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Effect of internal waves on the solar neutrino flux

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Abstract. The neutrino production rate strongly depends on the temperature, and a small variation of chemical composition in the central regions can imply a large variation of neutrino flux. For some values of their frequencies and wavenumbers, the internal waves e xcited at the bottom of the solar convective zone are able to propagate until the central region of the star, producing a macroscopic diffusive process (Press 1981, Schatzman & Montalbán 1995). In this paper we analyse the effect of this mixing mechani sm on the neutrino flux production. For that purpose, we introduce some improvements in the treatment of non-adiabatic internal waves in the central region of the Sun. We use the amplitude of these oscillations to determine the macroscopic diffusion coefficient linked to non-adiabatic propagation of gravity waves. We analyse the chemical mixing in the solar center and its effect on the decrease of solar neutrino production. We also study how the diffusion coefficient depends on the source function of internal waves, and we show that the amplitude of the diffusion coefficient in the internal regions is strongly affected by the characteristics of the source function at the boundary convective/stratified regimes.

Key words: diffusion – turbulence – Sun: interior – Sun: oscillations

1. Introduction

A number of well observed properties of the Sun cannot be explained in the framework of the Standard Solar Model (SSM). Their interpretation requires the introduction of a number of new physical processes in the description of the internal structure and evolution of the Sun, whose validity must be confirmed by the application to other stars.

Among these anomalous properties of the Sun we find the deficiency of the lithium abundance in the solar atmosphere and the deficiency of the solar neutrino flux, compared to the values predicted by the SSM. Given the fact that the nuclear reactions, responsible of lithium burning and neutrino production, take place at very different depths (near the bottom of the convective zone for the lithium, and in the central region for the neutrinos case), these controversial results provide information about the characteristics of the physical processes in these two very distant regions.

The lithium abundance in the solar atmosphere is about one hundred times smaller than the theoretical one and than the maximum observed in young clusters (Pleiades and αPer). The dependence on age, metallicity and mass of the lithium abundance has been well studied from the great number of observational data (see Rebolo 1990 for a review). Several mechanisms have been considered in order to produce chemical mixing in the stable region and then to explain these deviations from the standard model (Michaud & Charbonneau 1991). Those mechanisms, producing mixing by a transport process due to rotation, do not seem to provide very consistent results (Schatzman & Baglin 1991; Schatzman 1994; Montalbán 1995). Two diffusion processes linked to the internal waves have been considered by Press (1981). The first process, which has been studied by García López & Spruit (1991), assumes that the shear induced by random gravity waves generates turbulence. The effect is supposed to be limited to the region where the shear $k_{\rm V} \cdot u_{\rm h}$ is larger than the frequency of gravity waves. The consequence of that assumption is that the mixing process is limited to a region where the amplitude of gravity waves is sufficiently large. The second process is linked to the non-adiabatic propagation of internal waves in the stellar radiative region. The entropy losses during the oscillation allow to make an estimation of the mixing length and velocity, and provide a diffusion coefficient of the fourth order (Schatzman 1991, 1993). The transport process exists all over the region where gravity waves are propagating. Montalbán (1994), Schatzman & Montalbán (1995) and Schatzman (1996) introduced some improvements of the physical and mathematical treatment of the excitation and propagation of gravity waves, and they obtained a diffusion coefficient linked to the non-adiabatic propagation of internal waves. The application of this diffusion coefficient to predict light element abundances in the Sun and in solar type stars (Montalbán 1994; Montalbán & Schatzman 1996; Montalbán & Schatzman 1999) shown that this mixing mechanism can explain, in a consistent way, the features of lithium and beryllium abundances provided by the observational works. Therefore, it has been assumed that this mechanism is able to describe the transport process near the convective zone.

Recently, Fritzz et al. (1998) have proposed another transport process associated to the gravity waves generated by convec-

Send offprint requests to: J. Montalbán tion at the boundary between turbulent and stratified regions in the Sun. As in Schatzman & Montalbán (1995) and Schatzman (1996), the internal waves could be excitated by the penetration of convective plumes in the radiative interior of the star. The propagation, dissipation and filtering of these waves in a sheared and stratified layer seem to impose a residual circulation within the solar interior that could account for the observed lithium depletion in the Sun. The ability to mix seems to be restricted to the shallow layers close to the so urce layer. The model predict no Be depletion at the age of the Sun. Given the fast decrease of the transport efficiency with the distance to the internal waves source, no effect is expected in the regions where the neutrinos are produced.

Solar neutrino production and some features derived by helioseismologycal data depend strongly on the physical conditions in the central region of the Sun. Consequently, the neutrino flux for three different energy ranges and helioseismology data provide information on the internal region of the Sun. Since 1967 there have been several experiments (Chlorine; Kamiokande; Gallium: GALLEX, SAGE) to measure the value of these fluxes and to test the validity of the SSM. Despite large uncertainties in the solar model and in the experimental determination, the theoretically predicted values have always been larger than the experimental one, suggesting that some aspects of the problem have been forgotten. The neutrino deficiency could be explained by the MSW effect (Mikheyev & Smirnov, 1986; Wolfenstein, 1978), but there remains the possibility of considering a Non-Standard Solar Model (NSSM), taking into account the physical process of macroscopic diffusion, which is ignored in SSM. This is the aim of the present paper.

There has been several attempts to explain the neutrino deficiency by a partially mixed core, either without specifying the mechanism responsible of that mixing (Ezer & Cameron, 1968; Shaviv & Baudet, 1968; Bahcall et al., 1968; Shaviv & Salpeter, 1968; Iben, 1968,1969), or introducing a turbulent diffusion process (Schatzman & Maeder 1981; Lebreton & Maeder 1987). With the transport of more fuel to the center, the nuclear energy production rate is larger. Due to the calibration constraint, the mixed core would experience a decrease of the central temperature which leads to a smaller production of high energy neutrinos. On the other hand, helioseismological data (J. Provost 1984) determine the sound velocity in the central regions and therefore, give some constraints over the internal structure of the Sun. The coherence of these data implies that the gradient of the mean molecular weight has to be larger than a given value. Consequently, the models with a large mixing process in the central region, invoked to solve the solar neutrino problem (Schatzman & Maeder 1981; Lebreton & Maeder 1987) appears to be ruled out, although, a small mixing could be present and would be compatible with the values of the sound speed curve. The neutrino production depends strongly on the central temperature ($\sim T_c^{18}$) and a small variation of the chemical composition could imply a small diminution of temperature and a significant diminution of neutrino production.

On the other hand, in the models including microscopic processes, gravitational settling and thermal diffusion of ^{4}He and ^{1}H (Bahcall & Pinsonneault 1992), the computed neutrino fluxes are larger than in SSM. In fact, around the center, the abundance of ^{1}H is depleted by the microscopic diffusion, while the abundance of ^{4}He is increased. A direct consequence of this lowering of the available nuclear fuel, is a decrease of the thermonuclear energy production rate. The calibration process result is an increase of the central temperature and a concomitant increase of the neutrino flux. Therefore, the microscopic diffusion alone acts in a wrong direction; it is necessary to take into account the possible inhibition by other mixing processes.

Press (1981) noted that inward-propagating gravity waves could have an important effect over the neutrino production rate. In fact, these waves are focused as they approach the solar center, they therefore may become unstable and produce local mixing there, provided that they are generated with sufficiently large amplitude at the base of the convective zone and are not strongly damped along the way of propagation. Merryfield (1995) studied the likelihood that internal waves excitated by a classical convective zone, described by the mixing-length formalism, were able to generate turbulence in the center of the Sun. Gravity waves having wave numbers and frequencies given by time and space scales of convection are rapidly damped, and he concluded that turbulent mixing by convectively-driven internal waves is only possible if nonlinear interactions transform the emitted gravity waves into higher frequency waves that reach the core of the Sun.

Morel & Schatzman (1996) considered the effect on the center of the Sun of a non-turbulent macroscopic diffusion supposed to be induced by gravity waves. They used a phenomenological description based on the expression given by Schatzman & Montalbán (1995) (based on the second process noted by Press 1981). In the paper of Schatzman & Montalbán, a random field of internal waves is produced by inward-plumes of the convective zone (Rieutord & Zahn 1995) which are stopped at the bottom of the convective region. The transport process linked to internal waves is extended down to the central regions of the Sun, taking into account two dissipative effects: radiative damping, and work against the gradient of chemical composition generated during stellar evolution. The diffusion coefficient seems to have the necessary profile, respecting the helioseismological constraints and being able to reduce the solar neutrino flux. It presents first a fast decrease, due to radiative damping; when approaching the solar center, convergence of the waves produced a divergence of the diffusion coefficient, like r^{-12}. Near the center, due to the gradient of the mean molecular weight, the diffusion process has to work against gravity. The total flux of mechanical energy decreases and the diffusion coefficient vanishes like r^2 when r goes to zero. The result is that the diffusion coefficient presents a maximum at a distance R_0 to the solar center of the order of 0.2 $R_\odot$. This behavior corresponds to the one which, since the eighties, was considered as being able to explain simultaneously the light element abundances and the neutrino flux.

The theory of the diffusion process induced by stochastic gravity waves was still at that moment (and even now) in a preliminary state, and provides only indications of the magnitude and shape of the macroscopic diffusion coefficient curve. Morel & Schatzman (1996) presented the numerical experiment of a NSSM, with the macroscopic diffusion coefficient represented by a gaussian. They could neither reproduce the neutrinos flux nor the helioseismological data by changing only the parameters of the Gaussian representing the diffusion coefficient; it was necessary to introduce an asymmetric profile in the curve. They estimated that a diffusion coefficient with a maximum of the order of $10^3\,\mathrm{cm^2s^{-1}}$ at about $\sim 0.2R_\odot$ and a half-width of the order of $0.05R_\odot$ was needed. Schatzman (1997) has shown, using the numerical results by Morel & Schatzman (1996),that it may be possible to chose the parameters in such way (maximum, $10^3\,\mathrm{cm^2s^{-1}}$; center at $\sim 0.23R_\odot$, and half-width of the order of $0.04R_\odot$) that the frequency difference $\delta\nu_{nl}$ is correct, and the neutrino flux production appreciably decreased. However, this phenomenological, time independent, diffusion coefficient is certainly an oversimplification, and it must be improved, including the complete description of the diffusion coefficient due to internal waves, the effect of microscopic diffusion, and the time dependent model of the macroscopic diffusion process. In fact, they considered a time independent diffusion coefficient whereas the position of the maximum near the center depends on the gradient of the mean molecular weight, and that gradient must change with the evolution of the star. On the other hand, the diffusion coefficient derived by Schatzman & Montalbán (1995), using Press' approximation, is valid for the outer regions and the magnitude and location of the maximum depend strongly on the value of the diffusion coefficient below $0.2\mathrm{R}_\odot$ where the chemical gradient begins to appear. In fact, the diffusion coefficient which takes into account the dissipation of mechanical energy due to the work against ∇_μ was written as:

$$D_M = \frac{D(r)}{(1 + D(r) \times \Lambda(r))^2} \quad (1)$$

where $\Lambda(r)$ is a function of the stellar structure and is close to zero every place except near the center of the star where ∇_μ becomes important. Therefore, the correction term will be different to one only close to the stellar center, where it decreases rapidly. $D(r)$ is the diffusion coefficient without correction.

Another aspect of the effect of oscillatory motions on the diffusion process was considered by Knobloch & Merrifield (1992). They noted that the presence of an oscillatory motion can produce an increase of the microscopic diffusion, and that the increase in the case of gravity waves is very small. It is not clear how to compare the contribution of microscopic diffusion with recpect to the macroscopic diffusion studied here, as the oscillatory motion considered by Knobloch & Merrifield is not a random motion, whereas the macroscopic diffusion process we are studyding here is induced by macroscopic random motions. Remember that these random properties are induced by the stochastic motion present in plumes, with characteristics which are defined at the boundary of the convective zone.

On the other hand, Gough (1992) studied the effect on the solar model of small amplitude temperature oscillations produced by the g-modes (gravity waves that penetrate down to the center where they are reflected). His estimations show that ^{7}Be and ^{8}B neutrino fluxes are decreasing functions of the oscillation amplitude. The contribution of non-adiabatic g-modes to the diffusion procces described here will be treated in a future paper. Here we only consider the contribution from the progressive gravity waves.

We will derive an expression for the diffusion coefficient associated to the non-adiabatic propagation of internal waves that is valid till the central regions of the Sun. In Sect. 2 using the complete equations for non-adiabatic oscillations given by Unno et al. (1989) we obtain, in the quasi-adiabatic approximation, the internal wave amplitude as a function of solar radius, and a new expression for the radiative damping effect, both valid up to the central regions of the Sun. In Sect. 3 we follow the procedure given in Schatzman (1996) and using Sect. 2 results, we derive the diffusion coefficient in the central regions. It is assumed that these waves are produced by turbulent flow in the convective inward-plumes which are braked in the overshooting region, at the bottom of the external convective zone. In Sect. 4 we analyse the dependence of diffusion coefficient at the central region of the Sun on turbulence description. Finally, in Sect. 5 we present the summary and conclusions from these computings and we suggest other aspects of the problem that should be considered in the future.

2. Non-adiabatic internal waves

We follow here the complete analysis given by Unno et al. (1989) for non-adiabatic, non-radial oscillations, under the quasi-adiabatic and Cowling approximations. Real oscillations are inevitably non-adiabatic as there is always an energy exchange between mass elements during oscillations. Unlike Press (1981), we take into account the complete set of differential equations (21.11–21.15 in Unno et al. (1989)). In these equations the entropy perturbation $\delta S \neq 0$. The degree of non-adiabaticity is measured by a parameter c_4 defined as the ratio between the thermal ($\tau_{\rm th}$) and the dynamical ($\tau_{\rm dyn}$) times scales:

$$\tau_{\rm th} = \frac{4\pi r^3 \rho c_{\rm P} T}{L_{\rm R}}$$

$$\tau_{\rm dyn} = \sqrt{\frac{R^3}{GM}}$$

and

$$c_4 = \frac{\tau_{\rm th}}{\tau_{\rm dyn}} = \frac{4\pi r^3 \rho c_{\rm P} T}{L_{\rm R}}\,\sigma^*, \quad (2)$$

where $L_{\rm R}$ is the luminosity, σ^* is $\tau_{\rm dyn}^{-1}$. Other symbols have the usual meaning.

The equation of energy conservation is:

$$\imath\omega \frac{\tau_{\rm th}}{\tau_{\rm dyn}} \frac{\delta S}{c_{\rm P}} = \frac{4\pi r^3 \rho}{L_{\rm R}} \left(\delta\epsilon_{\rm N} - \frac{d\delta L_{\rm R}}{dM_r} \right) + \frac{l(l+1)}{d\ln T/d\ln r} \frac{T'}{T} + l(l+1) \frac{\xi_h}{r} \frac{4\pi r^3 \rho}{L_{\rm R}} \frac{dL_r}{dM_r}$$

where ω is the dimensionless frequency ($\omega = \sigma/\sigma^*$) and σ is the frequency, ξ_h the radial part of the displacement in the horizontal direction, $\delta\epsilon_{\rm N}$ the perturbation of nuclear energy generation rate. Consequently, if c_4 is very large, the entropy perturbation is, in most cases, so small that the real part of the angular frequency is essentially the same as its adiabatic value. Inside the Sun $c_4 \geq 5.10^{10}$. In this case we can use the quasi-adiabatic approximation, in which the terms of O(c_4^{-2}) are discarded. In quasi-adiabatic and Cowling approximations the entropy perturbation is expressed as a function of radial displacement ξ_r and pressure perturbation p':

$$\frac{\delta S}{c_{\rm P}} = \imath \frac{D_3}{\omega c_4}\xi_r + \imath \frac{D_4}{\omega c_4}\frac{p'}{\rho} + O(c_4^{-2}),$$

where $c_{\rm P}$ is the specific heat per mass unit at constant pressure. D_3 and D_4 are functions giving the radiative damping effect that we shall define below.

With these approximations, the equations of non-adiabatic oscillations are similar to those of adiabatic oscillations:

$$\frac{1}{r^2}\frac{d(r^2\xi_r)}{dr} - \left(\frac{g}{c^2} + \imath \frac{D_3 \upsilon_T}{\omega c_4}\right)\xi_r + \left[\left(1 - \frac{L_l^2}{\sigma^2}\right)\frac{1}{c^2} - \imath \frac{D_4 \upsilon_T}{\omega c_4}\right]\frac{p'}{\rho} \approx 0, \tag{3}$$

$$\frac{d}{dr}\left(\frac{p'}{\rho}\right) - \left(\frac{N^2}{g} + \imath \frac{D_4 g \upsilon_T}{\omega c_4}\right)\frac{p'}{\rho} + \left(N^2 - \sigma^2 - \imath \frac{D_3 g \upsilon_T}{\omega c_4}\right)\xi_r \approx 0, \tag{4}$$

where g is the gravity, c the sound velocity, σ the oscillation frequency, N the Brunt-Väisäla frequency:

$$N^2 = -g\left(\frac{d\ln\rho}{dr} + \frac{g}{c^2}\right), \tag{5}$$

L_l is the Lamb frequency:

$$L_l^2 \equiv l(l+1)c^2/r^2, \tag{6}$$

and

$$\upsilon_T \equiv -\left[\frac{\partial \ln\rho}{\partial \ln T}\right]_p = c_{\rm P}\,\rho T/p \cdot \nabla_{\rm ad}. \tag{7}$$

Eqs. (4) and (5) present a clear parallelism with the equivalent adiabatic equations (1) and (2) in Press (1981). The non-adiabatic contribution appears only by linear complex terms. Therefore, it is possible to follow the same procedure than in the adiabatic case. Introducing the new variables $\bar{\xi}_N$ and $\bar{\eta}_N$ as follows:

$$\bar{\xi}_N = r^2\xi_r \exp\left[-\int_0^r \left(\frac{g}{c^2} + \imath \frac{D_3 \upsilon_T}{\omega c_4}\right) dr\right] \tag{8}$$

and

$$\bar{\eta}_N = \frac{p'}{\rho}\exp\left[-\int_0^r \left(\frac{N^2}{g} + \imath \frac{D_4 g \upsilon_T}{\omega c_4}\right) dr\right], \tag{9}$$

the equations (4) and (5) become:

$$\frac{d^2\bar{\xi}_N}{dr^2} - \frac{1}{P_N}\frac{dP_N}{dr}\frac{d\bar{\xi}}{dr} - P_N\,Q_N\,\bar{\xi}_N = 0 \tag{10}$$

and

$$\frac{d^2\bar{\eta}_N}{dr^2} - \frac{1}{Q_N}\frac{dQ_N}{dr}\frac{d\bar{\eta}_N}{dr} - P_N\,Q_N\,\bar{\eta}_N = 0 \tag{11}$$

with P_N and Q_N being defined as

$$P_N = \frac{r^2 h_N(r)}{c^2}\left(\frac{L_l^2}{\sigma^2} - 1 + \imath \frac{D_4 c^2 \upsilon_T}{\omega c_4}\right) \tag{12}$$

and

$$Q_N = \frac{1}{r^2 h_N(r)}\left(\sigma^2 - N^2 + \imath \frac{D_3 g \upsilon_T}{\omega c_4}\right), \tag{13}$$

and

$$h(r) = \exp\left[\int^r \left\{\frac{N^2}{g} - \frac{g}{c^2} + \imath \frac{\upsilon_T}{\omega c_4}(g\,D_4 - D_3)\right\}{\rm d}r\right]$$

To solve the Eq. (10) we use the asymptotic or WKB approximation, and introducing for convenience a new variable υ as

$$\bar{\xi}_N = \upsilon\,P_N^{1/2}, \tag{14}$$

Eq. 10 appears as:

$$\upsilon'' + S_N(r)\cdot \upsilon = 0 \tag{15}$$

with

$$S_N = -P_N^{1/2}\frac{d^2}{dr^2}P_N^{-1/2} - P_N Q_N. \tag{16}$$

Far from the turning points ($S_N = 0$) we can write the solution of this equation with the WKBJ approximation:

$$\upsilon \sim S_N^{-1/4}\exp\left(\pm \int^r S_N^{1/2}\,dr\right), \tag{17}$$

and the amplitude of the vertical motion ξ_r:

$$\xi_r = \frac{1}{r^2}\exp\left\{\int_0^r \left(\frac{g}{c^2} + \imath \frac{D_3 \upsilon_T}{\omega\, c_4}\right){\rm d}r\right\} P_N^{1/2}\,\upsilon \tag{18}$$

In the expression of S_N the first term, $P_N^{1/2}\frac{d^2}{dr^2}P_N^{-1/2}$, is of the order of H_p^{-2}, and in the first order approximation this term is discarded because it is small compared with $k_r^2 = -P_N Q_N$. Therefore, the function S_N is reduced to:

$$S_N \simeq -\frac{1}{c^2}\left(\frac{L_l^2}{\sigma^2} - 1\right)(\sigma^2 - N^2) \left(1 + \imath \frac{\sigma^2}{L_l^2 - \sigma^2}\frac{D_4 c^2 \upsilon_T}{\omega c_4} + \imath \frac{1}{\sigma^2 - N^2}\frac{D_3 g \upsilon_T}{\omega c_4}\right). \tag{19}$$

Introducing Eq. (12) in Eq. (18) the solution is:

$$\xi_r \propto \frac{1}{r^2}\Lambda^{-\frac{1}{4}}\frac{r}{c}\left(\frac{L_l^2}{\sigma^2} - 1\right)^{\frac{1}{2}} \exp\left[\frac{1}{2}\int \left(\frac{N^2}{g} + \frac{g}{c^2}\right) dr\right] \exp\left[\frac{1}{2}\int \Lambda^{\frac{1}{2}}\frac{\upsilon_T}{\omega c_4}\left(\frac{\sigma^2 c^2 D_4}{L_l^2 - \sigma^2} + \frac{D_3 g}{\sigma^2 - N^2}\right) dr\right] \left(1 + \frac{\imath}{4}\frac{\upsilon_T}{\omega c_4}\left(\frac{\sigma^2 c^2 D_4}{L_l^2 - \sigma^2} - \frac{g D_3}{\sigma^2 - N^2}\right)\right) \exp\left[\imath \int \Lambda^{\frac{1}{2}}\,dr \frac{\imath}{2}\int \frac{\upsilon_T}{\omega c_4}(g D_4 + D_3) dr\right] \tag{20}$$

with

$$\Lambda = \frac{1}{c^2}\left(\frac{L_l^2}{\sigma^2} - 1\right)\left(N^2 - \sigma^2\right)$$

The variation of the amplitude of gravity waves as a function of depth is given by the real part of Eq. (20). Taking into account the fact that $L_l^2 \gg \sigma^2$, and the definitions (5) and (6), this variation can be written as:

$$\xi_r \;\propto\; \frac{\left(l(l+1)\right)^{1/4}}{r^{3/2}}\left(\frac{N^2}{\sigma^2} - 1\right)^{-1/4}\frac{1}{\sigma}\rho^{-1/2} \exp\left(-\frac{1}{2}A(r)\right). \tag{21}$$

The function $A(r)$ contains the radiative damping effect, whose expression is described in next section.

2.1. Radiative damping term

The full expressions of the funtions D_3 and D_4 which give the entropy perturbation are given by the Eqs. (22.9) and (22.10) of Unno et al. (1989).

We have computed the coefficients D_i in the solar case in order to find which terms give the largest contribution. The computation has been made with $l = 10$ and $\sigma = 4.10^{-5}\,\mathrm{s}^{-1}$. It appears that the functions D_3 and D_4 are essentially given by the following equations:

$$D_3 \;\simeq\; -\left(\frac{\nabla_{\rm ad}}{\nabla} - 1\right)\frac{l(l+1)}{r}\left(\frac{N^2}{\sigma^2} - 1\right), \tag{22}$$

$$D_4 \;\simeq\; \frac{l(l+1)}{\sigma^2\, r^2}\left[\left(\frac{\nabla_{\rm ad}}{\nabla} - 1\right)\left(\frac{K.r}{H_P} + \Delta\nabla - 2\right) + B\right] \tag{23}$$

with

$$K \;=\; \frac{\mathrm{d}\ln L_R}{\mathrm{d}\ln r}$$

Furthermore, the most important term in the expression of $A(r)$ is the D_3 term. The largest contribution to A is given, with a good approximation, by:

$$A \;\simeq\; \int_{r_b}^{r}\frac{1}{c}\left(L_l^2 - \sigma^2\right)^{\frac{1}{2}}\left(\frac{N^2}{\sigma^2} - 1\right)^{\frac{1}{2}}\frac{v_T}{\omega c_4}\frac{g}{\sigma^2 - N^2} \left(\frac{\nabla_{\rm ad}}{\nabla} - 1\right)\frac{l(l+1)}{r}\left(\frac{N^2}{\sigma^2} - 1\right)dr. \tag{24}$$

From the definitions (2) and (7) the term $v_T/\omega c_4$ is written as:

$$\frac{v_T}{\omega c_4} = \frac{\nabla_{\rm ad}\cdot L_R}{4\pi r^3 p\sigma} = \frac{1}{\sigma}4\nabla_{\rm ad}\frac{1/3aT^4}{p}\frac{c^*}{\kappa\rho}\frac{1}{r}\frac{\mathrm{d}\ln T}{\mathrm{d}r}. \tag{25}$$

where c^* is the light velocity. With the approximation $L_l^2 \gg \sigma^2$, and the definition (6) the damping term (24) is:

$$A \;\simeq\; \int_{r_b}^{r}\frac{\left(l(l+1)\right)^{3/2}}{r^3}\frac{1}{\sigma^4}N^2\left(N^2 - \sigma^2\right)^{\frac{1}{2}}D_{\rm th} \frac{\nabla_{\rm ad}}{\nabla}\left(\frac{\nabla_{\rm ad} - \nabla}{\Delta\nabla}\right)dr, \tag{26}$$

with

$$D_{\rm th} = 4\nabla\frac{c^*}{\kappa\rho}\frac{\frac{1}{3}aT^4}{p}, \tag{27}$$

and

$$\Delta\nabla = v_T\left(\nabla_{\rm ad} - \nabla\right) + \nabla\mu = \frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln p} - \frac{1}{\Gamma_1} \tag{28}$$

Using the approximation $k_{\rm H}^2 = l\cdot(l+1)/r^2$ for the horizontal wavenumber, and calling $k_{\rm H_b}$ its value at the boundary of the radiative zone ($r_{\rm b}$), we finally can write the function A as:

$$A \;=\; \frac{k_{\rm H_b}^3}{\sigma^4}\int_{r_{\rm b}}^{r}D_{\rm th}\left(N^2 - \sigma^2\right)^{1/2}N^2\left(\frac{r_b}{r}\right)^3 \frac{\nabla_{\rm ad}}{\nabla}\left(\frac{\nabla_{\rm ad} - \nabla}{\Delta\nabla}\right)dr. \tag{29}$$

We write later on:

$$A \;=\; \frac{k^3}{\omega^4}f \tag{30}$$

The difference between the expression (26) and the corresponding equation in Press (1981) is that here there appears two factors containing gradients that have disappeard in his Eqs. (31) and (32). These factors take values different than one as we go to the center of the Sun. This corresponds to the fact that Press combined the merit of spherical adiabatic equations in the inelastic approximation, with the plane non-adiabatic equation in the Boussinesq approximation, whereas the expression (24) given here is derived from the complete set of equations written by Unno et al. (1989). The properties of the damping integral f and the function under the integral symbol are shown in Figs. 1a,b, where we have also plotted the same curves using Press' approximation. One can see that the amplitude of radiative damping begins to decrease at $0.2\mathrm{R}_\odot$ with respect to the Press one. A large amplitude of A corresponds to a large damping of gravity waves of low frequency and relatively large wave number. As a consequence, we thought that we could consider that there is no reflection of gravity waves arriving to the stellar center, which means that we do not have to consider eigenvalues in this range of frequencies and wave numbers. However, in Fig. 2 we have plotted the dependence on radius of the amplitude of perturbation for several wave number-frequency couples. The wavenumbers considered correspond to spatial scales of the perturbation going from the plume dimension at the boundary of convective zone ($4\cdot 10^{-9}\,\mathrm{cm}^{-1}$), to the radius of the convective zone ($1.25\,10^{-10}\,\mathrm{cm}^{-1}$). Gravity waves, most likely to propagate from the convection zone to the core without appreciable damping are those having small horizontal wave numbers and high frequencies. Thus, we see that for $\omega = 1.\,10^{-6}\,\mathrm{s}^{-1}$ the perturbation does not penetrate for any of considered values of k. For $\omega = 1.\,10^{-5}\,\mathrm{s}^{-1}$, only waves with k of the order of $1.\,10^{-10}$ do succeed to penetrate below $0.5\mathrm{R}_\odot$. As the value of the frequency increases above $5.\,10^{-5}$, perturbations with larger horizontal wavenumbers are able to penetrate near the center. Oscillations with the characteristics

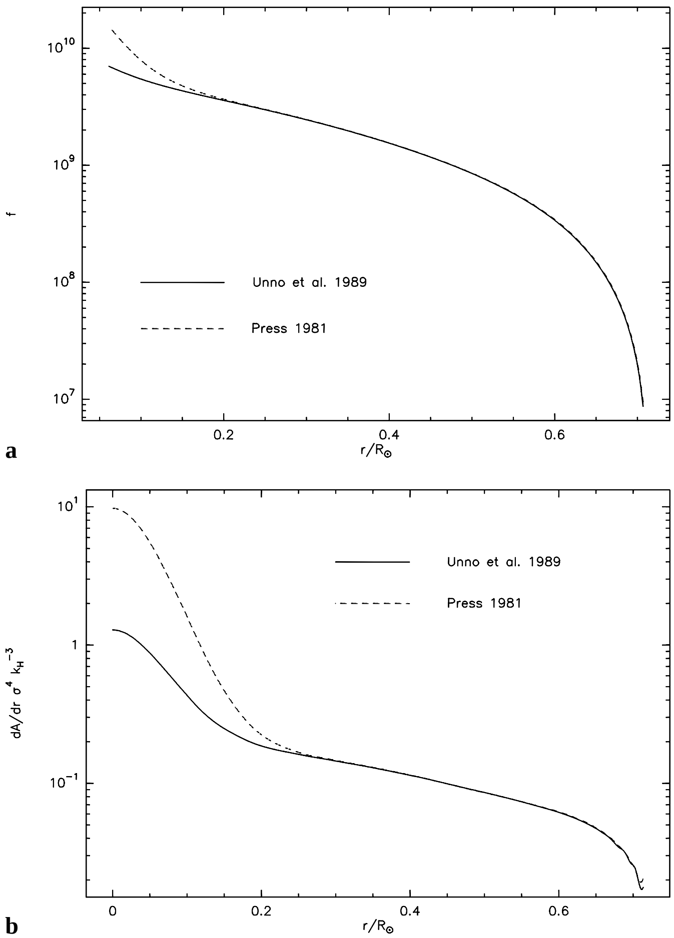


Fig. 1. a Radial dependence of radiative damping (f). **b** Derivative of f respect to r. Solid lines correspond to the quasi-adiabatic approach, and dashed line to Press' approach

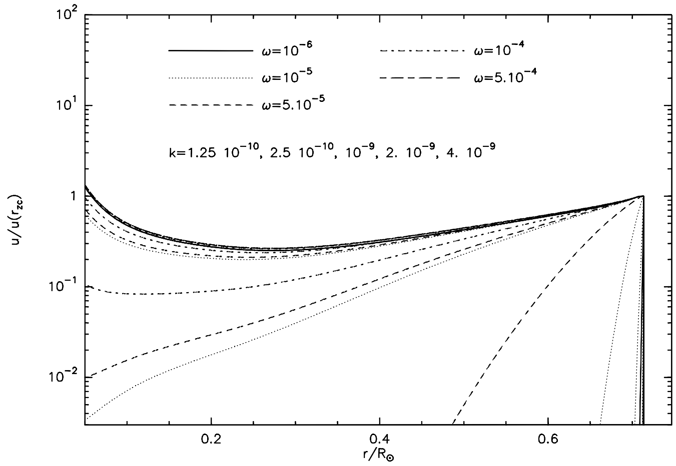


Fig. 2. Amplitude of internal waves (ω, k) as a function of radius, normalised to their amplitudes at the boundary of the convective zone

of ω and k given by the turbulence distribution inside the convective plumes, $(5.\,10^{-5}\,\mathrm{s}^{-1}, 4.\,10^{-9}\,\mathrm{cm}^{-1})$, disappear before arriving to $0.5\mathrm{R}_\odot$. However, in Fig. 2, it appears also that for frequencies larger than $5.\,10^{-5}\,\mathrm{s}^{-1}$ and wave number smaller than $1.\,10^{-9}\,\mathrm{cm}^{-1}$ the waves arrive to the center even with an amplitude larger than the initial one at the boundary of the convective zone. Therefore, to discard the possibility of g-modes in that range of frequency and wave number is only a first approximation, and the validity of that approximation depends on the source function distribution.

3. Diffusion process due to internal waves

A first approach of the diffusion problem is due to Press (1981). As mentioned by Bretherton (1969), adiabatic oscillations do not produce any diffusion process, as any piece of the fluid, in its vertical motion, comes always back to the same place. The diffusion process induced by random gravity waves results from radiative damping of waves.

As already mentioned, we deal with the problem of the generation of internal waves produced by the turbulent motion present inside the plumes, at the boundary of the convective zone. A description of the plumes, based on the picture given by Rieutord & Zahn (1995), is given by Schatzman (1996).

We shall describe the motion at the boundary by the sum of the motions of $N_{\rm pl}$ plumes,

$$u(x,y,t) \;=\; \sum_i u_i(x-x_i, y-y_i) \tag{31}$$

each of the velocities u_i having the same auto-correlation properties,

$$\big\langle u_i(x-x_i,\, y-y_i) \quad u_i(x'-x_i,\, y'-y_i)\big\rangle \;=\; u^2\,\Gamma(x'-x,\, y'-y) \tag{32}$$

We shall consider later (appendix) the problem of the auto-correlation function.

The velocity field at point (x,y,z,t) is the superimposition of all the contributions starting from $\mathrm{N}_{\rm pl}$ and arriving at points (x_i, y_i)

$$u(x,y,z,t) = \sum_{i=1}^{N_{\rm pl}} u_i(x-x_i, y-y_i, z, t) \tag{33}$$

The vertical diffusion coefficient produced by this velocity field is given in Schatzman (1996) using the results from Knobloch (1977):

$$D_v = \int_0^\infty \langle v(t), v(t')\rangle\, d(t-t') \tag{34}$$

where v is the Lagrangian velocity, and the relationship with our Eulerian velocity u is:

$$v = u_z(x,y,z;t) + \frac{\partial u_z(t)}{\partial z} \int^t u_z\, dt \tag{35}$$

Then, the diffusion coefficient is given by Eq. (4.13) in Schatzman (1996)

$$D_v = \int_0^\infty \left\langle \frac{\partial u_z(t)}{\partial z} \int^t u_z\, dt\, \frac{\partial u_z(t')}{\partial z} \int^{t'} u_z(t') \right\rangle d(t-t') \tag{36}$$

Later on, we shall use $u' \equiv u(t')$.

The field of velocity is described by the superimposition of "monochromatic" internal waves which propagate in the radiative interior of the star (Eqs. 18 and 24). The source function of these waves will be described using the the Fourier transform of the motion taking place at the boundary of the radiative zone ($z_{\rm b} = 0$). This allows us to introduce the spectrum of the turbulence generating the internal waves. In order to relate the amplitude of the movement at each side of the boundary convection/stratification, we chose the solution given by Press (1981), assuming a discontinuous behaviour of the Brunt-Väisälä frequency at the transition layer. It is well justified (Montalbán 1994) for stellar models using the treatment of the overshooting by Zahn (1991). Therefore, the vertical amplitude of a monochromatic component of the field is given by $\hat{u}_{\rm H}(\mathbf{k}, \omega, z_{\rm b})|\omega|/N_{\rm b}$. $\hat{u}_{\rm H}$, being the horizontal amplitude, $\hat{u}_{\rm H}^2$ is taken equal to the turbulent mean square velocity in the convective zone generating the waves. It is equivalent to the treatment of the transition used by García López & Spruit (1991) in considering the continuity of pressure fluctuations on both sides of the boundary. The variation of amplitude of monochromatic waves as a function of depth z is introduced using Eqs. 18 and 24.

The inverse Fourier transform at depth z, including this time $k_{\rm H}$-dependent terms, introduced by the z derivatives present in Eq. (37), provides the description of the motion at depth z:

$$u(x, y, t; z_b) = TF^{-1}\left(\hat{u}(\mathbf{k}, \omega)\right)$$

$$u(x, y, z, t) = TF^{-1}\left(\hat{u}(\mathbf{k}, \omega, z)\right)$$

where $\hat{u}$ is the Fourier transform of u.

Appendix gives the details of the way to obtain the expression of the diffusion coefficient. It is first expressed as a function of the variables in Fourier space. We follow the rule given by Knobloch(1977), which consists in replacing the ensemble avera ge of four terms by the product of two terms. We use the kinetic spectrum given by Lesieur and finally obtain the diffusion coefficient:

$$D_v = G^4(r)F^2(r)\left(\langle u^2\rangle \frac{N_{pl}}{N_b^2}\tau k_m^{-2}\right)^2 \left(\frac{3}{4}k_m^{\frac{11}{3}} f^{-\frac{4}{9}}\right)^2 \qquad (37)$$
$$\int_0^\infty d(t-t')\left[\int_0^\infty d\omega\, \omega^{-\frac{11}{9}} e^{-\tau\omega}\gamma\left(\frac{4}{9}, \frac{fk^3}{\omega^4}\right) e^{-\imath\omega(t-t')}\right]^2$$

In Fig. 3 we plot this diffusion coefficient and the curve obtained in Schatzman & Montalbán (1995) using Press approach. Both diffusion coefficients have been calculated considering the same model of plumes (Schatzman 1996) which generate the internal waves. The spatial scale of the turbulence inside the plumes, k_m is defined by the plume radius at the boundary of the convective zone (Eqs. 2.25, 3.11 of Schatzman 1996), and the velocity of turbulent flow in the plumes is given by the experimental relationship (List 1982) with the maximum velocity in the plume (Schatzman 1996 Eq. 2.26). τ is defined as $(k_m \cdot (\langle u^2\rangle)^{1/2})^{-1})$. The values of these parameters are: $k_m = 4.04\,10^{-9}\,{\rm cm}^{-1}$, $(\langle u^2\rangle)^{1/2} = 1.084\,10^4\,{\rm cm/s}$, and $\tau = 2.27\,10^4\,{\rm s}$, or $\omega_m = 4.4\,10^{-5}\,{\rm s}^{-1}$. We note that the new expression for the diffusion coefficient takes almost the same value near the boundary but decreases more rapidly as the depth increases than the old one. Furthermore, more significant for our present question, its value below 0.2R$_\odot$ is much smaller than in Schatzman & Montalbán (1995).

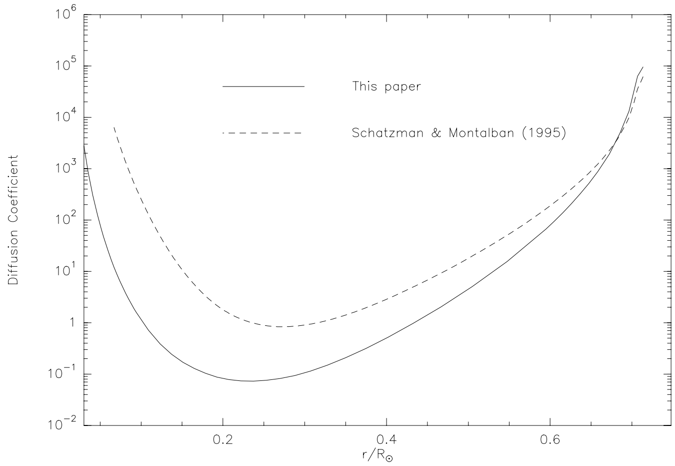


Fig. 3. Diffusion coefficient due to internal waves generated by a turbulence distribution characterized by $k = 4.04\,10^{-9}\,{\rm cm}^{-1}$ and $\omega = 4.4\,10^{-5}{\rm s}^{-1}$ (Schatzman & Montalbán 1995, Schatzman 1996). Solid line correspond to quasi-adiabatic approach, dashed line correspond to Press' (1981) approach.

The damping of internal waves has been decreased with respect to the old model and therefore, the waves go down more deeply. However, in the expression for the diffusion coefficient, appears also the radiative damping derivative as a factor F^2 (Eq. A9), and this magnitude has also decreased below 0.2R$_\odot$ as a consequence of new gradients appearing in the new expression of radiative damping. That behaviour implies that, when we take into account the effect of mean molecular weight gradient, the parameters of the maximum obtained by Schatzman & Montalbán (1995) will not correspond to the values required in Morel & Schatzman (1996) and Schatzman (1997). We would obtain a maximun located at lower radius and with l ower amplitude.

4. Role of turbulence description

Following the first paper of Press (1981), we consider that the internal gravity waves propagating in the radiative region are generated by the convective motions in the overshooting region, at the boundary between the convective and radiative regions. The problem is how to describe this "convective motion". Press (1981) considered that the motion could be described by only one eddy with a size given by the mixing length theory (MLT) of convection. Later on, some improvements were introduced (Garcia López & Spruit 1991; Schatzman 1993; Montalbán 1994) which did consist in considering a Kolmogorov spectrum of turbulence with characteristic scales given by the MLT. Laboratory experiments and numerical simulations have shown that in fact, the MLT is not appropriate for describing the convection movement, and that the convective transport of energy occurs through large structures. Therefore we have taken the model of plumes by Rieutord & Zahn (1995) to describe the convection. We consider that the gravity waves have been generated by the turbulent flow inside these plumes. Laboratory experiments provide the spatial evolution of these structures (*see* Turner 1986 for review) and their dimensions and velocity as they go down, as well as the relationship between the velocity of the mean flow and the mean square velocity of the turbulent flow inside the plume (List 1982).

However, the source of the gravity waves is the turbulent flow in the penetrative region, where the turbulent flow concentrated in the vertical plumes becomes horizontal and where the plumes disappear because of the strong stratification of the radiative zone. How can we describe this turbulent flow in that thin layer? What is the relationship with the turbulent flow in the plumes? We do not have the answer to these questions, so we shall try to approach the problem using some simple forms for the turbulence description.

In order to suggest forms of the velocity correlation, we guide ourselves by the results of isotropic, homogeneous, incompressible turbulence. These conditions certainly do not apply to our physical case, but it is only for this simple case that we have any guide as to the form of the correlation.

For incompressible, homogeneous, isotropic turbulence, the Fourier transform of the velocity correlation has the form (Batchelor 1953):

$$\Phi_{ij}(\mathbf{k},\omega) = \frac{E(\mathbf{k},\omega)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{38}$$

As we did above (appendix, Eq. A17), we consider that the turbulence energy spectrum $E(\mathbf{k},\omega)$ can be factorized into a spectrum $E(k)$ and a frequency-dependent factor $\Delta(\omega)$. For the k-spectrum we take a Kolmogorov spectrum $(k/k_m)^{-5/3}$, with a given k_m minimum wave number of the distribution. And for the frequency-dependent factor we have chosen an exponential distribution (Eq. A20) as indicated by some experimental studies (Anselmet et al., 1984). Furthermore, we consider that this frequency factor is independent of the wave number. Stein (1967) considered three different frequency-factors with the form of a smoothed delta function centered about $\omega = k\,u_k = (\tau_m (k/k_m)^{(2/3)})^{-1}$, with $\tau_m = (k_m\,u_m)^{-1}$. He considered an exponential distribution, and two Gaussian ones:

$$\Delta(\omega) = \tau_m \left(\frac{k}{k_m}\right)^{-2/3} \exp\left(-\omega\tau_m\left(\frac{k}{k_m}\right)^{-2/3}\right), \tag{39}$$

which is large for all frequencies smaller than ku_k and decreases rapidly for higher frequencies;

$$\Delta(\omega) = \frac{2}{\pi^{1/2}}\tau_m \left(\frac{k}{k_m}\right)^{-\frac{2}{3}} \exp\left(-\omega^2\tau_m^2\left(\frac{k}{k_m}\right)^{-\frac{4}{3}}\right), \tag{40}$$

which falls off faster at high frequencies, and

$$\Delta(\omega) = \frac{4}{\pi^{1/2}}\omega^2\tau_m^3 \left(\frac{k}{k_m}\right)^{-2} \exp\left(-\omega^2\tau_m^2\left(\frac{k}{k_m}\right)^{-\frac{4}{3}}\right), \tag{41}$$

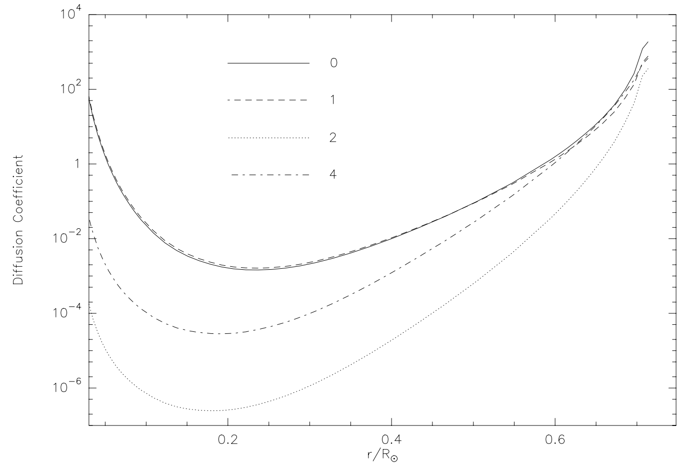


Fig. 4. Diffusion coefficient obtained using four different frequency spectrum, A20,39,40,41 and the same values of parameters k_m and ω_m.

which has a maximum at $\omega = ku_k$ and falls off for both lower and higher frequencies.

In order to explore the effects on the diffusion coefficient of the frequency dependence of the turbulent spectrum, we have represented the diffusion coefficient obtained with these four frequency-spectrum.

Let us now explore the form of the diffusion coefficient obtained with the different frequency dependences mentioned above. In Fig. 4 we represent the results obtained from Eq. A20, labeled by "0", and from Eqs. 39,40 and 41, labeled by "1", "2" and "3" respectively. We see that the fact of neglecting the dependence on k of frequency distribution is not very important, and the result is similar to the other exponential distribution (Eq. 39). However the behavior is very different when we take the Gaussian distributions. The diffusion coefficient decreases more rapidly in these cases, and its magnitude near the center of the Sun is very different for the three distributions proposed by Stein (1967).

Concerning the k-spectrum, we consider only the Kolmogorov spectrum, but there are uncertainties with respect to the characteristics scales. In previous works we considered the turbulent flow inside the plumes just before they find the boundary. Then, the largest eddy was determined by the radial dimension of the plume and the mean square velocity was related with the velocity at the center of the plume (see Schatzman & Montalbán 1995; Schatzman 1996). However, as we mentioned above, we do not know how large is the turbulent spectrum in the penetrative region. It is possible that the spatial dimension change with respect to the one inside the plume. In fact we could hope that the minimum k were given by the horizontal dimension, in that case, by the surface at the boundary of convective zone $(1.25\,10^{-10}\,\mathrm{cm}^{-1})$. We have made the computing of diffusion coefficient taking two different values of k_m and the corresponding $\omega_m = k_m\,u_m$ and with the frequency spectrum labeled "0".

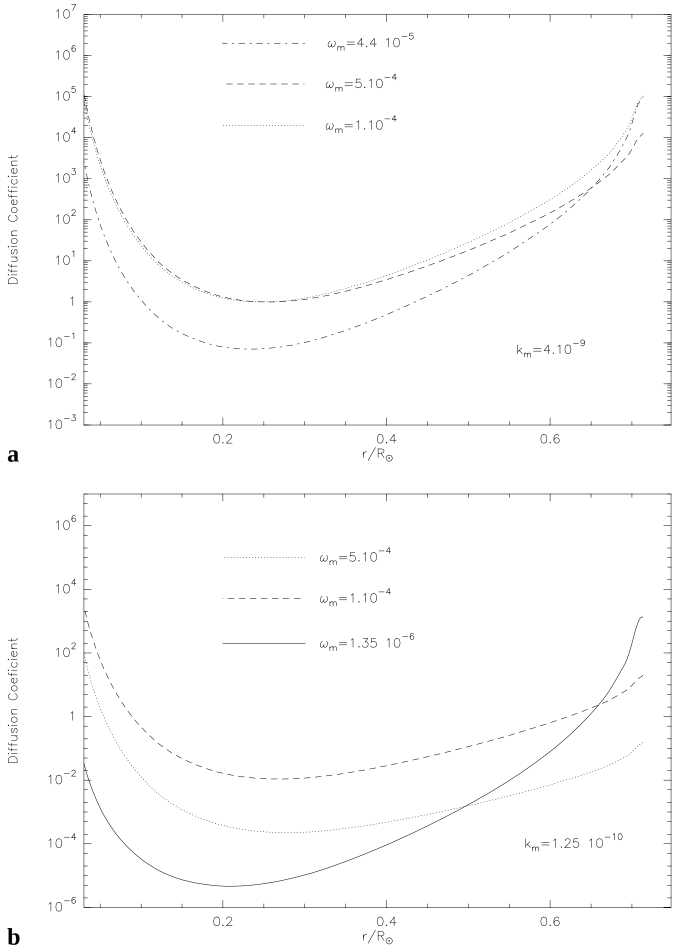
In Fig. 5. we show the effect of changing ω_m for a given value of k_m, and the effect of changing the maximum spatial dimension of the distribution of energy. We see that increasing the spatial scale from that one given by the plume to the one of the order of the radius of the convective zone, the diffusion coefficient decreases significatively, and that effect is even more important at the central region. The sensitivity of the diffusion coefficient to the minimum horizontal wavenumber of the turbulence distribution, and the fact that the light element abundances (sensitive to the mixing near the convective zone) were well justified by this transport process, taking into account the spatial scale of the perturbations associated to plumes dimension, could allow us to impose some constraints on spatial characteristics of the source functions of gravity waves. Concerning the time scale, the diffusion coefficient is much more sensitive to the frequency near the center than near the surface. So far we consider that the maximum of the frequency distribution was related also with the spatial scale by $k_m\, u_m$, but another relation is not impossible. In Fig. 5 it is also shown that when we change the value of ω_m from it value $4.4\,10^{-5}$ by a factor two, the diffusion coefficient change by a factor 50 at the depth 0.1 R$_\odot$.

5. Summary and conclusions

Diffusion coefficients obtained in previous works (Schatzman & Montalbán 1995; Schatzman 1996) are based on Press' (1981) approximation which is not completely valid in the most central region of the Sun. In order to approach accurately the problem of mixing in the regions where the nuclear reactions producing neutrinos take place, it is necessary to introduce an adequate treatment of the non-adiabatic propagation of internal waves in those regions. We have taken the quasi-adiabatic approach given by Unno et al. 1989, valid in the whole of the solar radiative region (Sec. 2). This treatment implies a decrease of radiative damping in the central region, and therefore, an increase of the internal waves amplitude in the solar center. However, the effect of radiative damping diminution on the diffusion coefficient at the solar center is opposite to what we need in order to reproduce Morel & Schatzman (1996) modelizations. In other words, the diffusion coefficients computed by Press' approximation and by quasi-adiabatic approach using the same turbulence description are very different in the neutrino production region. Instead of obtaining a diffusion coefficient that increases rapidly in order to allow a maximum at $r = 0.15 R_\odot$, we obtain one that decreases more rapidly and begin to increase more deeply in the Sun.

Since the damping term depends strongly on wave frequency and wavenumber, we have analyse the effect of the perturbation distribution at the convective boundary on the amplitude of movements at the solar center. We have considered several forms for the source of internal waves, taking into account different distributions of energy of perturbations as function of wavelength and frequency (Sect. 4). As we saw, at the stellar center, the value of diffusion coefficient linked to the internal waves propagation depends strongly on the turbulence description. The coherence with light elements depletion could impose some constraints on the turbulence description.

In this work we have only considered the contribution to the diffusion process of the propagating gravity waves and we have neglected the contribution from g-modes. In fact, we have con-



a

b

Fig. 5a and b. Dependence of Diffusion coefficient computed by Eq. (38) on characteristic frequency of turbulence distribution, for a fixed characteristic wavenumber **a** $k_m = 4.\,10^{-9}\mathrm{cm}^{-1}$ **b** $k_m = 1.25\,10^{-10}\mathrm{c}^{-1}$.

sidered that radiative damping can be so strong that reflection at the center is avoided, and in consequence there is no production of eigen-g-modes. However, we have seen in Sect. 3 that for several couples of wave number and frequency the amplitude of perturbations near the center is not negligible, and eigen-g-modes are possible. Therefore, their contribution should be taken into account. As we noted in Sect. 1, g-modes could have other effect in addition to the macroscopic diffusion. Gough (1992) took into consideration the possibility of a change of the central temperature when taking into account the non-linear effect due to g-modes. In his paper, he analysed the dependence of the neutrino flux on the amplitude of the the g-mode $l = 2$, that displays the maximum of the amplitude at $R = 0.15 R_\odot$. The result is that the neutrino flux is smaller when the amplitude of the mode is larger. A quantitative estimation of this effect implies to know the amplitude of these modes, and so, the characteristics of the excitation source.

We conclude that a complete analysis of the role of internal waves in the mixing in the central regions of the Sun should include not only the improvement of the treatment of internal waves propagation, but also a good characterization of the source function, the analyse of its ability to excitate g-modes, and the study of different effects of these modes.

Acknowledgements. We thank the anonymous referee for useful suggestions which helped to improve the presentation of the results.

Appendix A: derivation of the Eq. 38

The diffusion coefficient as functions of the variables in Fourier space is:

$$D_v = \int_0^\infty d(t'-t) \left\langle \frac{\partial}{\partial z}(TF^{-1}(\hat{u})) \int^t TF^{-1}(\hat{u})dt \right. \\ \left. \frac{\partial}{\partial z}(TF^{-1}(\hat{u}')) \int^{t'} TF^{-1}(\hat{u}')dt' \right\rangle \quad \text{(A1)}$$

The Fourier transform is given by the following expression:

$$TF(u(x,y,z,t)) = TF\left(\sum_i u_i(x-x_i, y-y_i, t, z)\right) = \\ \sum_i TF(u_i(x-x_i, y-y_i, t, z)) = \\ \sum_i \hat{u}_i(\mathbf{k},\omega,z)\exp(\imath\ell\cdot x_i + \imath m\cdot y_i) = \\ \sum_i \hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i} = \hat{u}(\mathbf{k},\omega,z) \quad \text{(A2)}$$

(with $\mathbf{x} \equiv (x,y)$).

The velocity can be expressed with the inverse Fourier transform:

$$u(x,y,z,t) = TF^{-1}\left(\sum_i \hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right) \quad \text{(A3)}$$

as well as the partial derivative of the velocity:

$$\frac{\partial u}{\partial z}(x,y,z,t) = TF^{-1}\left(\sum_i \frac{\partial\hat{u}_i}{\partial z}(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right) = \\ TF^{-1}\left(\sum_i \frac{k^3}{\omega^4}\frac{1}{2}\frac{\partial f}{\partial z}\hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right) = \\ \frac{1}{2}\frac{\partial f}{\partial z}TF^{-1}\left(\frac{k^3}{\omega^4}\sum_i \hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right) \quad \text{(A4)}$$

where the monochromatic component $\hat{u}_i(\mathbf{k},\omega,z)$ (using Eqs. 21 and 30) is written as:

$$\hat{u}_i \ (\mathbf{k},\omega,z) \sim \hat{u}_{\rm H}(\mathbf{k},\omega,z_b)\frac{|\omega|}{N_b}\left(\frac{r_b}{r}\right)^{\frac{3}{2}}\left(\frac{\rho_b}{\rho}\right)^{\frac{1}{2}} \quad \text{(A5)} \\ \left(\frac{(N^2/\omega^2-1)^{-\frac{1}{4}}}{(N_b^2/\omega^2-1)^{-\frac{1}{4}}}\right)\exp\left(-\frac{1}{2}f\frac{k^3}{\omega^4}\right)e^{-\imath(\mathbf{k}\cdot\mathbf{r}_i+\omega t)}.$$

$\hat{u}_{\rm H}(\mathbf{k},\omega,z_b)|\omega|/N_b$ is the vertical amplitude at the boundary. $\hat{u}_{\rm H}$, the horizontal amplitude, is taken equal to the turbulent mean square velocity in the convective zone generating the waves.

$$\int^t u\,dt = \int^t u(x,y,z,t) = TF^{-1}\left(\sum_i \hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right)dt \\ = TF^{-1}\left(\frac{1}{\imath\omega}\sum_i \hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right) \quad \text{(A6)}$$

We therefore have the expression of the vertical diffusion coefficient:

$$D_v = G^4F^2 \int_0^\infty d(t'-t) \quad \text{(A7)} \\ \left\langle TF^{-1}\left(\frac{k^3}{\omega^4}\sum_i \hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right)_t \cdot \right. \\ TF^{-1}\left(\frac{1}{\imath\omega}\sum_i \hat{u}_i(\mathbf{k},\omega,z)e^{\imath\mathbf{k}\cdot\mathbf{x}_i}\right)_t \cdot \\ TF^{-1}\left(\frac{k'^3}{\omega'^4}\sum_j \hat{u}_j(\mathbf{k}',\omega',z)e^{\imath\mathbf{k}'\cdot\mathbf{x}_j}\right)_{t'} \cdot \\ \left. TF^{-1}\left(\frac{1}{\imath\omega'}\sum_j \hat{u}_(\mathbf{k}',\omega',z)e^{\imath\mathbf{k}'\cdot\mathbf{x}_j}\right)_{t'} \right\rangle$$

where

$$G(r) = \left(\frac{r_b}{r}\right)^{3/2}\left(\frac{\rho_b}{\rho}\right)^{1/2}\left(\frac{(N^2/\omega^2-1)^{-1/4}}{(N_b^2/\omega^2-1)^{-1/4}}\right) \quad \text{(A8)}$$

and

$$F(r) = \frac{\partial f}{\partial r} = D_{\rm th}\left(N^2-\omega^2\right)^{1/2}N^2\left(\frac{r_b}{r}\right)^3 \\ \frac{\nabla_{\rm ad}}{\nabla}\left(\frac{\nabla_{\rm ad}-\nabla}{\Delta\nabla}\right) \quad \text{(A9)}$$

From now on $\hat{u}_i(\mathbf{k},\omega,z) \equiv \hat{u}_H(\mathbf{k},\omega,z_b)$, the turbulent motion at the boundary. In order to calculate the ensemble average, we follow the rule given by Knobloch (1977), which consists in replacing the ensemble average of four terms by the products of two ensemble averages of two terms.

This implies the presence of three products, but the integral taken over two of them does vanish. Therefore, the ensemble average of the four terms in Eq. (37) can be obtained as the product of two ensemble averages:

$$D_v = \int_0^\infty d(t' - t)\underbrace{\left\langle\frac{\partial u_z(t)}{\partial z}\int^{t'} u_z(t'){\rm d}t'\right\rangle}_{\langle 1\rangle} \\ \left\langle\frac{\partial u_z(t')}{\partial z}\int^t u_z(t){\rm d}t\right\rangle \quad \text{(A10)}$$

The first ensemble average is given by the following equation:

$$\langle 1\rangle \equiv \langle TF^{-1}|_t \cdot TF^{-1}|_{t'}\rangle \equiv \quad \text{(A11)}$$

$$\left\langle \int_{\infty}^{\infty} \cdots \int \frac{|k|^3}{\omega^4} \frac{|\omega|}{N_b} \frac{|\omega'|}{N_b} \frac{1}{\imath\omega'} \exp\left(-\frac{1}{2} f\left(\frac{|k|^3}{\omega^4} + \frac{|k'|^3}{\omega'^4}\right)\right) \right.$$
$$\sum_i \sum_j \hat{u}_i(\mathbf{k},\omega,z_b)\hat{u}_j(\mathbf{k}',\omega',z_b) e^{\imath\left(\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x})+\mathbf{k}'\cdot(\mathbf{x}_j-\mathbf{x})\right)}$$
$$\left. \exp\left(-\imath(\omega+\omega')t + \imath(t-t')\omega'\right) d\mathbf{k}\cdot d\mathbf{k}'\cdot d\omega\cdot d\omega' \right\rangle =$$
$$\int_{\infty}^{\infty} \cdots \int \frac{|k|^3}{\omega^4} \frac{|\omega|}{N_b} \frac{|\omega'|}{N_b} \frac{1}{\imath\omega'} \exp\left(-\frac{1}{2} f\left(\frac{|k|^3}{\omega^4} + \frac{|k'|^3}{\omega'^4}\right)\right)$$
$$\left\langle \sum_i^{N_{pl}} \hat{u}_i(\mathbf{k},\omega,z_b) e^{\imath\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x})} \cdot \sum_j^{N_{pl}} \hat{u}_j(\mathbf{k}',\omega',z_b) e^{\imath\mathbf{k}'\cdot(\mathbf{x}_j-\mathbf{x})} \right\rangle$$
$$\exp\left(-\imath(\omega+\omega')t + \imath(t-t')\omega'\right) d\mathbf{k}\cdot d\mathbf{k}'\cdot d\omega\cdot d\omega'$$

with

$$\left\langle \sum_i^{N_{pl}} \hat{u}_i(\mathbf{k},\omega,z_b) e^{\imath\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x})} \cdot \sum_j^{N_{pl}} \hat{u}_j(\mathbf{k}',\omega',z_b) e^{\imath\mathbf{k}'\cdot(\mathbf{x}_j-\mathbf{x})} \right\rangle$$
$$= \langle \hat{u}(\mathbf{k},\omega)\cdot\hat{u}(\mathbf{k}',\omega')\rangle = \qquad \text{(A12)}$$
$$\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\langle u^2\rangle TF\left(R(\mathbf{x}_i-\mathbf{x}',t-t')\right)\cdot$$
$$\left\{\sum_{i=j} \exp\left(\imath(\mathbf{k}+\mathbf{k}')(\mathbf{x}_i-\mathbf{x})\right) + \sum_{i\neq j} \exp\left(\imath(\mathbf{x}_i-\mathbf{x})\mathbf{k} + \imath(\mathbf{x}_j-\mathbf{x})\mathbf{k}'\right)\right\}$$

the term between brackets tends to N_{pl} when N_{pl} is large and when $\mathbf{k} = -\mathbf{k}'$:

$$\{\cdots\} \equiv N_{pl} + \sum_{i\neq j}^{N_{pl}} \exp\left(\imath(\mathbf{x}_i-\mathbf{x}_j)\mathbf{k}\right) \to N_{pl}. \qquad \text{(A13)}$$

So,

$$\begin{aligned} \langle \hat{u},\hat{u}\rangle &= \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\langle u^2\rangle N_{pl} \qquad \text{(A14)} \\ &\quad TF\left(R(\mathbf{x}-\mathbf{x}',t-t')\right) = \\ &\quad \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')N_{pl} \\ &\quad TF\left(\langle (u(\mathbf{x},t),u(\mathbf{x}',t'))\rangle\right), \end{aligned}$$

and

$$\begin{aligned} \langle (u(\mathbf{x},t),u(\mathbf{x}',t'))\rangle &= \langle u^2\rangle R(\mathbf{x}-\mathbf{x}',t-t') = \\ &\quad TF^{-1}\left(\Phi(\mathbf{k},\omega)\right), \qquad \text{(A15)} \end{aligned}$$

where $\Phi(\mathbf{k},\omega)$ is the spectrum of turbulence. If we can factorize the auto-correlation function as:

$$R(\mathbf{x}-\mathbf{x}',t-t') \equiv \Gamma(\mathbf{x}-\mathbf{x}')\,\Gamma(t-t'), \qquad \text{(A16)}$$

and

$$\Phi(\mathbf{k},\omega) \equiv \hat{U}(\mathbf{k})\cdot\Psi(\omega) \qquad \text{(A17)}$$

$$\langle (\hat{u}(\mathbf{k},\omega)\ ,\ \hat{u}(\mathbf{k}',\omega')\rangle = \delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')\frac{N_{pl}}{\left(\sqrt{2\pi}\right)^{n+1}}$$
$$\int_{\infty}^{\infty} \cdots \int \Gamma(\mathbf{x}-\mathbf{x}') e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')} d(\mathbf{x}-\mathbf{x}')$$
$$\int_{\infty}^{\infty} \cdots \int \Gamma(t-t') e^{-\imath\omega(t-t')} d(t-t') \qquad \text{(A18)}$$

from Lesieur (V-5-4):

$$\langle u^2\rangle \int_{\infty}^{\infty} \cdots \int \Gamma(\mathbf{x}-\mathbf{x}') e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')} d(\mathbf{x}-\mathbf{x}') \equiv \hat{U}(\mathbf{k}), \qquad \text{(A19)}$$

and the mean square velocity for 3D and 2D turbulence is given by:

$$\langle u^2\rangle = \int_{\infty}^{\infty} \hat{U}(\mathbf{k}) d\mathbf{k} = \begin{cases} \int_0^\infty 4\pi k^2 \hat{U}(\mathbf{k}) dk & \text{3D} \\ \int_0^\infty 2\pi k \hat{U}(\mathbf{k}) dk & \text{2D} \end{cases}$$

Furthermore, for the dependence on frequency of the turbulence spectrum, we chose an exponentiel one:

$$\int_{\infty}^{\infty} \cdots \int \Gamma(t-t') e^{-\imath\omega(t-t')} d(t-t') = \tau e^{-\tau|\omega|} \equiv \Psi(\omega), \qquad \text{(A20)}$$

and,

$$\Gamma(t-t') \equiv \left(\frac{2}{\pi}\right)^{1/2} \frac{\tau^2}{\tau^2+(t-t')^2} = \frac{(2/\pi)^{1/2}}{1+\left(\frac{t-t'}{\tau}\right)^2}. \qquad \text{(A21)}$$

So, the expression $\langle 1\rangle$ takes the following aspect:

$$\langle 1\rangle \equiv \int_{\infty}^{\infty} \cdots \int \langle u^2\rangle \frac{|k|^3}{\omega^3} \quad \frac{N_{pl}}{N_b^2} \exp\left(-f\frac{|k|^3}{\omega^4}\right)$$
$$\hat{U}(\mathbf{k})\Phi(\omega) e^{-\imath\omega(t-t')} d\mathbf{k}\, d\omega \qquad \text{(A22)}$$

from Lessieur (p.111), for 3D case, $\hat{U}(k) \propto E(k)/2\pi k^2$, with $E(k)$ given by a Kolmogorov law $E(k) \propto (k/k_m)^{-5/3}$. Then the expression for $\hat{U}(k)$ is:

$$\hat{U}(k) \propto \frac{1}{2\pi}(k_m)^{5/3} k^{-11/3} \equiv \frac{1}{2\pi} k_m^{-2} \left(k/k_m\right)^{-11/3}$$

which leads to a new expression of $\langle 1\rangle$:

$$\langle 1\rangle \equiv \langle u^2\rangle \frac{N_{pl}}{N_b^2} k_m^{-2} \tau \qquad \text{(A23)}$$
$$\underbrace{\int_0^{\infty} \int \frac{k^3}{\omega^3} \exp\left(-f\frac{k^3}{\omega^4}\right)\left(\frac{k}{k_m}\right)^{-\frac{11}{3}} e^{-\tau\omega} e^{-\imath\omega(t-t')} k dk\, d\omega}_{\langle 2\rangle}$$

From Eq. (A8) and Eq. (A10) we hve a new expression of the diffusion coefficient:

$$\begin{aligned} D_v &= G^4(r)F^2(r)\int_0^{\infty} d(t-t')[\langle 1\rangle]^2 \equiv \qquad \text{(A24)} \\ &\quad G^4(r)F^2(r)\left(\langle u^2\rangle \frac{N_{pl}}{N_b^2}\tau k_m^{-2}\right)^2 \int_0^{\infty} [\langle 2\rangle]^2 d(t-t') \end{aligned}$$

$$\langle 2\rangle \equiv k_m^{11/3} \int_0^{\infty} \frac{e^{-\tau\omega}}{\omega^3} \left(\frac{3}{4}\int_{k_m}^{\infty} d\left(k^{4/3}\right) \exp\left(-f\frac{k^3}{\omega^4}\right)\right)$$
$$e^{-\imath\omega(t-t')} d\omega. \qquad \text{(A25)}$$

Introducing a new variable $x \equiv fk^3/\omega^4$, its value at k_m, $x_m \equiv fk_m^3/\omega^4$; and using the function $\gamma(4/9,x_m)$:

$$\gamma(4/9,x_m) = \int_{x_m}^{\infty} d\left(x^{4/9}\right) e^{-x} \equiv \frac{4}{9}\Gamma(4/9,x_m)$$

with Γ being here the incomplete Gamma Function, the diffusion coefficient takes the following expression:

$$D_v = G^4(r)F^2(r)\left(\langle u^2\rangle \frac{N_{pl}}{N_b^2}\tau k_m^{-2}\right)^2 \left(\frac{3}{4}k_m^{\frac{11}{3}} f^{-\frac{4}{9}}\right)^2$$
$$\int_0^\infty d(t-t')\left[\int_0^\infty d\omega\, \omega^{-\frac{11}{9}} e^{-\tau\omega}\gamma\left(\frac{4}{9},\frac{fk^3}{\omega^4}\right) e^{-\imath\omega(t-t')}\right]^2$$

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