$

one gets

$$\begin{aligned} R_3 &= (c_{20}\cos g_1 - ic_{21}\sin g_1)\mathsf{P}_I + \\ &\quad +(c_{21}\cos g_1 - ic_{20}\sin g_1)\mathsf{P}_Q + \\ &\quad +c_{22}\mathsf{e}^{ig_1}\mathsf{P}_A + c_{23}\mathsf{e}^{-ig_1}\mathsf{P}_B = \\ &= c_{30}\mathsf{P}_I + c_{31}\mathsf{P}_Q + c_{32}\mathsf{P}_A + c_{33}\mathsf{P}_B. \end{aligned} \tag{A9}$$

Now the process is to be repeated for L_i to obtain R_4, R_5 and R_6. Being the L_i the complex conjugated of H_i, every expression is immediate just by using the corresponding complex conjugated coefficients and by substituting the functions g_4, g_5 and g_6 for g_1, g_2 and g_3 respectively. We successively obtain

$$\begin{aligned} R_4 &= (c_{30} + 2g_6c_{33})\mathsf{P}_I + (c_{31} - 2g_6c_{33})\mathsf{P}_Q + \\ &\quad +(-c_{30}g_6 - c_{31}g_6 + c_{32})\mathsf{P}_A + c_{33}\mathsf{P}_B \\ &= c_{40}\mathsf{P}_I + c_{41}\mathsf{P}_Q + c_{42}\mathsf{P}_A + c_{43}\mathsf{P}_B, \end{aligned} \tag{A10}$$

and

$$\begin{aligned} R_5 = &= (c_{40}T_0 - c_{41}T_1 + 2c_{42}T_2 - 2c_{43}T_3)\mathsf{P}_I + \\ &\quad +(c_{40}T_1 + c_{41}T_0 - 2c_{42}T_3 + 2c_{43}T_2)\mathsf{P}_Q + \\ &\quad +(c_{40}T_2 + c_{41}T_3 + c_{42}T_0 + c_{43}T_1)\mathsf{P}_A + \\ &\quad +(c_{40}T_3 - c_{41}T_2 + c_{42}T_1 + c_{43}T_0)\mathsf{P}_B \\ &= c_{50}\mathsf{P}_I + c_{51}\mathsf{P}_Q + c_{52}\mathsf{P}_A + c_{53}\mathsf{P}_B, \end{aligned} \tag{A11}$$

where

$$T_0 = \cosh g_5 \cdot \cos g_5, \tag{A12}$$
$$T_1 = \sinh g_5 \cdot \sin g_5, \tag{A13}$$
$$T_2 = -\frac{1}{2}(\cosh g_5 \cdot \sin g_5 + \cos g_5 \cdot \sinh g_5), \tag{A14}$$
$$T_3 = \frac{1}{2}(\cosh g_5 \cdot \sin g_5 - \cos g_5 \cdot \sinh g_5). \tag{A15}$$

The final result is

$$\begin{aligned} R_6 &= (c_{50}\cos g_4 + ic_{51}\sin g_4)\mathsf{P}_I + \\ &\quad +(c_{51}\cos g_4 + ic_{50}\sin g_4)\mathsf{P}_Q + c_{52}\mathrm{e}^{-ig_4}\mathsf{P}_A + \\ &\quad +c_{53}\mathrm{e}^{ig_4}\mathsf{P}_B = \\ &= c_{60}\mathsf{P}_I + c_{61}\mathsf{P}_Q + c_{62}\mathsf{P}_A + c_{63}\mathsf{P}_B. \end{aligned} \quad \text{(A16)}$$

And we are only left with the dilatation operator, for which the operations are at this point almost immediate and give:

$$\left(\mathsf{O}^{-1}\mathsf{P}\mathsf{O}\right) = \mathrm{e}^{g_7}\left(c_{60}\mathsf{P}_I + c_{61}\mathsf{P}_Q + c_{62}\mathsf{P}_A + c_{63}\mathsf{P}_B\right). \quad \text{(A17)}$$

The D_i coefficients at Sect. 5, can straightforwardly be obtained from this expression as

$$\begin{aligned} D_0(\tau) &= \mathrm{e}^{g_7(\tau)}c_{60}(\tau) \\ D_1(\tau) &= \mathrm{e}^{g_7(\tau)}c_{61}(\tau) \\ D_2(\tau) &= \mathrm{e}^{g_7(\tau)}\frac{1}{2}(c_{62}(\tau) + c_{63}(\tau)) \\ D_3(\tau) &= \mathrm{e}^{g_7(\tau)}\frac{1}{2}i(c_{63}(\tau) - c_{62}(\tau)). \end{aligned} \quad \text{(A18)}$$

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