Astron. Astrophys. 350, 1100–1106 (1999)

ASTRONOMY
AND
ASTROPHYSICS

Properties of longitudinal flux tube waves

I. Shock amplitude relations

M. Cuntz[1,2]

[1] Center for Space Plasma, Aeronomy, and Astrophysics Research (CSPAAR), TH S101, University of Alabama in Huntsville, Huntsville, AL 35899, USA (cuntzm@cspar.uah.edu)

[2] Institut für Theoretische Astrophysik der Universität Heidelberg, Tiergartenstrasse 15, D-69121 Heidelberg, Germany

Received 5 July 1999 / Accepted 26 August 1999

Abstract. We derive relations between the pressure, density, magnetic field strength, velocity, and other quantities across shocks for the case of longitudinal tube waves. Due to the extreme coupling of the Rankine-Hugoniot relations for this type of waves, these relationships cannot be given separately as function of a given shock strength M_s, contrary to the case of acoustic waves. In case of weak shocks, however, those relationships can successfully be decoupled and evaluated. In this paper, the analytic expansion for these amplitude relations are given. We also compare our analytical results with numerical results for shocks of small and moderately large strength. Comparisons are given for cases with different values of plasma-β.

Key words: Magnetohydrodynamics (MHD) – shock waves – methods: analytical – Sun: chromosphere – stars: chromospheres

1. Introduction

Magnetoacoustic wave propagation in thin magnetic flux tubes in the last two decades has been extensively studied (Defouw 1976; Roberts & Webb 1978, 1979; Wilson 1979, 1980, 1981; Parker 1979; Wentzel 1979; Spruit 1981a, b, 1982; Roberts 1983; Edwin & Roberts 1983) particularly in the case of negligible gravity. These treatments usually start from the linearized MHD equations valid for tube geometries. In the case of zero gravity, the wave equation is solved in terms of a wave moving along the vertically-directed tube axis, where the horizontal variation is described in terms of Bessel functions both inside and outside the tube. Suitable matching at the tube boundary allows the construction of the complete wave solution. There are several wave types possible (see Spruit 1982; Edwin & Roberts 1983): a torsional mode, kink modes and a longitudinal mode.

The longitudinal tube wave is the only one of these modes which is compressive and therefore closely resembles acoustic tube waves (i.e., acoustic waves in prescribed generalized geometry) in its propagation characteristics. For situations of non-negligible gravity, these waves have first been studied by Defouw (1976), who assumed an isothermal atmosphere. This study was subsequently generalized by Roberts & Webb (1978) to non-isothermal atmospheres. In both investigations the pressure perturbation and thus the wave in the medium outside the flux tube was neglected to make the problem feasible.

A systematic comparison of longitudinal tube waves, acoustic tube waves and ordinary acoustic waves in plane-parallel geometry has been given by Rae & Roberts (1982) and by Roberts (1981, 1983) for the linear regime. In the work of Herbold et al. (1985) such a comparison was made for the non-linear regime. As the previous authors, Herbold et al. found that the longitudinal tube waves are similar to plane acoustic waves and particularly to acoustic tube waves. A main property of all three waves is that they easily form shocks. The main difference between the longitudinal tube waves and the acoustic tube waves is the variation of the tube cross section. While for acoustic tube waves the tube cross section remains constant, the cross section of the magnetic tube varies during passage of longitudinal waves with the wave phase. It expands or contracts in a time-dependent way while maintaining pressure balance with its surroundings. As shown by Lighthill (1978), the distensibility of the tube augments the pressure as restoring force resulting in a decrease of the "spring constant", which in turn reduces the propagation speed of longitudinal waves compared to acoustic tube waves.

The shock formation property of longitudinal tube waves is particularly interesting for the problem of chromospheric and coronal heating. Observations of solar and stellar UV and X-ray fluxes (e.g., Linsky 1980; Vaiana et al. 1981) have shown that for chromospheres, transition regions and coronae of rapidly rotating stars magnetic heating mechanisms must be present. Recent results from Doppler imaging further indicate that conglomerates of flux tubes may also be responsible for stellar magnetic spots, an important contribution to inhomogeneous magnetic surface structure (e.g., Solanki 1996). With respect to the heating of magnetic flux tubes, longitudinal tube waves are a prime candidate (e.g., Narain & Ulmschneider 1990, 1996). Fawzy et al. (1998) recently presented heating models for the solar magnetic chromosphere assuming flux tubes of different spreading. These flux tube models indicate the heating potential of longitudinal tube waves for different wave parameters and different spreadings for the tubes as function of height. Recent results for the heating of two-component chromospheres (i.e., acoustic and magnetic) for K2V stars with different levels of magnetic activity are given by Cuntz et al. (1999). The results show that the calculated models can nicely reproduce the observed Ca II emission — stellar rotation relationship. The close similarity between acoustic waves and longitudinal tube waves also extends to the mode of wave energy generation in the stellar convection zone (Musielak et al. 1989, 1994, 1995; Ulmschneider & Musielak 1998).

Because of these reasons, it is very important to understand the heating properties of longitudinal tube waves in analytic approximation. In this paper, we study the relationships between pressure, density, magnetic field strength, velocity, and other quantities across shocks for the case of longitudinal tube waves for prescribed values of the shock strengths. We find that these relationships can only be given as decoupled equations in case of weak shocks. In Sect. 2, we give the magnetohydrodynamic equations. The amplitude relations are derived in Sect. 3. In Sect. 4, we compare our analytical results with numerical results for shocks of small and moderate strength assuming different values for plasma-β. The conclusions are given in Sect. 5.

2. Magnetohydrodynamic equations

We consider a thin, vertically directed magnetic flux tube in a solar/stellar atmosphere, which is embedded in a non-magnetic external medium. Following Herbold et al. (1985) the magnetohydrodynamic equations in the thin flux tube approximation can be written

$$\frac{\partial}{\partial t}\left(\frac{\rho}{B}\right)+\frac{\partial}{\partial z}\left(\frac{\rho u}{B}\right)=0 \quad , \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}+g=0 \quad , \tag{2}$$

$$p+\frac{B^2}{8\pi}=p_e\left(z\right) \quad , \tag{3}$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial z}=\left.\frac{dS}{dt}\right|_{Rad} \quad . \tag{4}$$

Here ρ is the density, p the gas pressure, S the entropy, u the gas velocity and B the magnetic field strength in the tube. They are functions of height z and time t. p_e is the gas pressure outside the tube and g is the gravity, which are both functions of z only. The Alfvén wave speed c_A is given by

$$c_A=\frac{B}{\sqrt{4\pi\rho}} \quad . \tag{5}$$

By introducing plasma-β, we find

$$\beta=\frac{p}{p_e-p}=\frac{1}{\epsilon-1}=\frac{2p}{\rho c_A^2} \quad , \tag{6}$$

where $\epsilon=p_e:p$ is the ratio between the external and internal gas pressure. With Eqs. (1), (3), and (5) the B derivatives can be eliminated, and one obtains

$$\frac{\partial\rho}{\partial t}+\frac{1}{c_A^2}\frac{\partial p}{\partial t}+\rho\frac{\partial u}{\partial z}+u\frac{\partial\rho}{\partial z}-\frac{u}{c_A^2}\left(\frac{dp_e}{dz}-\frac{\partial p}{\partial z}\right)=0\,. \tag{7}$$

Eliminating B, the system of four equations (1) to (4) is reduced to three differential equations (2), (4) and (7) for three unknowns: two thermodynamic quantities, e.g. ρ and S, and the velocity u. These equations can be used to find time-dependent solutions of longitudinal wave propagation which can be obtained by the method of characteristics (Herbold et al. 1985). For recent examples see Fawzy et al. (1998) and Cuntz et al. (1998, 1999). These calculations have been based on the ideal gas law

$$p=\rho\frac{\Re T}{\mu} \quad . \tag{8}$$

Here $\Re$ is the gas constant and μ the mean molecular weight taken as 1.3 g mol^{-1} (neutral gas). The adiabatic sound speed c_S and the tube speed c_T are given by

$$c_S^2=\gamma\frac{\Re T}{\mu}=\gamma\frac{p}{\rho} \quad , \tag{9}$$

$$\frac{1}{c_T^2}=\frac{1}{c_S^2}+\frac{1}{c_A^2} \quad , \tag{10}$$

respectively, where γ is the ratio of the specific heats. Eqs. (1), (2), and (4) are however not valid across shocks, which are essential for the energy balance of flux tubes. Here the Rankine-Hugoniot relations must be solved instead.

3. Amplitude relations for shocks

3.1. Rankine–Hugoniot relations

Now let us discuss the shock properties of longitudinal flux tube wave. Shocks are formed when waves propagate through a medium of decreasing density, a phenomenon also found for acoustic waves. The Rankine–Hugoniot relations for longitudinal tube waves are given by (Herbold et al. 1985)

$$\rho_1 A_1 v_1 \;=\; \rho_2 A_2 v_2 \;=j \;, \tag{11}$$

$$A_1(\rho_1 {v_1}^2-2p_e+2p_1)\;=\;A_2(\rho_2 {v_2}^2-2p_e+2p_2)\;, \tag{12}$$

$$\frac{1}{2}{v_1}^2+H_1=\frac{1}{2}{v_2}^2+H_2 \;. \tag{13}$$

Here ρ_1 is the density, A_1 is the tube cross section, v_1 the gas velocity, p_1 the gas pressure and H_1 the enthalpy in front of the shock. Values with index 2 refer to the state behind the shock. j is the mass flux and p_e the gas pressure of the field free region outside the tube. The velocities v_1 and v_2 are measured in the frame that moves with the shock. The transformation of the gas velocity from the laboratory (Euler) frame into the shock frame is given by

$$v_1=u_1-U_{\rm sh} \quad , \quad v_2=u_2-U_{\rm sh} \quad , \tag{14}$$

where $U_{\rm sh}$ is the shock speed in the laboratory frame. We also assume for the gas velocities u_1, u_2 the identity $u_2=-u_1=u_m$ with u_m as velocity amplitude, implying that the mean velocity at the shocks (Euler frame) is zero.

The shock strength (or Mach number) M_s is given by

$$M_s = \frac{U_{\rm sh} - u_1}{c_{T1}} = 1 + \hat{\alpha} \quad , \tag{15}$$

where u_1 and c_{S1} is the flow speed and sound speed in front of the shock, respectively, and $\hat{\alpha}$ is the residual shock strength for weak shocks. This definition deviates slightly from the case of acoustic waves (e.g., Landau & Lifshitz 1987), because here the Mach number is given in terms of c_{T1} instead of c_{S1}. The reason is that for very weak shocks, i.e. $\hat{\alpha} \to 0$, it is found that the shock speed $U_{\rm sh}$ approaches the tube speed c_{T1} instead of the adiabatic sound speed c_{S1} as for acoustic waves (see Appendix A).

The enthalpy H_1 in front of the shock is given by

$$H_1 \;=\; \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} \;=\; \frac{c_{S1}^2}{\gamma-1} \tag{16}$$

with an equivalent expression holding for H_2. Because of the magnetic flux conservation

$$B_1 A_1 = B_2 A_2 = BA = \phi = \ const \quad , \tag{17}$$

we find by using Eqs. (5), (6)

$$p_e - p_1 \;=\; \frac{\phi^2}{8\pi A_1^2} \;=\; \frac{1}{2}\,\rho_1 c_{A1}^2 \;=\; \frac{p_1}{\beta_1} \tag{18}$$

with c_{A1} as Alfvén speed and β_1 the plasma-β in front of the shock. Again, equivalent expressions hold for the state behind the shock.

3.2. Analytic expansions for weak shocks

It is the main goal of this paper to relate the jumps in pressure, density, temperature, and other quantities across shocks to the shock strength M_s. As in the case of acoustic waves[1], we introduce $\Phi = p_2/p_1$, $\Theta = \rho_2/\rho_1$, $\Xi = T_2/T_1$, and $\Omega = A_2/A_1$ (see Appendix A). However, contrary to acoustic waves, it is *not possible* for longitudinal tube waves to give separate expressions for Φ, Θ, Ξ, and Ω because of the strong coupling of the Rankine-Hugoniot relations introduced by the distensibility (i.e., $A_1 \neq A_2$). Intriguing examples of expressions attainable include

$$M_s^2 \;=\; \frac{4}{\gamma-1}\,\frac{c_{S\rm o}^2}{c_{T\rm o}^2}\,\frac{\Phi-\Theta}{\Phi+\Theta}\,\frac{1}{1-\Theta^{-2}\Omega^{-2}} \tag{19}$$

and

$$\Xi \;=\; \frac{\Phi}{\Theta} \tag{20}$$

obtained from Eqs. (11) and (13) to (16). Note that Eq. (20) is a direct consequence of the ideal gas law, which is used both for longitudinal tube waves and acoustic waves (see Appendix A).

[1] Since we study the behavior of variables across shocks but not the *propagation* of shocks, there is no necessity to distinguish between acoustic waves and acoustic tube waves. The shock amplitude relations are identical in both cases.

In Eq. (19) we assumed that $c_{T\rm o} \simeq c_{T1}$ with $c_{T\rm o}$ as tube speed of the mean atmosphere. Also, $c_{S\rm o}$ is the adiabatic sound speed of the mean atmosphere.

It is nevertheless possible to obtain separate expressions for Φ, Θ, Ξ, and Ω and other quantities as function of M_s, if we restrict ourself to the case of weak shocks. The reason is that in this case the relevant equations can be linearized and decoupled. For convenience we first write the Rankine-Hugoniot relations (12) and (13) in different forms

$$({v_1}^2 - {c_{A1}}^2)\, v_2 \;=\; ({v_2}^2 - {c_{A2}}^2)\, v_1 \quad , \tag{21}$$

$${v_1}^2 + \frac{2{c_1}^2}{\gamma-1} \;=\; {v_2}^2 + \frac{2{c_2}^2}{\gamma-1} \quad , \tag{22}$$

using Eqs. (11), (18) and (9), (16), respectively. The equations to be linearized are (11), (21), (22), (17), (9), (5), (10), and (8), which are the three Rankine-Hugoniot relations, the equation of magnetic flux conservation, the definitions for c_S, c_A, and c_T, and the ideal gas law. We thus have eight equations with eight variables, which are $\rho_m, p_m, T_m, B_m, A_m, c_{Sm}, c_{Am}$, and c_{Tm}. These variables are the amplitudes of the different quantities at the shock, e.g., $\rho_m = (\rho_2 - \rho_1)/2$. As all amplitudes must be positive, B_1 and B_2 as well as c_{A1} and c_{A2} need to be exchanged in that definition.

For the velocity v_1 we have with Eq. (15)

$$v_1 \;=\; c_{T1}(1+\hat{\alpha}) \quad . \tag{23}$$

We also need to adopt a relationship between v_1 and v_2. Technically, this would require an expansion for $U_{\rm sh}$, for which we first would have to solve the set of linearized equations. Alternatively, we can use an *Ansatz* akin to the result for acoustic waves Eq. (A10) (see Appendix A):

$$v_2 \;=\; v_1 - c_{T1}\frac{4}{\gamma+1}\xi\hat{\alpha} \quad . \tag{24}$$

Here we replaced c_{S1} by c_{T1} and α by $\hat{\alpha}$. The factor ξ ensures that (24) is an exact equation even in case that the $v_2 - v_1$ relations for acoustic waves and longitudinal tube waves turn out to be not completely equivalent. In Appendix B, the factor ξ will be determined self-consistently, and it will be found that ξ solely depends on plasma-β.

Noting that all quantities with subscript "o" refer to mean values at the shocks, e.g., $\rho_{\rm o} = (\rho_1 + \rho_2)/2$, we obtain the following set of linearized equations:

$$A_{\rm o}\rho_m + \rho_{\rm o}A_m \;=\; \frac{2}{\gamma+1}\,\rho_{\rm o}A_{\rm o}\,\xi\hat{\alpha} \tag{25}$$

$$c_{A\rm o}c_{Am} \;=\; \frac{1}{\gamma+1}\,(c_{A\rm o}^2 + c_{T\rm o}^2)\,\xi\hat{\alpha} \tag{26}$$

$$c_{S\rm o}c_{Sm} \;=\; \frac{\gamma-1}{\gamma+1}\,c_{T\rm o}^2\,\xi\hat{\alpha} \tag{27}$$

$$A_{\rm o}B_m - B_{\rm o}A_m \;=\; 0 \tag{28}$$

$$c_{S\rm o}^2\rho_m + 2\rho_{\rm o}c_{S\rm o}c_{Sm} - \gamma p_m \;=\; 0 \tag{29}$$

$$2\pi c_{A\text{o}}^2\rho_m + B_\text{o}B_m - 4\pi\rho_\text{o}c_{A\text{o}}c_{Am} \;=\; 0 \tag{30}$$

$$(c_{A\text{o}}^2 - c_{T\text{o}}^2)c_{S\text{o}}c_{Sm} \;-\; (c_{S\text{o}}^2 + c_{A\text{o}}^2)c_{T\text{o}}c_{Tm} - (c_{S\text{o}}^2 - c_{T\text{o}}^2)c_{A\text{o}}c_{Am} \;=\; 0 \tag{31}$$

$$\frac{\Re}{\mu}T_\text{o}\rho_m - p_m + \frac{\Re}{\mu}\rho_\text{o}T_m \;=\; 0 \tag{32}$$

Now we are able to express the shock amplitudes in terms of $\hat{\alpha}$ by solving the system of Eqs. (25) to (32) via successive elimination. Therefore, we introduce the normalized shock amplitudes ρ'_m, p'_m, T'_m, B'_m, A'_m, c'_{Sm}, c'_{Am}, and c'_{Tm} given as $\rho'_m = \rho_m/\rho_\text{o}$ and so on. We find the following results:

$$\rho'_m \;=\; \frac{2}{\gamma+1}\Big(1 - \frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big)\,\xi\hat{\alpha} \tag{33}$$

$$p'_m \;=\; \frac{2\gamma}{\gamma+1}\Big(1 - \frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big)\,\xi\hat{\alpha} \tag{34}$$

$$T'_m \;=\; \frac{2(\gamma-1)}{\gamma+1}\Big(1 - \frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big)\,\xi\hat{\alpha} \tag{35}$$

$$B'_m \;=\; \frac{2}{\gamma+1}\frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\,\xi\hat{\alpha} \tag{36}$$

$$A'_m \;=\; \frac{2}{\gamma+1}\frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\,\xi\hat{\alpha} \tag{37}$$

$$c'_{Sm} \;=\; \frac{\gamma-1}{\gamma+1}\Big(1 - \frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big)\,\xi\hat{\alpha} \tag{38}$$

$$c'_{Am} \;=\; \frac{1}{\gamma+1}\Big(1 + \frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big)\,\xi\hat{\alpha} \tag{39}$$

$$c'_{Tm} = \left[\frac{\gamma-1}{\gamma+1}\Big(1-\frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big)^2 - \frac{1}{\gamma+1}\frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big(1+\frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2}\Big)\right]\xi\hat{\alpha} \tag{40}$$

Here we ensured that only $c_{T\text{o}}$ and $c_{A\text{o}}$ appear in the amplitude relations. The variables ρ_o, T_o, A_o, B_o, and $c_{S\text{o}}$ have been eliminated using Eqs. (5), (8), (9), (10), and (17). In fact, it is also possible to eliminate both $c_{T\text{o}}$ and $c_{A\text{o}}$ by introducing plasma-β. With Eqs. (6), (9), (10) and (33) we obtain

$$\frac{{c_{T\text{o}}}^2}{{c_{A\text{o}}}^2} \;=\; 1 - \Big(1 + \frac{1}{2}\beta\gamma\Big)^{-1} \quad , \tag{41}$$

which shows that the shock amplitudes of the different quantities in fact only depend on plasma-β and $\hat{\alpha}$.

A further variable of interest is the velocity amplitude u_m. Here we assume that the mean velocity u_o is zero yielding $u_m = u_2 = -u_1$. By introducing $u'_m = u_m/c_{T\text{o}}$ we find

$$u'_m \;=\; \frac{2}{\gamma+1}\,\xi\hat{\alpha} \tag{42}$$

with Eqs. (14), (15), (23), and (24). For acoustic waves, we find $u'_m \;=\; 2\alpha/(\gamma+1)$ with $u'_m = u_m/c_{S\text{o}}$, which is equivalent to Eq. (42). With $\rho' = (\rho_2 - \rho_1)/\rho_1$ and similar expressions holding for p', T', and A' we also obtain

$$\rho' \;=\; \Theta - 1 \;=\; 2\rho'_m\rho_\text{o}/\rho_1 \tag{43}$$

$$p' \;=\; \Phi - 1 \;=\; 2p'_m p_\text{o}/p_1 \tag{44}$$

$$T' \;=\; \Xi - 1 \;=\; 2T'_m T_\text{o}/T_1 \tag{45}$$

$$A' \;=\; \Omega - 1 \;=\; 2A'_m A_\text{o}/A_1 \tag{46}$$

Note that in Eqs. (43) to (46) terms like ρ_o/ρ_1 can in fact be omitted because ρ_o and ρ_1 are identical in zeroth order and $\hat{\alpha}$ is of first order. Note again that we still have to determine ξ to obtain complete solutions in Eqs. (33) to (40) and (42) to (46) (see Appendix B).

Similar to the case of acoustic waves, we introduce the shock parameter $\hat{\eta}$ by

$$\hat{\eta} \;=\; \frac{\rho_2 - \rho_1}{\rho_1} \;=\; \rho' \quad . \tag{47}$$

For acoustic waves, we find $\eta = 4\alpha/(\gamma+1)$, whereas for longitudinal flux tube waves we obtain

$$\hat{\eta} \;=\; \frac{4}{\gamma+1}\Big(1 + \frac{1}{2}\beta\gamma\Big)^{-1}\,\xi\hat{\alpha} \tag{48}$$

with Eqs. (33) and (41).

4. Comparison with numerical results

We now compare the results given by the analytic expansions for weak shocks (see Eqs. 33–40 and 42) with results from numerical computations. Our aim is to check the validity of these analytic relations for weak shocks and shocks of moderate strength and to find out at which values of $\hat{\alpha}$ the weak shock approximation becomes invalid. In addition, the results for longitudinal tube waves are compared with those for acoustic waves. For the magnetic flux tube computations we consider the two cases $p_e : p_i = 4$ and 10 with p_e and p_i as external and internal gas pressure, respectively. These pressure ratios correspond to plasma-β values of 0.33 and 0.11 and to amplitude parameters ξ of 1.24 and 1.08, respectively. In all calculations, the mean values of pressure, temperature, magnetic field strength and flux tube area are can kept constant and are set to $p_\text{o} = 1$ dyn cm^{-2}, $T_\text{o} = 5000$ K, $B_\text{o} = 1500$ G, and $A_\text{o} = 7.85\cdot10^{13}$ cm^2, respectively. The tube area is that of a flux tube with radius 50 km.

The different panels of Fig. 1 show the relative shock amplitudes of the flow speed, density, pressure, temperature, tube area, and Alfvén speed of our example. The analytic results for weak shocks (Eqs. 42, 33–35, 37, and 39) are compared with numerical results given by solving the Rankine-Hugoniot relations (Eqs. 11–14). The results for acoustic shocks are also included. Note that in the acoustic case we have $A'_m = 0$ and c'_{Am} is undefined (i.e., c'_{Am} would formally be infinity). Fig. 1 includes the two cases plasma-β = 0.33 and 0.11. The solid / dotted line pair for plasma-β = 0.11 is that closest to the acoustic case (also applicable to c'_{Am}) since the choice of a very high $p_e : p_i$ ratio mimics the acoustic case. A detailed analysis shows that for

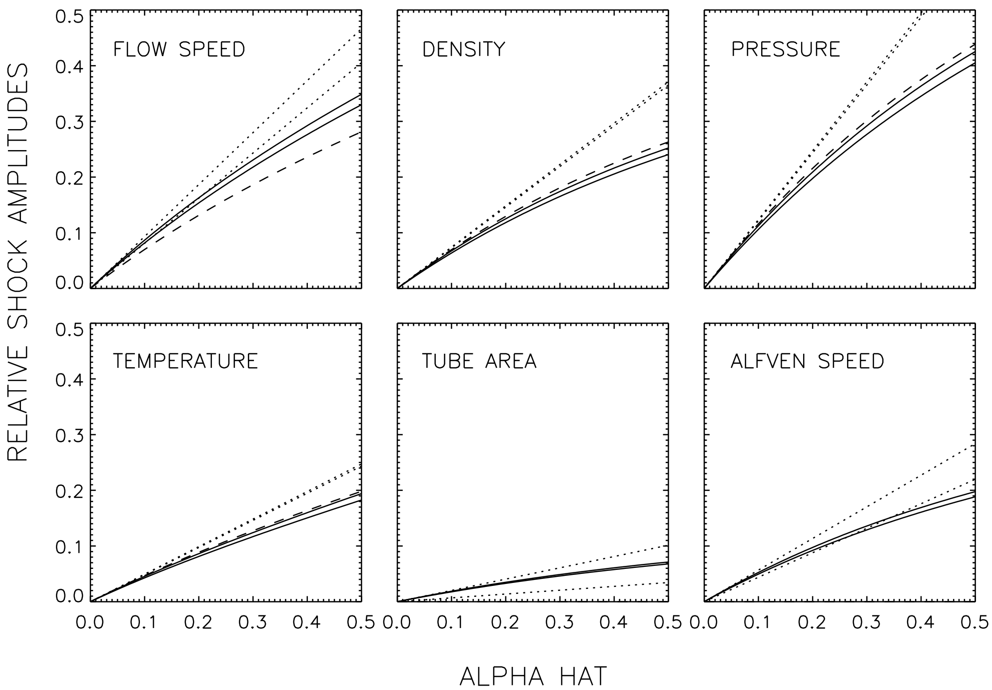

Fig. 1. Panel showing the behavior of the relative shock amplitudes for the flow speed, density, pressure, temperature, tube area, and Alfvén speed given by u'_m, ρ'_m, p'_m, T'_m, A'_m, and c'_{Am}, respectively, as function of $\hat{\alpha}$. Here we show the results from the analytic formulas for weak shocks (dotted lines) together with numerical results for the shocks (solid lines). Results are given for plasma-β = 0.33 and 0.11 (for identification see text). Results for acoustic shocks are given for comparison (dashed lines).

very small values of $\hat{\alpha}$ the numerical and analytical results of the various shock amplitudes are in perfect agreement, which is strong evidence that the derived analytic weak shock formulas are correct. For larger shock strengths (i.e., $\hat{\alpha} > 0.15$) significant differences between the analytical and numerical results occur, as expected. It is found that the increase of the relative shock amplitudes from the numerical results is typically smaller than the linear increase given by the analytic formulas, which indicates that the second term in the amplitude expansions (i.e., the $\hat{\alpha}^2$ term) would be negative.

Taking the shock computations discussed in Fig. 1, we can also explore the relationship between $\hat{\alpha}$ and α (see Eqs. 15, A5). Here α denotes the residual shock strength as given for acoustic shocks. It is found that α and $\hat{\alpha}$ do not agree, as expected (see Fig. 2). Also, for very small values of $\hat{\alpha}$, α is negative, which is particularly evident for relative high values of plasma-β. For high values of $\hat{\alpha}$, i.e. high shock strengths, it is found that α approaches $\hat{\alpha}$. Notice that α and $\hat{\alpha}$ essentially agree for very small values of plasma-β, even for small $\hat{\alpha}$, as in this case the shocks of longitudinal tube waves are to be identified with acoustic shocks.

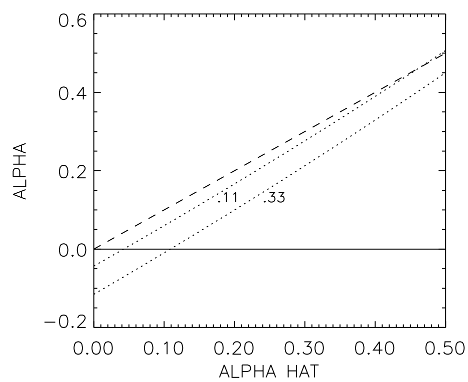

Fig. 2. Relationship between α and $\hat{\alpha}$ for the shock computations of Fig. 1 with plasma-β = 0.33 and 0.11 (dotted lines). The case $\alpha = \hat{\alpha}$ is given as reference (dashed line).

5. Conclusions

We derived relations between the pressure, density, magnetic field strength, velocity, and other quantities across shocks for longitudinal tube waves. Furthermore, we calculated examples of shocks with small and moderate strength allowing us to compare the analytical results with detailed calculations. We found the following results:

1. Contrary to acoustic waves it is not possible to expand the shock amplitude relations for different quantities as separate expressions as function of M_s due to the extreme coupling of the Rankine-Hugoniot relations through the distensibility (i.e., $A_1 \neq A_2$). In case of weak shocks, however, the shock amplitude relations can be successfully decoupled and evaluated.
2. For consistency reasons, the definition of shock strength was slightly changed compared to acoustic waves as it was found that the shock speed $U_{\rm sh}$ of very weak shocks approaches the tube speed c_{T1} rather than the adiabatic sound speed c_{S1}. Consequently, $\hat{\alpha}$ was introduced as residual shock strength rather than α.
3. A detailed analysis showed that the analytic expressions for the various shock amplitudes, i.e., u'_m, ρ'_m, p'_m, T'_m, B'_m, A'_m, c'_{Sm}, c'_{Am}, and c'_{Tm} solely depend on plasma-β and $\hat{\alpha}$. Most expressions show significant similarities with acoustic shocks.

4. The comparison between numerical and analytic results for the relative shock amplitudes showed perfect agreement for very weak shocks, but revealed significant discrepancies for stronger shocks, as expected. The limit of applicability of the analytic amplitude formulas is at $M_s \lesssim 1.15$.
5. Results were obtained for different values of plasma-β. It was found that $\beta \to 0$ closely resembles the limiting case of acoustic waves. In this case, the flux tube is almost completely evacuated, and the ratio between external and internal gas pressure is most extreme.

Acknowledgements. I am grateful to P. Ulmschneider for helpful discussions. This work has been supported by the NASA Astrophysical Theory Program under NAG5-3027 to the University of Alabama in Huntsville and by the UAH/USRA/NASA Cooperative Agreement for Research in Space Plasma Physics.

Appendix A: shock amplitude relations for acoustic waves

For acoustic waves, the amplitude relations for the pressure, density, temperature, and the cross section across the shocks (e.g., Mihalas & Mihalas 1984; Landau & Lifshitz[2] 1987) are given by

$$\frac{\rho_2}{\rho_1} = \Theta = \frac{(\gamma+1)M_s^2}{(\gamma-1)M_s^2+2} \quad \text{(A1)}$$

$$\frac{p_2}{p_1} = \Phi = \frac{2\gamma M_s^2-(\gamma-1)}{\gamma+1} \quad \text{(A2)}$$

$$\frac{T_2}{T_1} = \Xi = \frac{\Phi}{\Theta} \quad \text{(A3)}$$

$$\frac{A_2}{A_1} = \Omega = 1 \quad \text{(A4)}$$

Here M_s denotes the shock strength given as

$$M_s = \frac{U_{\rm sh}-u_1}{c_{S1}} = 1+\alpha \quad . \quad \text{(A5)}$$

where $U_{\rm sh}$ is the shock speed and u_1 and c_{S1} is flow speed and adiabatic sound speed in front of the shock, respectively, and α is the residual shock strength of weak shocks. Note that this definition deviates slightly from Eq. (15) because here we have c_{S1} instead of c_{T1} in the denominator. Eqs. (A1) to (A3) allow to calculate the jumps in pressure, density, and temperature for shocks of given strength M_s, whereas Eq. (A4) indicates the absence of distensibility for acoustic tube waves, contrary to longitudinal MHD flux tube waves (e.g., Herbold et al. 1985).

Although Eqs. (A1) to (A4) are valid for any value of M_s, those expressions can again be expanded for weak shocks, i.e. $\alpha \to 0$ (e.g., Ulmschneider 1970; Gonczi et al. 1977). For early results see also Brinkley & Kirkwood (1947). With $\rho' = (\rho_2 - \rho_1)/\rho_1$ and similar expressions for p', T', and A', we find

$$\rho' = \Theta - 1 = \rho_1 \frac{4}{\gamma+1}\alpha \quad \text{(A6)}$$

[2] Note a typo in the expression for T_2/T_1 in Fluid Mechanics by Landau and Lifshitz (1st and 2nd edition)

$$p' = \Phi - 1 = p_1 \frac{4\gamma}{\gamma+1}\alpha \quad \text{(A7)}$$

$$T' = \Xi - 1 = T_1 \frac{4(\gamma-1)}{\gamma+1}\alpha \quad \text{(A8)}$$

$$A' = \Omega - 1 = 0 \quad . \quad \text{(A9)}$$

Note that Eqs. (A6) to (A9) can now directly be compared to Eqs. (43) to (46) obtained for longitudinal tube waves and they prove to be fully equivalent. Ideal equivalence is attained for Eqs. (A6) to (A8), which differ only by a scaling factor from their acoustic counterparts which solely depends on plasma-β.

Using the Rankine-Hugoniot relations for acoustic shocks, we can also derive the relationship between v_1 and v_2, i.e. the gas velocities in the shock frame, as

$$v_2 = v_1 - c_{S1}\frac{4}{\gamma+1}\alpha \quad \text{(A10)}$$

with c_{S1} as adiabatic sound speed in front of the shock. Following Eqs. (15) and (A5), we introduced two different versions of the residual shock strength for acoustic waves and longitudinal tube waves, given as α and $\hat\alpha$, respectively. For the relationship between α and $\hat\alpha$ we find

$$\alpha = (1+\hat\alpha)\frac{c_{T1}}{c_{S1}} - 1 \quad \text{(A11)}$$

with

$$\frac{c_{T1}}{c_{S1}} = \frac{c_{T\rm o}}{c_{S\rm o}}\left(1 + c'_{Sm} - c'_{Tm}\right) \quad . \quad \text{(A12)}$$

Here c'_{Sm} and c'_{Tm} are given by Eqs. (38) and (40), respectively, and are calculated from $\hat\alpha$ and plasma-β also using Eq. (41) with $c_{T\rm o}$ and $c_{S\rm o}$ being known. Note that for very small values of $\hat\alpha$, α is found negative.

Appendix B: derivation of the shock amplitude parameter ξ

To derive the shock amplitude parameter ξ we write the velocity variable v_1^2 in three different ways. Using Eq. (15) we have

$$v_1^2 = c_{T1}^2\,(1+2\hat\alpha) \quad \text{(B1)}$$

and from the momentum and energy flux across shocks we obtain

$$v_1^2 = \frac{c_{A1}^2 - c_{A2}^2\frac{\rho_2 A_2}{\rho_1 A_1}}{1-\frac{\rho_1 A_1}{\rho_2 A_2}} \quad , \quad \text{(B2)}$$

$$v_1^2 = \frac{2}{\gamma-1}\,\frac{c_2^2-c_1^2}{1-\frac{\rho_1^2 A_1^2}{\rho_2^2 A_2^2}} \quad . \quad \text{(B3)}$$

In Eqs. (B2) and (B3) we eliminated v_2^2 with Eq. (24) and also took advantage of Eqs. (11), (16), and (18). We now expand Eqs. (B1) to (B3) in terms of $\hat\alpha$ by using the shock amplitude relations (33) and (37) to (40). In the process, all pre-shock and post-shock variables (subscripts 1, 2) are eliminated and the respective mean variables (subscript o) are introduced. This step yields a system of three linear equations of the form

$$v_1^2 \;=\; \mathcal{A}_i + \mathcal{B}_i\hat{\alpha} + \mathcal{C}_i\xi\hat{\alpha} \tag{B4}$$

with i = 1, 2, 3 with the variables v_1^2, $\hat{\alpha}$, and ξ. The coefficients $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i$ are given by

$$\mathcal{A}_1 \;=\; c_{T\mathrm{o}}^2 \tag{B5}$$

$$\mathcal{B}_1 \;=\; 2c_{T\mathrm{o}}^2 \tag{B6}$$

$$\mathcal{C}_1 \;=\; -2c_{T\mathrm{o}}^2\left[\frac{\gamma-1}{\gamma+1}\left(1-\frac{{c_{T\mathrm{o}}}^2}{{c_{A\mathrm{o}}}^2}\right)^2 - \frac{1}{\gamma+1}\frac{{c_{T\mathrm{o}}}^2}{{c_{A\mathrm{o}}}^2}\left(1+\frac{{c_{T\mathrm{o}}}^2}{{c_{A\mathrm{o}}}^2}\right)\right] \tag{B7}$$

$$\mathcal{A}_2 \;=\; c_{T\mathrm{o}}^2 \tag{B8}$$

$$\mathcal{B}_2 \;=\; 0 \tag{B9}$$

$$\mathcal{C}_2 \;=\; \frac{4}{\gamma+1}\,c_{T\mathrm{o}}^2 \tag{B10}$$

$$\mathcal{A}_3 \;=\; c_{T\mathrm{o}}^2 \tag{B11}$$

$$\mathcal{B}_3 \;=\; 0 \tag{B12}$$

$$\mathcal{C}_3 \;=\; \frac{4}{\gamma+1}\,c_{T\mathrm{o}}^2 \quad . \tag{B13}$$

Interestingly, it is found that Eqs. (B2) and (B3) render the same linear equation. However, it is still possible to eliminate both v_1^2 and $\hat{\alpha}$ in order to find an expression for ξ. Hence, ξ can be determined *independently* of $\hat{\alpha}$, which is a necessary condition for the system of shock amplitude relations (33) to (40) to be well-defined. For ξ we obtain

$$\xi \;=\; \left[\frac{2}{\gamma+1}+\frac{\gamma-1}{\gamma+1}\left(1-\frac{{c_{T\mathrm{o}}}^2}{{c_{A\mathrm{o}}}^2}\right)^2 - \frac{1}{\gamma+1}\frac{{c_{T\mathrm{o}}}^2}{{c_{A\mathrm{o}}}^2}\left(1+\frac{{c_{T\mathrm{o}}}^2}{{c_{A\mathrm{o}}}^2}\right)\right]^{-1} \quad , \tag{B14}$$

which can be simplified to

$$\xi \;=\; \frac{(\gamma+1)(1+\frac{1}{2}\beta\gamma)^2}{1+\frac{3}{2}\beta\gamma+\gamma} \tag{B15}$$

with Eqs. (6), (9), and (10) by introducing plasma-β. For $\beta \to 0$, we find $\xi \to 1$ and furthermore $c_T \to c_S$. This corresponds to the case of a flux tube being almost completely evacuated. In this case, the behavior of a longitudinal flux tube wave is essentially that of an acoustic wave. For large values of β, we find $\xi \to \beta\gamma(\gamma+1)/6$.

References

Brinkley Jr. S.R., Kirkwood J.G., 1947, Phys. Rev. 71 (9), 606

Cuntz M., Rammacher W., Ulmschneider P., Musielak Z.E., Saar S.H., 1999, ApJ 522, 1053

Cuntz M., Ulmschneider P., Musielak Z.E., 1998, ApJ 493, L117

Defouw R.J., 1976, ApJ 209, 266

Edwin P.M., Roberts B., 1983, Solar Phys. 88, 179

Fawzy D.E., Ulmschneider P., Cuntz M., 1998, A&A 336, 1029

Gonczi G., Mangeney A., Souffrin P., 1977, A&A 54, 689

Herbold G., Ulmschneider P., Spruit H.C., Rosner R., 1985, A&A 145, 157

Landau L.D., Lifshitz E.M., 1987, Fluid Mechanics. Course of Theoretical Physics Vol. 6, 2nd Ed., Pergamon Press, Oxford

Lighthill J., 1978, Waves in Fluids. Cambridge University Press, Cambridge

Linsky J.L., 1980, ARA&A 18, 439

Mihalas D., Mihalas B.W., 1984, Foundations of Radiation Hydrodynamics. Oxford University Press, New York

Musielak Z.E., Rosner R., Ulmschneider P., 1989, ApJ 337, 470

Musielak Z.E., Rosner R., Gail H.P., Ulmschneider P., 1995, ApJ 448, 865

Musielak Z.E., Rosner R., Stein R.F., Ulmschneider P., 1994, ApJ 423, 474

Narain U., Ulmschneider P., 1990, Space Sci. Rev. 54, 377

Narain U., Ulmschneider P., 1996, Space Sci. Rev. 75, 453

Parker E.N., 1979, Cosmical Magnetic Fields. Clarendon Press, Oxford, Chap. 8

Rae I.C., Roberts B., 1982, ApJ 256, 761

Roberts B., 1981, In: Cram L.E., Thomas J.H. (eds.) Physics of Sunspots. Sacramento Peak Observatory, Sunspot, New Mexico, p. 369

Roberts B., 1983, Solar Phys. 87, 77

Roberts B., Webb A.R., 1978, Solar Phys. 56, 5

Roberts B., Webb A.R., 1979, Solar Phys. 64, 77

Solanki S.K., 1996, In: Strassmeier K.G., Linsky J.L. (eds.) Stellar Surface Structure. IAU Symp. 176, Dordrecht, Kluwer, p. 201

Spruit H.C., 1981a, In: Jordan S. (ed.) The Sun as a Star. NASA, Washington, NASA SP-450, p. 385

Spruit H.C., 1981b, A&A 98, 155

Spruit H.C., 1982, Solar Phys. 75, 3

Ulmschneider P., 1970, Solar Phys. 12, 403

Ulmschneider P., Musielak Z.E., 1998, A&A 338, 311

Vaiana G.S., Cassinelli J.P., Fabbiano G., et al., 1981, ApJ 245, 163

Wentzel D.G., 1979, A&A 76, 20

Wilson P., 1979, A&A 71, 9

Wilson P., 1980, ApJ 237, 1008

Wilson P., 1981, ApJ 251, 756