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Propagation of nonlinear longitudinal-transverse waves along magnetic flux tubes in the solar atmosphere

III. Modified equation of motion

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Abstract. In this series of papers the time-dependent propagation of nonlinear longitudinal-transverse waves in thin vertical magnetic flux tubes embedded in the solar atmosphere is investigated numerically using the (one-dimensional) thin tube approximation. As in the last decade the particular form of the backreaction term in the transverse equation of motion has been under considerable dispute we investigate this issue once again and suggest a new expression for the backreaction term in the local approximation. This new expression intends to avoid criticisms leveled at the previous terms. In the present paper we numerically compare the actual effects these different terms produce in a number of cases including the situations where the reported discrepancies are prominent. We find that noticeable discrepancies between the various proposed backreaction terms occur only, when strong longitudinal fluid flows are present simultaneously with the swaying of the tube and that for weak flows these discrepancies disappear. If only weak longitudinal flows occur it thus appears that the particular choice of the backreaction term is not important.

Key words: Magnetohydrodynamics (MHD) – methods: numerical – Sun: magnetic fields – Sun: chromosphere – Sun: corona

1. Introduction

A powerful approach to study the behavior of waves in intense magnetic flux tubes observed on the Sun is the *thin tube approximation*, which considers the tube as a one-dimensional (1D) sequence of mass elements, where the physical variables do not change much across the tube from their values on the tube axis. In this approach one usually assumes further that the flux tube is embedded in an otherwise nonmagnetic, static solar atmosphere obeying horizontal pressure balance and that the motion of the tube, apart of displacing matter, does not perturb much the outside atmosphere. With these assumptions a formidable time-dependent problem to treat magnetic tubes in three dimensions in an external atmosphere which varies in all three spatial directions can be reduced to a more easily manageable one-dimensional problem. The properties of waves in thin flux tubes and the use of the thin tube approximation have been extensively discussed both in time-independent (e.g. Defouw, 1976; Roberts & Webb, 1978, 1979; Wilson, 1979; Parker, 1979; Wentzel, 1979; Spruit, 1981, 1982; Rae & Roberts, 1982; Edwin & Roberts, 1983) and time-dependent investigations (e.g. Herbold et al., 1985; Molotovshchikov & Ruderman, 1987; Ferriz-Mas et al., 1989; Ulmschneider et al., 1991, henceforth called Paper I, Zhugzhda et al., 1995, henceforth called Paper II, and Moreno-Insertis et al., 1996).

In Paper I, following Spruit (1981), the magnetohydrodynamic equations in the thin tube approximation were solved using the method of characteristics in order to study the time-dependent propagation of longitudinal-transverse waves along solar magnetic flux tubes. In this treatment the backreaction term in the equation of motion developed by Spruit (1981) was used which subsequently has been criticized in several ways and by different authors. Choudhuri (1990), Cheng (1992), Fan et al. (1994) as well as Moreno-Insertis et al. (1996) pointed out that the centrifugal and Coriolis force terms in the transverse equation of motion should be discarded. The latter authors moreover claim that the whole local 1D treatment of the backreaction is inappropriate. In addition to these criticisms, three-dimensional simulations of kink waves in magnetic flux tubes by Ziegler & Ulmschneider (1997) find that even for very slender tubes one always has important internal structure. This would be in disagreement with the basic assumption of the thin tube approximation that for slender enough tubes the physical state in a tube section is given essentially by the values at the tube axis. Yet the work of Ziegler & Ulmschneider (1997) did not investigate the low plasma $\beta = 8\pi p/B^2$ tube cases such that more detailed three-dimensional (3D) studies must be carried out to settle the above questions.

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While clearly a fully 3D treatment of the magnetic flux tube dynamics would make many of the above discussions obsolete, it should be kept in mind that the elegant thin flux tube approximation, applied in an appropriate setting, may well give a reasonable description of the essential physics of a given problem, without requiring the very much larger computational effort of the 3D simulation. This is particularly so because it may not be wise to concentrate on the one hand on an extremely accurate magnetohydrodynamics, when on the other hand the radiative emission, which in some situations is often of much greater importance, is badly treated or even neglected. The radiation from thin magnetic flux tubes in the outer stellar atmospheres unavoidably is a 3D problem which is compounded by the fact that these tubes are prime candidates for departures from thermodynamic equilibrium. To get a reasonable physical picture it might be much better to spend the available computational power for the modelling of the NLTE line and continuum radiation losses, rather than for a more accurate magnetohydrodynamic simulation. There are thus powerful reasons to continue to investigate the feasibility and accuracy of the thin tube approximation.

From the above discussion of the various forms of the backreaction term, one might get the impression that all computations which have used these terms are unreliable or obsolete. This picture is incorrect. In the present paper we want to show that the various proposed backreaction terms in the transverse equation of motion do not differ much in their effect on the wave motion if the axial mass flows in the tubes are minor. We thus feel that the thin tube approximation may provide a reasonable method for the treatment of waves along magnetic flux tubes in low β cases and situations where longitudinal flows are not very important. Naturally, this approximation and the best way to represent the backreaction must be critically tested with more extensive 3D calculations and in various astrophysical situations. The paper is organized as follows: In Sect. 2 we briefly review the backreaction force terms suggested by different authors, while in Sect. 3 we compare the importance of these terms in a number of transverse wave computations. Our conclusions are presented in Sect. 4. Some additional discussion on backreaction is given in the Appendix.

2. Proposed backreaction terms

2.1. Definitions and tube geometry

Embedded in a gravitational nonmagnetic atmosphere, as shown in Fig. 1, consider a very thin, roughly vertically oriented magnetic flux tube with unresolved cross-section, which at various times t_k is shown as a solid line. Following Moreno-Insertis et al. (1996) one can also draw a system of curves $\{s_k\}$ (dashed) which is orthogonal to the system $\{t_k\}$. The three-dimensional motion of a gas element inside this tube can be specified by the radius vector $\boldsymbol{r}(a,t)$. Here a is the *Lagrange-length*, which is the arc-length of the fluid element along the tube measured from a reference point $a=0$ at the initial time $t=t_0$.

Although the *arc-length* l along the tube is easily determined for tubes at any given time, it is difficult to use l as a basic variable. Holding l constant in a differentiation with respect

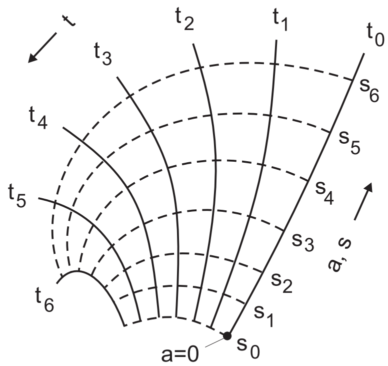

Fig. 1. Thin magnetic tube at time t_0 and at a later times $t_1, \cdots, t_6$. The lines $s_i = \mathrm{const}, i = 0, \cdots, 6$, are orthogonal to the momentary tube axis.

to time, for instance, has various meanings, since there is no obvious way to define a zero reference level $l=0$ at different times. The zero reference position $l=0$ may be taken at the point where the tube crosses the stellar surface $z=0$, or at the position where the moving mass element $a=0$ is at a given time, or it could be taken at the line $s_0 = const$. Since $l=0$ cannot be defined in an obvious manner we decided to avoid expressions used e.g. by Cheng (1992), where one has to hold $l = const$. On the other hand, differences Δl are well defined, such that differentiations with respect to l, holding $t = const$, are unambiguous.

We also use a third variable, which henceforth is called *stall-length* s, (see Fig. 1). It is identical to the parameter A of Moreno-Insertis et al. (1996). The families of curves $\{t_k\}$ and $\{s_k\}$ can be considered as the coordinate lines of an orthogonal curvilinear coordinate system in 3D space. The stall-length s is taken to be identical to the Lagrange-length a at the initial time $t=t_0$, where $s_0 = 0$ is identified with $a=0$. The stall-length s identifies geometrical points while the Lagrange-length a denotes mass elements. Note that the stall-length s has a well-defined zero line s_0, and that the distance Δl between two $s = const$ lines can decrease to zero. This occurs when the tube attains a strong bend which ultimately leads to a point with a discontinuity in the slope. Such transverse shock forming situations have been considered in Paper II. Therefore, s is a purely geometrical quantity, very different from l. The reason for calling s stall-length is that in the absence of parallel flows, that is, in 'stalled' fluids, one has $\boldsymbol{v}^{\|} = 0$ and the lines $s = const$ become fluid paths $a = const$.

The tube orientation is described by the three unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, where the vector $\mathbf{e}_1$ along the tube axis is defined by

$$\mathbf{e}_1 = \left(\frac{\partial \boldsymbol{r}}{\partial l}\right)_t \quad . \tag{1}$$

Note that in Paper I this vector was denoted by $\mathbf{l}$. Vector $\mathbf{e}_2$ is the unit vector in the direction of the local center of curvature, the principal normal direction $\boldsymbol{k} = \kappa \mathbf{e}_2$, and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ is in the binormal direction.

The variation of the unit vectors $\mathbf{e}_i$ with the arc-length l is described by the Frenet-Serret relations (Spiegel, 1959)

$$\left(\frac{\partial \mathbf{e}_1}{\partial l}\right)_t = \kappa \mathbf{e}_2 \ ,$$
$$\left(\frac{\partial \mathbf{e}_2}{\partial l}\right)_t = -\kappa \mathbf{e}_1 + \tau \mathbf{e}_3 \ , \quad (2)$$
$$\left(\frac{\partial \mathbf{e}_3}{\partial l}\right)_t = -\tau \mathbf{e}_2 \ .$$

Here $1/\kappa$ is the radius of curvature and $1/\tau$ is the radius of torsion. Using the Darboux vector

$$\boldsymbol{\omega}_D = \tau \mathbf{e}_1 + \kappa \mathbf{e}_3 \quad (3)$$

the Frenet-Serret relations can be written as

$$\left(\frac{\partial \mathbf{e}_i}{\partial l}\right)_t = \boldsymbol{\omega}_D \times \mathbf{e}_i \ . \quad (4)$$

In a similar fashion we can introduce $\boldsymbol{\omega}$, the angular velocity of a fluid particle

$$\left(\frac{\partial \mathbf{e}_i}{\partial t}\right)_a = \boldsymbol{\omega} \times \mathbf{e}_i \ . \quad (5)$$

2.2. *Backreaction terms employed in previous work*

There are two distinct types of external forces which arise when a magnetic flux tube moves through the external medium, one is a function of velocity and the other is due to the acceleration of the moving tube. The two forces are independent and should be treated on an equal footing. One pictures a flow with a developed wake behind the moving tube and a laminar *shell flow* around the tube-plus-wake system. The total force is then the sum of the drag caused by the generation of wake vortices (the *drag force*) and the force due to the fact that the external medium has to be accelerated to move out of the way (the *acceleration reaction force*). As in all treatments discussed in this paper the drag forces are neglected, we concentrate on the acceleration reaction forces.

Spruit (1981), assuming the external flow to be potential and incompressible, took for the backreaction of the fluid the expression

$$\mathbf{f}_{\rm acc} = -\rho_e \left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} \ , \quad (6)$$

where d/dt is the substantial derivative, ρ_e the density outside the tube and $\perp$ denotes vector components perpendicular to the tube. This expression is correct for the simple transverse motion of a straight flux tube. However, it is important to distinguish between the motion of the fluid inside the tube and that of the tube through the outside medium. Only the latter causes a backreaction. This important difference was overlooked when using Eq. (6) in Paper I.

Choudhuri (1990) noticed that internal fluid motion along a curved section of the tube at rest or during uniform translation would give a contribution when using Eq. (6), although such flows should not produce a backreaction of the surrounding medium. As Eq. (6) thus overestimates the backreaction force, Choudhuri suggested to subtract a term corresponding to the '*fluid turning around the arc*':

$$\mathbf{f}_{\rm acc} = -\left[\rho_e \left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} - \rho_e v_C^2 \boldsymbol{k}\right] \ . \quad (7)$$

where $v_C = -\kappa^{-2}\mathbf{e}_1 \cdot (\partial^2 \boldsymbol{v}/\partial l^2)$ and $\boldsymbol{k} = \kappa \mathbf{e}_2$.

Cheng (1992) later pointed out that in some cases Choudhuri's correction is incomplete and proposed a more general expression for the backreaction term. Cheng distinguished between the 'velocity of the tube element $\boldsymbol{v}_T = (\partial \boldsymbol{r}/\partial t)_l$', supposing that the tube element can be defined by $l = const$, and the 'velocity of the fluid element relative to the tube $\boldsymbol{v}_R$'. It can be said that while Choudhury pointed out that the *centrifugal acceleration* of a fluid particle in the tube should not be included in the backreaction, Cheng noted that also the *Coriolis acceleration* should not contribute. On the basis of Cheng's expressions (32) and (38) we conclude that Cheng (1992) suggested to take

$$\mathbf{f}_{\rm acc} = -\rho_e \left(\frac{\partial \boldsymbol{v}_T}{\partial t}\right)_l^{\perp} \ . \quad (8)$$

Fan et al. (1994) in their study of the dynamics of emerging flux loops noted that only the transverse velocity should be taken into account in the derivation of the added inertia term and suggested to include a term in the form

$$\mathbf{f}_{\rm acc} = -\rho_e \left(\frac{d\boldsymbol{v}^{\perp}}{dt}\right)^{\perp} \ . \quad (9)$$

Here we have written their term for a nonrotating star, while in case of rotation the velocity $\boldsymbol{v}$ in Eq. (9) is replaced by $\boldsymbol{v} - \boldsymbol{v}_e$ where $\boldsymbol{v}_e = \boldsymbol{\omega} \times \boldsymbol{r}$.

Moreno-Insertis et al. (1996) considered a backreaction term in the local approximation which in the case of external fluid at rest ($\boldsymbol{v}_e = 0$) can be written as

$$\mathbf{f}_{\rm acc} = -\rho_e \left(\frac{d_{\perp}\boldsymbol{v}^{\perp}}{dt}\right) \ , \quad (10)$$

where $\mathrm{d}_{\perp}/\mathrm{dt} = (\partial/\partial t)_s$ is the perpendicular derivative (see Eq. (A4) in the Appendix). Note that in Eqs. (6) - (8) it was also assumed that the tube moves in an outside medium at rest ($\boldsymbol{v}_e = 0$). Moreno-Insertis et al. subsequently discredited their expression because the longitudinal component present in their backreaction term cannot be physically supported. This, they believe, proves that the entire local approximation approach is inadequate.

Finally, as discussed in the Appendix (see also Roberts & Ulmschneider 1997) we suggest a backreaction term

$$\mathbf{f}_{\rm acc} = -\rho_e \left(\frac{d_{\perp}\boldsymbol{v}^{\perp}}{dt}\right)^{\perp} \ . \quad (11)$$

3. Method

To compare the different versions of backreaction terms we now derive the equations which were used to implement these terms.

3.1. Transverse equation of motion

From the full equation of motion

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla p - \frac{1}{4\pi}\boldsymbol{B}\times(\nabla\times\boldsymbol{B}) + \rho\mathbf{g}\ , \qquad (12)$$

the transverse component is given by

$$\rho\left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} = \rho\left[\frac{d\boldsymbol{v}}{dt} - \mathbf{e}_1\cdot\frac{d\boldsymbol{v}}{dt}\mathbf{e}_1\right] = \frac{B^2}{4\pi}\kappa\mathbf{e}_2 + \mathbf{e}_1\times\left[\rho\mathbf{g} - \nabla\left(p+\frac{B^2}{8\pi}\right)\right]\times\mathbf{e}_1\ . \qquad (13)$$

Assuming horizontal pressure balance

$$p + \frac{B^2}{8\pi} = p_e \quad , \qquad (14)$$

taking the atmosphere outside the tube to be in hydrostatic equilibrium ($\nabla p_e = \rho_e\mathbf{g}$) and using the Alfvén speed $c_A = B/\sqrt{4\pi\rho}$, we have

$$\rho\left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} = \rho c_A^2\kappa\mathbf{e}_2 + (\rho-\rho_e)\mathbf{g}^{\perp} \ . \qquad (15)$$

This equation does not include forces due to the backreaction of the external medium. The complete Eq. (15) neglecting drag forces thus reads

$$\rho\left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} = \rho c_A^2\kappa\mathbf{e}_2 + (\rho-\rho_e)\mathbf{g}^{\perp} + \mathbf{f}_{\rm acc}\ , \qquad (16)$$

where for $\mathbf{f}_{\rm acc}$ we take one of the expressions of Eqs. (6) to (11). With our suggested backreaction term Eq. (16) now reads

$$\rho\left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} + \rho_e\left(\frac{d_\perp\boldsymbol{v}^{\perp}}{dt}\right)^{\perp} = \rho c_A^2\kappa\mathbf{e}_2 + (\rho-\rho_e)\mathbf{g}^{\perp} \ . \qquad (17)$$

Since we use a Lagrangean description, we are interested in the substantial derivative

$$\frac{d\boldsymbol{v}}{dt} = \frac{d_\perp\boldsymbol{v}}{dt} + v^{\|}\frac{\partial\boldsymbol{v}}{\partial l} = \frac{d_\perp\boldsymbol{v}^{\perp}}{dt} + \frac{d_\perp v^{\|}}{dt}\mathbf{e}_1 + v^{\|}\frac{d_\perp\mathbf{e}_1}{dt} + v^{\|}\frac{\partial\boldsymbol{v}}{\partial l} \ . \qquad (18)$$

Taking into account that

$$\left(\frac{\partial\boldsymbol{v}}{\partial l}\right)_t = \frac{1}{l_a}\frac{\partial^2\boldsymbol{r}(a,t)}{\partial a\partial t} = \left(\frac{\partial\mathbf{e}_1}{\partial t}\right)_a = \frac{d\mathbf{e}_1}{dt} = \boldsymbol{\omega}\times\mathbf{e}_1 \ , \qquad (19)$$

where $l_a = (\partial l/\partial a)_t$ is the scale factor defined in Eq. (12) of Paper I, and introducing the angular velocity $\boldsymbol{\omega}_s$ of the 'tube element' we find

$$\left(\frac{\partial\mathbf{e}_1}{\partial t}\right)_s \stackrel{\rm def}{=} \boldsymbol{\omega}_s\times\mathbf{e}_1 = \frac{d_\perp\mathbf{e}_1}{dt} = \frac{d\mathbf{e}_1}{dt} - v^{\|}\frac{\partial\mathbf{e}_1}{\partial l} = \boldsymbol{\omega}\times\mathbf{e}_1 - v^{\|}\kappa\mathbf{e}_2 \ . \qquad (20)$$

With this, Eq. (18) can now be written as

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{d_\perp\boldsymbol{v}^{\perp}}{dt}\right) + \left(\frac{d_\perp v^{\|}}{dt}\right)\mathbf{e}_1 + v^{\|2}\boldsymbol{k} + 2(\boldsymbol{\omega}_s\times\boldsymbol{v}^{\|})\ , \qquad (21)$$

or

$$\frac{d\boldsymbol{v}}{dt} = \left(\frac{d_\perp\boldsymbol{v}^{\perp}}{dt}\right) + \left(\frac{d_\perp v^{\|}}{dt}\right)\mathbf{e}_1 - v^{\|2}\boldsymbol{k} + 2(\boldsymbol{\omega}\times\boldsymbol{v}^{\|})\ . \qquad (22)$$

Note that Eq. (21) demonstrates that the total absolute acceleration of a fluid particle, using the frame of reference co-moving with a tube element (at $s = const$), is expressed as the sum of the *translational, relative (tangential plus centrifugal)* and *Coriolis* accelerations. Comparing Eq. (21) with (11) one can also say that the fluid backraction is based on the normal component of the translational acceleration. Finally, Eq. (17) becomes

$$(\rho+\rho_e)\left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} = (\rho c_A^2 + \rho_e v^{\|2})\kappa\mathbf{e}_2 + 2\rho_e v^{\|}(\boldsymbol{\omega}_s\times\mathbf{e}_1) + (\rho-\rho_e)\mathbf{g}^{\perp}, \qquad (23)$$

or, with Eq. (22)

$$(\rho+\rho_e)\left(\frac{d\boldsymbol{v}}{dt}\right)^{\perp} = (\rho c_A^2 - \rho_e v^{\|2})\kappa\mathbf{e}_2 + 2\rho_e v^{\|}(\boldsymbol{\omega}\times\mathbf{e}_1) + (\rho-\rho_e)\mathbf{g}^{\perp}. \qquad (24)$$

Although Eq. (23) looks essentially identical to Eq. (38) of Cheng (1992), this is only a formal resemblance as the parameter s used to define the backreaction force in our case is quite different from the arc-length l used by Cheng (see Appendix).

3.2. Characteristics

We now write down the full set of equations and show them in the characteristic form taken for our time-dependent computations using the method of characteristics. As only the transverse equation of motion (Eq. (24)) is new, the remaining equations can be taken from Paper I. We want to have the set of equations in the Lagrangean frame with a and t as independent variables. Using scale factor l_a defined in Eq. (12) of Paper I we can get from Eqs. (2), (5)

$$\kappa\mathbf{e}_2 = \frac{1}{l_a}\left(\frac{\partial\mathbf{e}_1}{\partial a}\right)_t \ . \qquad (25)$$

$$\boldsymbol{\omega}\times\mathbf{e}_1 = \left(\frac{\partial\mathbf{e}_1}{\partial t}\right)_a \ . \qquad (26)$$

With these relations the *transverse* component of the equation of motion (Eq. (24)) is written

$$\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_a - \mathbf{e}_1\cdot\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_a\mathbf{e}_1 - \frac{(\rho c_A^2 - \rho_e v^{\|2})}{(\rho+\rho_e)l_a}\left(\frac{\partial\mathbf{e}_1}{\partial a}\right)_t - \frac{2\rho_e v^{\|}}{(\rho+\rho_e)}\left(\frac{\partial\mathbf{e}_1}{\partial t}\right)_a - \frac{(\rho-\rho_e)}{(\rho+\rho_e)}\mathbf{g}^{\perp} = 0\ . \qquad (27)$$

The *longitudinal* component of the equation of motion is taken from Eq. (27) of Paper I

$$\mathbf{e}_1\cdot\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_a + \frac{1}{l_a}\left[\frac{2c_S}{\gamma-1}\left(\frac{\partial c_S}{\partial a}\right)_t - \frac{c_S^2\,\mu}{\gamma\mathcal{R}}\left(\frac{\partial S}{\partial a}\right)_t\right] + gl_z = 0\ . \qquad (28)$$

Here, as in Paper I, l_x, l_y, l_z are the direction cosines of $\mathbf{e}_1$ with respect to the unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. The *transverse and longitudinal* components of the combined induction and continuity equations are given by Eqs. (29), (32) of Paper I

$$\left(\frac{\partial \mathbf{e}_1}{\partial t}\right)_a = \frac{1}{l_a}\left[\left(\frac{\partial \boldsymbol{v}}{\partial a}\right)_t - \mathbf{e}_1 \cdot \left(\frac{\partial \boldsymbol{v}}{\partial a}\right)_t \mathbf{e}_1\right] , \tag{29}$$

$$\frac{\mathbf{e}_1}{l_a} \cdot \left(\frac{\partial \boldsymbol{v}}{\partial a}\right)_t + \frac{2c_S}{\gamma - 1}\frac{1}{c_T^2}\left(\frac{\partial c_S}{\partial t}\right)_a - \frac{v_z}{\rho c_A^2}\frac{dp_e}{dz} - \frac{\mu}{\gamma \mathcal{R}}\left(\frac{c_S^2}{c_A^2} + \gamma\right)\left(\frac{\partial S}{\partial t}\right)_a = 0 \ . \tag{30}$$

where the tube speed c_T is defined by

$$\frac{1}{c_T^2} = \frac{1}{c_S^2} + \frac{1}{c_A^2} \ . \tag{31}$$

Here we have eliminated thermodynamic variables in favor of c_S and entropy S using the relations

$$d\rho = \rho\left(\frac{2}{\gamma - 1}\frac{dc_S}{c_S} - \frac{\mu}{\mathcal{R}}dS\right) \ , \tag{32}$$

$$dp = \rho\left(\frac{2c_S}{\gamma - 1}dc_S - \frac{\mu c_S^2}{\gamma \mathcal{R}}dS\right) \ . \tag{33}$$

In addition to the above equations we use the *entropy equation* (34) from Paper I

$$\left(\frac{\partial S}{\partial t}\right)_a = \left.\frac{dS}{dt}\right|_{\mathrm{Rad}} , \tag{34}$$

where for our present adiabatic case $dS/dt|_{\mathrm{Rad}} = 0$.

Combining the Eqs. (28), (30) for the longitudinal components and Eq. (34) we find

$$\mathbf{e}_1 \cdot d\boldsymbol{v} \pm \frac{2}{\gamma - 1}\frac{c_S}{c_T}dc_S \mp \frac{\mu c_S^2}{\gamma \mathcal{R} c_T}dS \mp \left[\frac{\mu c_T}{\gamma \mathcal{R}}(\gamma - 1)\left.\frac{dS}{dt}\right|_{\mathrm{Rad}} + \frac{v_z c_T}{\rho c_A^2}\frac{dp_e}{dz} \mp g l_z\right] dt = 0 , \tag{35}$$

valid along the characteristics C_1^+, C_1^- given by

$$C_1^{\pm} = \pm\frac{c_T}{l_a} \ . \tag{36}$$

where the top sign in the last two equations is for the C_1^+ and the bottom sign for the C_1^- characteristic. Note that for purely vertical propagation, when $l_z = 1$, Eqs. (35), (36) reduce to the longitudinal tube wave equations of Herbold et al. (1985, their Eqs. 53, 54).

Combining Eqs. (27), (29) for the transverse components we find after some algebra the two equations

$$\left(1 - l_x^2\right) dv_x - l_x l_y dv_y - l_x l_z dv_z - c_k^{\pm} dl_x - \frac{\rho - \rho_e}{\rho + \rho_e} g l_x l_z dt = 0 , \tag{37}$$

$$\left(1 - l_y^2\right) dv_y - l_x l_y dv_x - l_y l_z dv_z - c_k^{\pm} dl_y - \frac{\rho - \rho_e}{\rho + \rho_e} g l_y l_z dt = 0 , \tag{38}$$

both along the characteristics C_2^+,C_2^- given by

$$C_2^{\pm} = \frac{c_k^{\pm}}{l_a} \ . \tag{39}$$

where c_k is the propagation speed of the pure transverse waves given by

$$c_k^{\pm} = -\frac{\rho_e}{\rho + \rho_e}v^{\|} \pm \sqrt{\left(\frac{\rho_e}{\rho + \rho_e}\right)^2 v^{\|2} + \frac{\rho c_A^2 - \rho_e v^{\|2}}{\rho + \rho_e}} , \tag{40}$$

or, introducing $\eta = \rho/\rho_e$,

$$c_k^{\pm} = (1 + \eta)^{-1}\left(-v^{\|} \pm \sqrt{\eta(1 + \eta)c_A^2 - \eta v^{\|2}}\right) . \tag{41}$$

Note that the pure longitudinal wave propagates with the tube speed c_T and in our approximation is not affected by the backreaction of the external medium, whereas the propagation of the transverse mode is strongly affected by the backreaction of the external fluid. Propagation of the transverse mode is in general asymmetric, $c_k^- \neq -c_k^+$. The symmetry is restored when $v^{\|} = 0$ and the propagation speed is then

$$c_k^{\pm} = \pm c_A\sqrt{\frac{\rho}{\rho + \rho_e}} \ . \tag{42}$$

which is the same as Eq. (39) of Paper I. The presence of square root in Eq. (40) points to the possibility of hyperbolicity violation and hence to the development of a wave instability. This will happen if the longitudinal fluid velocity $v^{\|}$ is large enough

$$v^{\|} > \sqrt{\frac{\rho + \rho_e}{\rho_e}}c_A \ . \tag{43}$$

This expression defines the threshold of the so-called *fire hose instability*.

3.3. Boundary and initial conditions

In this paper we consider the same type of boundary conditions as those used in Paper I. In general the fluid is displaced by a longitudinal piston and at the same time the tube is swayed sideways with specified transverse velocities. The case where all three prescribed velocities are sinusoidal functions corresponds to the *closed boundary case* of Paper I:

$$v_{iB} = -v_{i0}\sin\left(2\pi\frac{t}{P_i} + \Phi_i\right) . \tag{44}$$

where v_{iB} are the velocity components in the locally orthogonal system $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ at the bottom of the tube, v_{i0} are specified velocity amplitudes, P_i wave periods and Φ_i phase shifts of the excitation.

Another type of boundary condition, called the *open boundary case* in Paper I, allows the longitudinal excitation to be transmitting. Here in Eqs. (56) of Paper I we take

$$v_{1B} = v_{1S1} \ , \tag{45}$$

(see Fig. 2 of Paper I) where v_{1S1} is the velocity at the intersection point S_1 of the C_1^- characteristic from the bottom boundary and the old time level. In the present paper, same as in Papers I and II, we consider swaying in a plane only, that is $v_{30} = 0$.

The *top boundary* is assumed to be *transmitting* with the longitudinal v_{1T} and transverse v_{2T}, v_{3T} velocities given by:

$$v_{1T} = v_{1R1} \ , \ v_{2T} = v_{2R2} \ , \ v_{3T} = v_{3R2} \ . \tag{46}$$

Here $R1$ and $R2$ are the intersection points of the C_1^+ and C_2^+ characteristics from the boundary and the old time level. Note the change of notation compared with Paper I, $\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3$ are now $\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1$.

3.4. Comparing different terms

We will now try to assess the actual differences between various expressions for the external backreaction force suggested by different authors. Namely, we will consider the four terms given by Spruit (1981), Choudhury (1990), Cheng (1992) and our new term (OVU, see Eq. (11)). The only difference between these approaches is the transversal equation (Eq. (24))

$$\gamma_+ \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_a^\perp - \frac{\alpha}{l_a}\left(\frac{\partial \mathbf{e}_1}{\partial a}\right)_t - \beta\left(\frac{\partial \mathbf{e}_1}{\partial t}\right)_a - \gamma_- \mathbf{g}^\perp = 0 \tag{47}$$

where $\gamma_\pm = (\rho \pm \rho_e)$ and

$$\begin{array}{llll} a) & \alpha = \rho c_A^2 & ; \ \beta = 0 & (Spruit) \\ b) & \alpha = \rho c_A^2 + \rho_e v_C^2 & ; \ \beta = 0 & (Choudhury) \\ c) & \alpha = \rho c_A^2 - \rho_e v_R^2 & ; \ \beta = 2\rho_e v_R & (Cheng) \\ d) & \alpha = \rho c_A^2 - \rho_e v_\parallel^2 & ; \ \beta = 2\rho_e v_\parallel & (OVU) \end{array} \tag{48}$$

To perform numerical computations we need equations for v_C and v_R. As to the term v_R, from the definition (Cheng 1992, Eq. (6))

$$\boldsymbol{v} = \boldsymbol{v}_T + v_R \mathbf{e}_1 \tag{49}$$

we obtain

$$\frac{\partial \boldsymbol{v}}{\partial l} = \frac{\partial \boldsymbol{v}_T}{\partial l} + \frac{\partial v_R}{\partial l}\mathbf{e}_1 + v_R \frac{\partial \mathbf{e}_1}{\partial l} \ . \tag{50}$$

By multiplying the above by $\mathbf{e}_1$ and taking into account that

$$\frac{\partial \boldsymbol{v}_T}{\partial l} = \boldsymbol{\omega}_T \times \mathbf{e}_1 \tag{51}$$

(Cheng 1992, Eqs. (8),(9)) and $\partial/\partial l = l^{-1}\partial/\partial a$, we finally obtain

$$\frac{\partial v_R}{\partial a} = \mathbf{e}_1 \cdot \frac{\partial \boldsymbol{v}}{\partial a} \tag{52}$$

Using this first-order differential equation one easily calculates v_R once the $\mathbf{e}_1$ and $\boldsymbol{v}$ on the new time level are known.

Choudhury's v_C requires additional discussion. As has already been mentioned by Cheng (1992, Sect. 6) the derivation of v_C by Choudhury (1990, Eqs. (3),(4)) is incorrect. We are now going to rederive v_C following Choudhury's basic idea that fluid motion inside the tube undergoing uniform translation does not produce any backreaction. Consider a bent part of a flux tube undergoing *uniform translation* $\boldsymbol{v}_T = \boldsymbol{v}_T(t)$. In this case the tube moves as a rigid body and there is no problem to identify reference point for v_C, the velocity of the '*fluid turning around the arc*'. The fluid velocity at any point is thus

$$\boldsymbol{v} = \boldsymbol{v}_T + v_C \mathbf{e}_1 \tag{53}$$

It is quite clear that in such a case v_C actually coincides with Cheng's v_R, relative fluid velocity. Taking derivatives with respect to a we obtain

$$\mathbf{e}_1 \cdot \frac{\partial \boldsymbol{v}}{\partial a} = \frac{\partial v_C}{\partial a} \tag{54}$$

which is identical to Eq. (52). When the tube moves in a more general fashion including arbitrary rotation but still as a rigid body, this derivation is not valid, while there is still no problem with Cheng's definition of v_R. Summarizing our analysis we can say that the above three backreaction terms (48abc) are correct if used within proper areas of applicability (for special types of tube motion). The Eq. (47) with case (48a) is valid for the *translational motion of a straight tube*, case (48b) (with our corrections) applies to the *translational motion of a bent rigid tube*, case (48c) can be used for an *arbitrary motion of a rigid tube* while our new term, corresponding to case (48d), is valid for the *general tube motion*. In order to estimate the importance of the backreaction terms on tube dynamics we have carried out a series of calculations for the four different cases (48a) - (48d).

4. Results

The aim of the present investigation is to obtain a feeling for the errors introduced by using previous versions of the transverse equation of motion which did not adequately take into account the external fluid backreaction. By repeating our calculations of Paper I we want to see how much our previous results have to be corrected.

In order to reproduce the wave calculations of Paper I we first reconstructed the correct tube model. At height $z = 0$ (where in the external nonmagnetic atmosphere one has optical depth $\tau = 1$) the model is specified by a magnetic field strength $B = 1500\ G$ and a tube radius of $r = 50$ km. Up to a height of $z = 400$ km (note here a printing error in Paper I) the tube then spreads exponentially in an H^- radiative equilibrium atmosphere, while at greater height only linear spreading is permitted. This model reproduces rather well Table 1 of Paper I.

By pure transverse sinusoidal shaking with an amplitude of 0.4 km s^{-1} we now excited waves, assuming a downward transmitting boundary condition for the longitudinal component (open boundary condition), and followed the wave until a time of 600 s. At the top we use transmitting boundary conditions. The wave calculations were performed for periods of 45, 90, 180 and 300 s and we reproduced Figs. 3 to 6 of Paper I. Using the backreaction terms suggested by different authors and the corrected term of the present paper, we found very small differences. As an example we show in Fig. 2 the horizontal displacement x as a function of height z for the wave with 90 s period at time 600 s. We attribute the small differences to the rather low flow velocities v_z and $v^\parallel$ in all calculations.

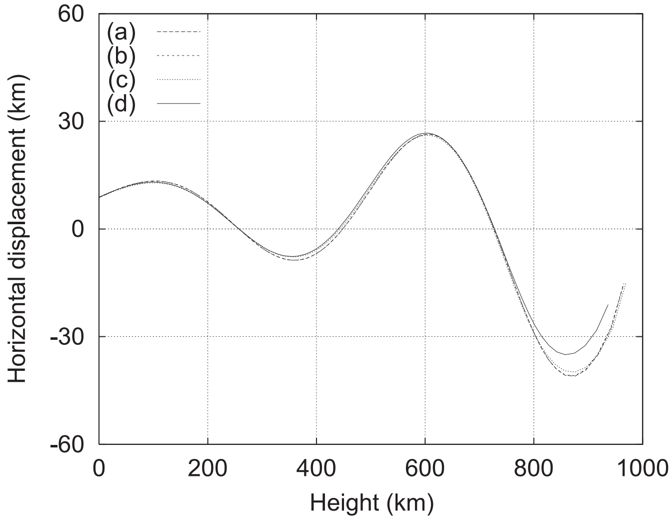

Fig. 2. Horizontal displacement x versus height z of the magnetic flux tube axis for a longitudinal-transverse wave with period 90 s, 600 s after the start of the sinusoidal shaking at the bottom. Curves are labelled according to Eq. (48). The curve labeled '(a)' reproduces a calculation of Paper I.

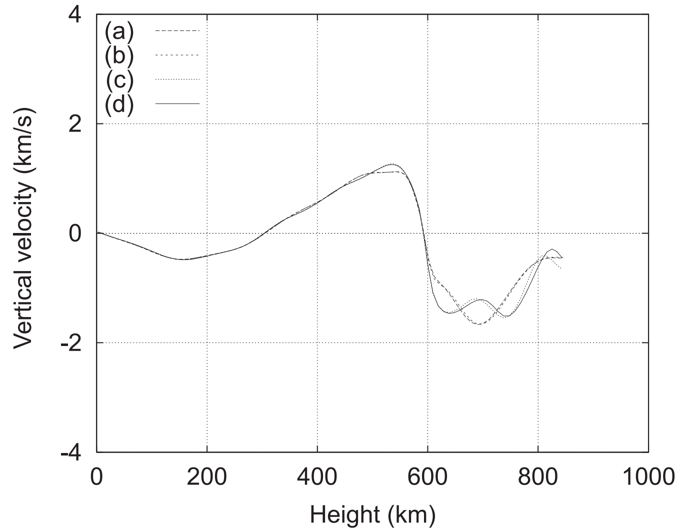

Fig. 3. Vertical velocity v_z versus height of a longitudinal-transverse wave, 180 s after introducing longitudinal and transverse velocity perturbations at the bottom. The velocity amplitude of the longitudinal perturbation was $v_{10} = 0.35\,\mathrm{km\,s^{-1}}$, with a period $P_1 = 90\,s$, the transverse velocity amplitude $v_{20} = 0.4\,\mathrm{km\,s^{-1}}$ with a period $P_2 = 45\,s$. There is zero phase shift between the perturbations.

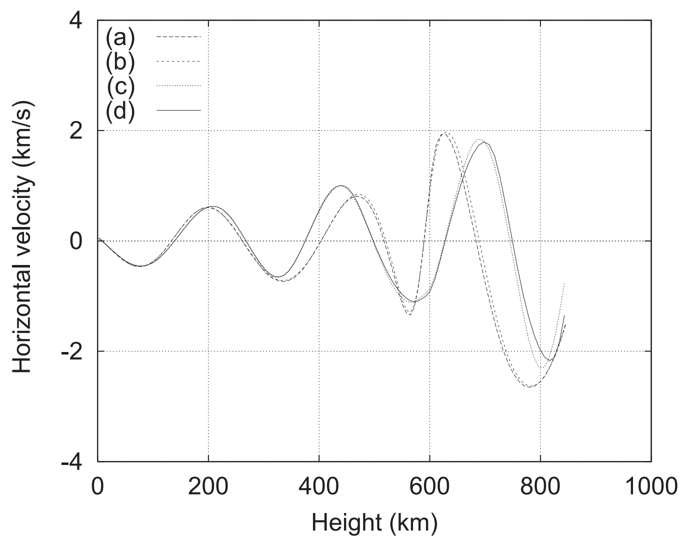

Fig. 4. Same as in Fig. 3, however for the horizontal velocity component v_x.

The reason for the small differences is understood from the Eq. (23) or (24). The new equation differs from the old one by the two terms in Eq. (23), the centrifugal force $v^{\|2}\kappa\mathbf{e}_2$ and Coriolis force $2v^{\|}(\boldsymbol{\omega}_s \times \mathbf{e}_1)$. The first term is large when the fluid moves fast enough along a heavily curved tube section, while the second term is large when there is a fast flow in a rapidly rotating tube element. In the case of purely transverse shaking the longitudinal velocities, generated by the nonlinear mode-coupling, are relatively small and thus do not make more than a slight effect. This shows that the results and conclusions of Paper I, despite of the necessary corrections in the momentum equation, are basically unchanged. A comparison for the shock computations of Paper II will be carried out separately.

We now look for a situation, where the differences between the old and new methods are more clearly visible. This is found in cases where a combined longitudinal (closed boundary condition) and transverse excitation is applied. This calculation, where the longitudinal velocity amplitude was $v_{10} = 0.35\,\mathrm{km\,s^{-1}}$ with period $P_1 = 90\,\mathrm{s}$ and the transverse velocity amplitude $v_{20} = 0.4\,\mathrm{km\,s^{-1}}$ with period $P_2 = 45\,\mathrm{s}$, is shown in Figs. 3 to 5. The figures show calculations both with the old force terms and the new, corrected one (solid).

As the tube, despite of the transverse shaking is fairly straight it is seen in Fig. 3, that the longitudinal velocity v_z is essentially unchanged in the new method as opposed to the old method. The reason for this is that for this velocity component the mass, momentum and energy equations are unchanged. However, the mode-coupling from the transverse wave, does influence the v_z component, leading to small, but noticeable variations. As the corrections were made in the transverse momentum equation, it is seen in Fig. 4 that in the horizontal velocities v_x noticeable changes occur. These differences in v_x are reflected in turn in the horizontal displacements x as seen in Fig. 5.

To directly see the influence of the correction terms in the transverse momentum equation, the centrifugal force $v^{\|2}\kappa\mathbf{e}_2$ and Coriolis force $2v^{\|}\boldsymbol{\omega} \times \mathbf{e}_1$ terms are compared with the total transverse force term in Fig. 6. The results of the calculations indicate that the Coriolis force plays major role in the flux tube dynamics, whereas the centrifugal force in all considered examples produces only a negligible effect. However, the centrifugal force is responsible for the onset of the fire hose instability.

The result can be summarized as follows. In all cases the particular form of the backreaction term has only a minor effect on the tube dynamics while our new term, suggested in

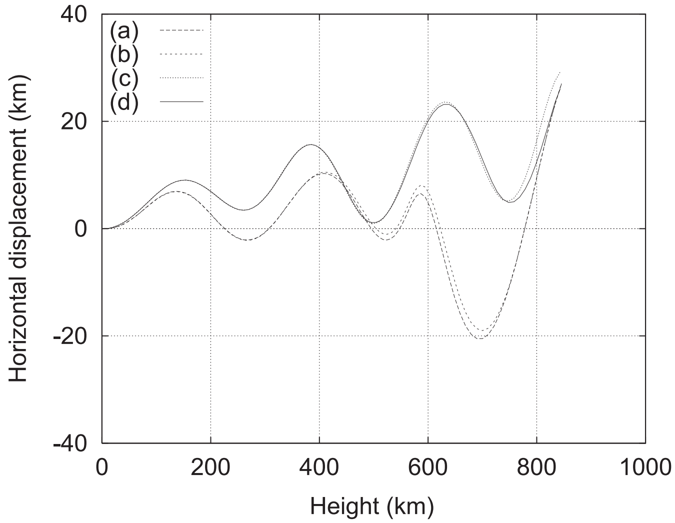

Fig. 5. Same as in Fig. 3, however for the horizontal displacement x.

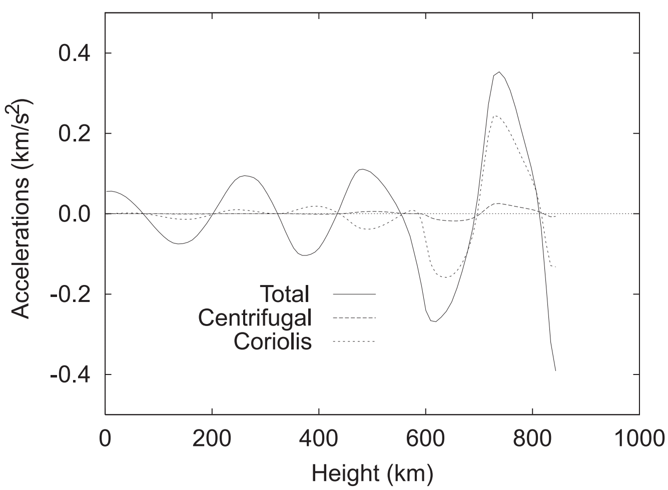

Fig. 6. The total acceleration $(d\boldsymbol{v}/dt)^{\perp}$, compared with the correction terms - the centrifugal $v^{\|2}\kappa$ and Coriolis $2v^{\|}\omega\times\mathbf{e}_1$ terms, for the same case as shown in Figs. 2 to 4.

the present paper, gives the largest deviation from the original (Spruit's) term.

5. Conclusions

We have discussed the question of the correct expression for the backreaction of the external fluid on a moving thin magnetic flux tube. Several examples have been given (see Appendix) where previously published expressions do not agree with the general properties of the interaction of moving bodies in an ideal fluid (d'Alemberts paradox).

1. We believe that the criticism of the local approach must be substantiated by a comparison with a full 3D computation and since these computations are presently missing, we assume together with most previous authors, that the local approach represents a reasonably good approximation of the backreaction phenomenon. We suggest a new local expression for the backreaction of the external fluid which avoids criticisms leveled at previous expressions. The new backreaction force is based on the *normal component of the translational acceleration* of a tube element, while the accelerations connected with the longitudinal fluid motion (tangential, centrifugal and Coriolis) are avoided.

2. A threshold value is obtained for the onset of the fire hose instability.

3. A number of numerical calculations has been carried out to check and illustrate the effects introduced when using the published expressions and the new backreaction force term. No significant differences have been found with the earlier published results concerning propagation of purely transverse waves. This fact is attributed to the particular choice of boundary conditions used in these calculations, which produced small longitudinal flow velocities. We find that the differences between the published expressions and the new backreaction term are minor when the longitudinal flows are small.

However, an example where in addition to the transverse excitation a sizeable longitudinal perturbation was considered, showed more noticeable effects of the corrected transverse momentum equation.

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Appendix A: the acceleration reaction term

In this Appendix we want to suggest a backreaction term which hopefully is free of the limitations and inconsistences of previous results. We assume, together with the authors of Eqs. (6) to (9), that a local approximation of the fluid backreaction can be accepted in certain situations.

Following the mentioned authors we assume that for the backreaction of the external fluid, the results of the theory of enhanced inertia of an infinite circular cylinder moving in a potential flow of an ideal fluid can be applied. This approach allows to obtain simple local equations to describe the external fluid backreaction. The limits of validity of such an approximation can only be found by a full 3D simulation and is the subject of a forthcoming separate investigation. We thus defer the criticism of Moreno-Insertis et al. (1996) about the validity of the local approach to the discussion of the comparison of the 1D and 3D simulations.

In the present work our scope is thus reduced to finding a consistent backreaction term assuming the local approach. Guided from the well-known solution of the classical problem of a 2D circular cylinder in an ideal incompressible fluid we suggest the following term

$$\mathbf{f}_{\rm acc} = -\rho_e \left(\frac{\partial \boldsymbol{v}^{\perp}}{\partial t}\right)^{\perp}_{s} \quad . \tag{A1}$$

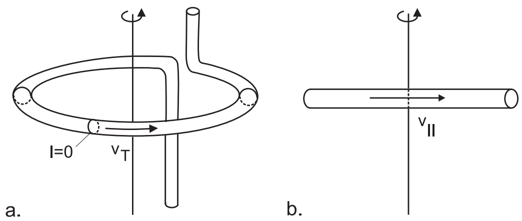

Fig. A1. **a** Rigid ring-shaped tube segment rotating around the ring axis. **b** Tube rotating around a minor axis with internal flow.

Note that neither the longitudinal velocity nor the longitudinal acceleration enter the Eq. (A1). This incorporates the valid criticisms by Choudhuri and Cheng that the backreaction of the external fluid does not care about what goes on inside the tube. The additional $^{\perp}$ sign excludes longitudinal component of the (centrifugal) acceleration which has nothing to do with the fluid backreaction.

Eq. (A1) can be written in a more convenient form. Since the function $s = s(a, t)$ clearly exists (see Fig. 1), we can also consider the function $a = a(s, t)$, the Lagrange coordinate of the fluid particle expressed as a function of s and t. We thus have

$$\left(\frac{\partial}{\partial t}\right)_s = \left(\frac{\partial}{\partial t}\right)_a + \left(\frac{\partial a}{\partial t}\right)_s \left(\frac{\partial l}{\partial a}\right)_t \left(\frac{\partial}{\partial l}\right)_t \quad . \tag{A2}$$

Applying this to the radius-vector $\boldsymbol{r}(a, t)$ and using Eq. (1),

$$\mathbf{e}_1 \left(\frac{\partial l}{\partial a}\right)_t \left(\frac{\partial a}{\partial t}\right)_s = \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_s - \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)_a = \boldsymbol{v}^{\perp} - \boldsymbol{v} = -\boldsymbol{v}^{\parallel} \quad , \tag{A3}$$

where $\boldsymbol{v}^{\parallel} = v^{\parallel}\mathbf{e}_1$. Now substituting Eq. (A3) into (A2) we obtain

$$\left(\frac{\partial}{\partial t}\right)_s = \frac{d}{dt} - v^{\parallel} \left(\frac{\partial}{\partial l}\right)_t \stackrel{\text{def}}{=} \frac{d_{\perp}}{dt} \quad . \tag{A4}$$

Using this we can express our new suggested backreaction term in Eq. (A1) also as

$$\mathbf{f}_{\text{acc}} = -\rho_e \left(\frac{d_{\perp}\boldsymbol{v}^{\perp}}{dt}\right)^{\perp} \quad . \tag{A5}$$

We now show that our new term survives criticism directed to the terms proposed in Eqs. (8) to (10).

The problem with Cheng's expression is that it retained the longitudinal component of the tube velocity and that it did not clearly specify how the arc-length l along the tube is measured. Even when some natural choice for the reference point exists, his expression still gives the wrong result. To show a situation where Cheng's backreaction term fails consider a rigid ring-shaped tube section of large radius between arc-length l_0 and l_1, rotating around the axis passing through the center of curvature, as shown in Fig. A1. Then, writing Cheng's term as

$$\left(\frac{\partial \boldsymbol{v}_T}{\partial t}\right)_l^{\perp} = \left(\frac{\partial \boldsymbol{v}_T^{\perp}}{\partial t}\right)_l^{\perp} + \left(\frac{\partial \boldsymbol{v}_T^{\parallel}}{\partial t}\right)_l^{\perp} \quad . \tag{A6}$$

one can see that while the first term on the right hand side is identically zero (because $\boldsymbol{v}_T^{\perp} = 0$), the second term is not and, hence, will give a backreaction of the external fluid although the external medium is not disturbed in any way by this ring element.

For the term suggested by Fan et al. (1994) consider a tube, which in the presence of internal flows, rotates with constant angular velocity around a minor axis, see Fig. A1. We have

$$\left(\frac{d\boldsymbol{v}^{\perp}}{dt}\right)^{\perp} = \left(\frac{d_{\perp}\boldsymbol{v}^{\perp}}{dt}\right)^{\perp} + v^{\parallel} \left(\frac{\partial \boldsymbol{v}^{\perp}}{\partial l}\right)_t^{\perp} \quad , \tag{A7}$$

where $\boldsymbol{v}^{\perp}$ arises from the angular velocity. The first term on the right hand side is the one we suggest in the present paper. This term gives $(\partial \boldsymbol{v}^{\perp}/\partial t)_s^{\perp} = 0$ in agreement with the fact that we should not have a backreaction force in this situation of uniform rotation. However, the second term on the right hand side is not zero because both $v^{\parallel} \neq 0$ and $(\partial \boldsymbol{v}^{\perp}/\partial l)_t^{\perp} \neq 0$.

Finally, we discuss the term Eq. (10) by Moreno-Insertis et al. (1996). It is seen from Fig. 1 that the velocities $\boldsymbol{v}^{\perp}$ are tangential along the lines $s = const$. The time derivative of this curved vector field therefore will have a longitudinal component which should not contribute to the backreaction force. This longitudinal (centrifugal) acceleration comes from the rotation of the frame of reference and is not responsible for any momentum exchange between the fluid and the tube. Compared to their term Eq. (10) our term Eq. (A1) has another $^{\perp}$ which kills this longitudinal component.

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