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# The influence of the Reynolds stress on the solar p-mode frequencies

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**Abstract.** We examine the influence of the Reynolds stress as the main contribution of turbulence on the oscillation frequencies of the solar p-modes. We regard the radial oscillations as well as the nonradial ones in order to study the influence depending on the degree $l$ of the modes.

For the radial modes we find that the influence of the turbulence gives corrections that reduce significantly the difference between observed and theoretical calculated frequencies. Regarding the nonradial modes we find, additionally, an increase in the corrections with the harmonic degree $l$ which is in good agreement with the observations.

We consider turbulent corrections appearing in the oscillation equations as well as the changes of the model itself for consistency. For radial and nonradial modes we find that the changes of the model are of minor importance compared to the turbulence terms appearing directly in the equations of the oscillation.

**Key words:** turbulence – Sun: oscillations

## 1. Introduction

In the last years, while observing the solar oscillations on longer timescales and with higher precision, it became evident that these oscillations play an important role in understanding the solar interior, bearing more or less the only information of the deeper parts of the sun which can not be probed otherwise.

The enormous progress that has been made in measuring the solar p-mode frequencies, up to a relative accuracy of $10^{-5}$, places new restrictions on the theoretical models. In spite of very exact calculations, the theoretical calculated frequencies of the solar p-modes are still systematically a few $\mu$Hz higher than the observed ones, thus indicating that parts of the assumptions used in the solar models are still too inexact. Many features of the theoretical models have already been improved in order to reduce the gap to the observed frequencies. Different tables for the equation of state (Noels et al. 1984; Guzik et al. 1996) or the effects of helium or hydrogen diffusion (Guenther et al. 1993; Gabriel & Carlier 1997) have been examined.

In any case, the properties of the difference indicate that the corrections must be searched for in the outer layers of the sun. Two reasons illustrate this fact. First, the difference increases with increasing radial order $n$. The higher the radial order $n$, the closer the upper turning point of the confined waves lies to the surface. Because the frequencies very sensibly depend on the location of this turning or reflection point, this is a first hint that the solar model has to be improved in this outer region. Moreover it can be found that the difference between the observed and the calculated frequencies (O-C-difference) increases with increasing harmonic degree $l$, i.e. the more the waves are confined again in the outer "convective" part of the sun. Thereby, the difference is as low as a few $\mu$Hz for low degree modes ($l < 10$) and increases up to the order of a few tens $\mu$Hz for high degrees ($l > 100$) (Guzik & Swenson 1997).

Therefore, turbulence has become one of the most important explanations for the p-modes (Stein et al. 1988; Rosenthal 1998) as well as for the f-modes (Murawski & Roberts 1993). The turbulent pressure modifies the pressure in the convective region and this change is largest in the outermost part of the sun, where the convective particles have the highest velocities.

In earlier papers (Rüdiger et al. 1997; Böhmer & Rüdiger 1998 (paper I)) we already dealt with the influence of the turbulence on radial p-modes. As mentioned above the pressure and, therefore, the structure of the solar model are modified (corrections in zeroth order of the oscillation). This of course has a direct influence on the frequencies of the modes. Additionally, this changed physics must be taken into account in making up the oscillation equations (corrections in first order of the oscillation), resulting in an extra term due to the turbulence. In this paper, we extend our calculations to the nonradial oscillations in order to examine the dependency on the degree $l$, considering again effects in zeroth and first order. The latter contributions are included via perturbation theory as has already been done for the radial modes. We restrict our calculations to the Reynolds stress, that is, convection is modelled in anelastic approximation (Gough 1969, Latour et al. 1976).

## 2. Improving the oscillation equations

The linearised basic equations of the oscillation, including the first order perturbation due to the turbulent pressure, have the following form. For the conservation of mass we have

$$\delta\rho + \rho_0 \,\mathrm{div}\, \boldsymbol{\xi} = 0 \quad (1)$$

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where $\delta\rho$ is the Lagrangian perturbation of the density, while $\rho_0$ denotes the equilibrium value. It was shown in the previous paper that this equation is only affected by the meridional flow, whose influence is much smaller than that due to the Reynolds stress. Therefore in this consideration the equation of mass conservation remains unchanged.

For the conservation of momentum there is a correction due to the Reynolds stress in

$$\frac{\partial \tilde{p}}{\partial x_i} + \tilde{\rho} g_i - \omega^2 \rho_0 \xi_i = -\frac{\partial}{\partial x_j}(\tilde{\rho} Q_{ij}), \tag{2}$$

where the tilded quantities denote the Eulerian perturbations and equilibrium values as in (1). The righthand side of Eq. (2) gives the perturbation term of the turbulence, where

$$Q_{ij} = \langle u'_i(\boldsymbol{x}, t) u'_j(\boldsymbol{x}, t) \rangle \tag{3}$$

is called the Reynolds stress. The eddy viscosities, as part of the Reynolds, stress are assumed to have influence only on the line profiles (Stix et al. 1993).

We now treat the case of a total anisotropic turbulence, which means we neglect the horizontal movement of the turbulent particles. This approximation makes the equations much more feasible. The difference in the radial part of the perturbation term between anisotropic and isotropic turbulence consists in an additional term of the form $(-2\tilde{\rho} Q_{rr}/r)$ that, because of the strong gradient of the velocity, is of small importance, so that for radial oscillations there should be no great error in the use of an anisotropic turbulence. For nonradial modes one should nevertheless be aware of the neglect of the nonradial terms.

The perturbation term (right side of Eq. (2)) has then only a radial part of the form

$$\mathcal{Q}_r = -\frac{1}{r^2}\frac{d}{dr}\left(r^2 \tilde{\rho} Q_{rr}\right) \tag{4}$$

while the Eulerian perturbation of the density can be expressed, following Eq. (1),

$$\tilde{\rho} = -\frac{1}{r^2}\frac{d}{dr}\left(\rho_0 r^3 X\right) - \rho_0 Y \tag{5}$$

with

$$X = \frac{\xi_r}{r} \tag{6}$$

including the radial part of the displacement $\xi_r$ and

$$Y = -l(l+1)\frac{\xi_h}{r} \tag{7}$$

including the horizontal part of the displacement $\xi_h$.

Then, we have for the perturbation term

$$\mathcal{Q}_r = \frac{1}{r^2}\frac{d}{dr}\left(r^2 Q_{rr}\left(\frac{1}{r^2}\frac{d}{dr}\left(\rho_0 r^3 X\right) + \rho_0 Y\right)\right). \tag{8}$$

The solutions of the unperturbed oscillation equations, i.e. setting the Reynolds stress to zero in Eqs. (1) and (2), together with the boundary conditions

$$\xi_r(0) = 0 \qquad \delta p(R) = 0 \tag{9}$$

form a complete and orthogonal base of eigenfunctions, which can easily be seen for radial modes, but can also be shown for nonradial oscillations (Unno et al. 1989). They can therefore be used to apply perturbation theory to calculate the corrections in first order of the oscillations (in the following called first order corrections) due to the turbulent influence (Eq. (8)). This is a very commonly used method, because the eigenfunctions change only slightly under the influence of a small perturbation. Other corrections, as the influence of magnetic fields and rotation on the p-mode frequencies (Gough & Thompson 1990), or the turbulent influence on the f-modes (Gruzinov 1998), have been treated in the same way.

## 3. Improving the solar model

In zeroth order of the oscillations, i.e. dealing with the equations of the solar model, we have to include the turbulent corrections as well for consistency (in the following called zeroth order corrections). We use a solar evolution model based on the method described by Kippenhahn et al. (1967) and by Stix & Skaley (1990) using the OPAL 95 opacity tables. In contrary to paper I, where the convection zone was matched to a fixed core solution, this model is solved consistently. The outer layer solution, obtained by integrating the equations for the convective envelope, acts as a boundary condition for the inner solution of the core equations at $M = 0.97 M_\odot$. The quantity $\log p$ is chosen as the independent variable of the solar model.

The convection is described by the well known mixing length theory (Böhm-Vitense 1958). Due to the Reynolds stress, we have a correction in the integration of the radial variable $\log r$

$$\frac{d\log r}{d\log p} = -\frac{p}{\rho r g}\left(1 + \frac{p_{\rm T}}{p}\right) \tag{10}$$

and hence in the integration of the mass

$$\frac{d\log m}{d\log p} = \frac{4\pi\rho r^3}{m}\frac{d\log r}{d\log p}, \tag{11}$$

where $p_{\rm T}$ is

$$p_{\rm T} = \frac{1}{r^2}\frac{d}{d\log p}\left(r^2 \rho_0 Q_{rr}\right). \tag{12}$$

Comparing two frequencies calculated with solar models, where one is slightly changed in its physics, is always somewhat ambiguous. The question arises which constraints we set on the models in order to compare them. For our calculations we choose the two models to have the same radius and luminosity, which appear to be the better known quantities compared with the mixing length or the initial helium concentration.

It is well known that the velocity field obtained from the mixing length theory has a large gradient at the outer boundary, where the velocity of the rising particles decreases nearly abruptly to zero. This implies that $p_{\rm T}$ (cf. Eq. (12)) has very large values in this region. This leads to some numerical difficulties, because $p_{\rm T}/p$ in Eq. (10) is no longer much smaller than unity.

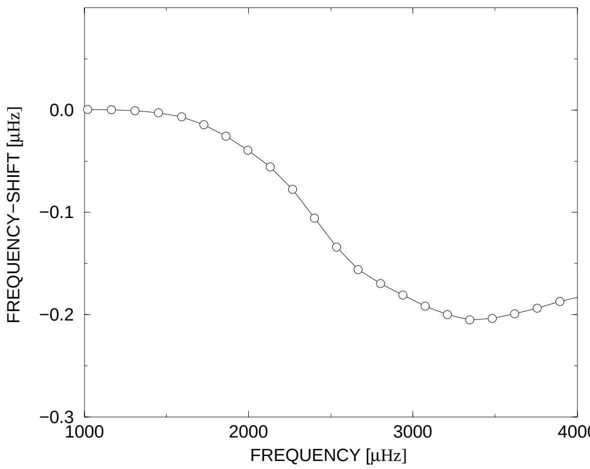


**Fig. 1.** Zeroth order corrections due to the Reynolds stress, arising from changes in the solar model, for frequencies of radial p-modes

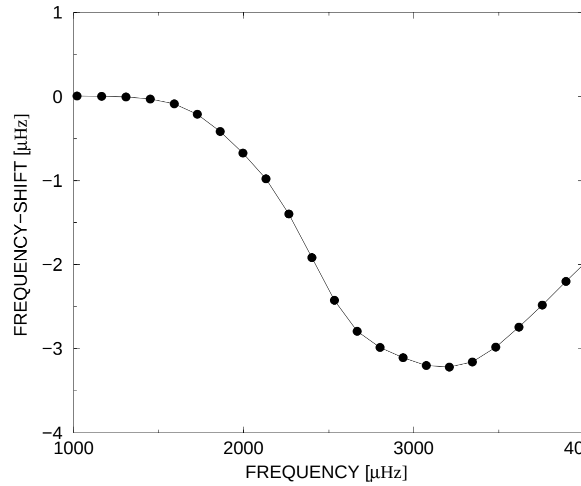


**Fig. 2.** First order corrections due to the Reynolds stress, arising from additional terms in the oscillation equations, for frequencies of radial p-modes

Comparisons of the velocity field, calculated with the Mixing Length Theory (MLT) and a velocity field, obtained from numerical calculations of the convection zone (Abbet et al. 1997), show that a smaller decrease of the velocity at the outer boundary is more realistic, so that the velocity curve is smoothed out and also not decreasing to zero but to some finite value. This applies also to the decrease towards the lower boundary of the convection zone, although the difference in the slope between the velocities is not as large. The velocity field, resulting from the MLT, appears to have a gradient that is about 10 times larger than that received by numerical simulations. We decided therefore to reduce $p_{\rm T}$ by a factor of 10, to a more tractable value. In order to have no irregularity, and for simplicity, this is done over the whole convection zone, although the decrease of the slope towards the inner boundary of the convection zone may therefore be underestimated. The correction, in this part, is small enough that this error may be negligible.

## 4. Results

To examine the influence of the turbulent pressure on the frequencies, we first calculated the frequencies of the unperturbed model as a reference. Then, we included the turbulent terms both in the model and in the oscillation equations. In the figures, we show the difference between the frequencies calculated with and without the turbulent corrections, comparable to "with turbulence observed" minus "so far laminar computed" frequencies.

Our first interest was in recalculating the radial modes including a complete solar model. As has been shown previously, the Reynolds stress with a realistic velocity field tends to reduce the frequencies in the interesting range between $2000$–$4000$ $\mu$Hz.

We find it instructive, instead of summing up the zeroth order and first order corrections for the total deviation, to show both contributions separately.

In the Figs. 1 and 2 the frequency-shifts are shown (in the sense observed minus computed) for the radial modes in zeroth and first order respectively. Although both corrections have similar features, it is remarkable that the zeroth order correction is about one order of magnitude smaller than the correction due to the first order. The total deviation, i.e. simply the sum of both, is therefore dominated by the first order correction.

It can be seen that in both cases the difference is increasing with the radial order $n$ of the modes. The difference in Fig. 2 is of the order of $\mu$Hz, which is in reasonable agreement with what is known from the O-C-difference. We conclude that the Reynolds stress is a good explanation for the deviation of the radial modes producing the right dependence on the radial order with the right order of magnitude.

The question remains whether also the dependence of the difference on the harmonic degree $l$ of the modes can be explained with the help of turbulent corrections, in order to have an explanation consistent with both $n$- and $l$-dependencies. To answer this question the calculations must be extended to nonradial modes.

In Fig. 3 we show the first order corrections for three values of $l$ ($l = 1$, $l = 10$ and $l = 100$). The $n$ dependence of the difference is similar to that of the radial modes, but we find an increase of the corrections with the harmonic degree $l$ of the modes. Although the difference between $l = 1$ and $l = 10$ is only small, it can already be seen that the correction is increasing. For $l = 100$ the correction is remarkably larger then for the lower degrees.

As for the radial modes, we found here that the zeroth order corrections (Fig. 4) are smaller than the first order corrections. Thereby we have to take into account that in contrast to the first order corrections where, via perturbation theory, the difference is calculated directly, here we calculated the difference in subtracting the unperturbed from the perturbed frequencies. Therefore, the zeroth order correction is very sensitive to the number of grid points we chose to calculate the frequencies. As the matrix to compute the nonradial oscillations is twice the matrix for radial oscillations, for numerical reasons, we could

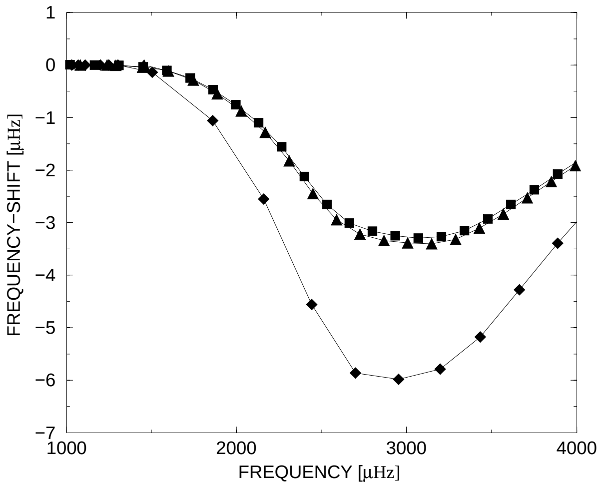


**Fig. 3.** First order corrections due to the Reynolds stress, arising from additional terms in the oscillation equations, for nonradial p-mode frequencies with $l = 1$ (squares), $l = 10$ (triangles), $l = 100$ (diamonds)

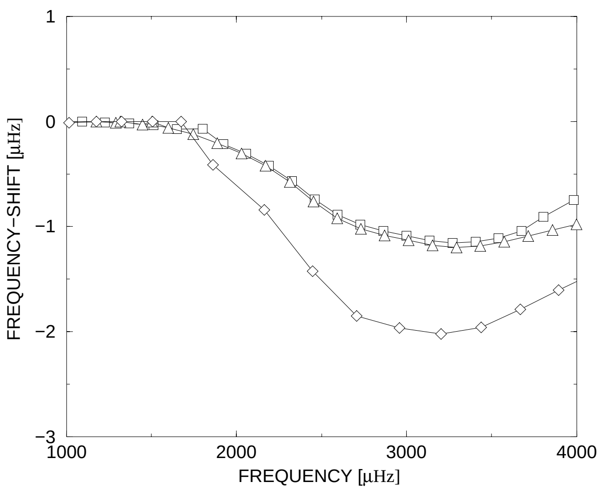


**Fig. 4.** Zeroth order corrections due to the Reynolds stress, arising from changes in the solar model, for nonradial p-mode frequencies with $l = 1$ (squares), $l = 10$ (triangles), $l = 100$ (diamonds)

not use as many grid points as in the radial case. With increasing grid points, however, the zeroth order difference tends to a smaller value, so that the values in Fig. 4 are still too large and tend to some values that have only small influence compared with the first order corrections. Nevertheless, it can be seen that the $l$-dependence of the difference is similar to that in first order, increasing again with the degree $l$ of the mode.

In Fig. 5 the difference between observed and computed frequencies before and after applying turbulent corrections is plotted, for the radial mode and the one with $l = 100$. The consideration of turbulence decreases this difference significantly. Also shown are the curves determined with the unchanged, original velocity field. Here we calculated only the first order corrections (cf. Sect. 2), but the contribution from the zeroth order corrections should again be small. It is obvious that the correction is sensibly depending on the form of the velocity field – the greater increase leading to a greater correction. The difference between observed and computed frequencies can be further reduced by other effects, as mentioned in the introduction, or by regarding non-adiabatic oscillations (Demarque et al. 1999).

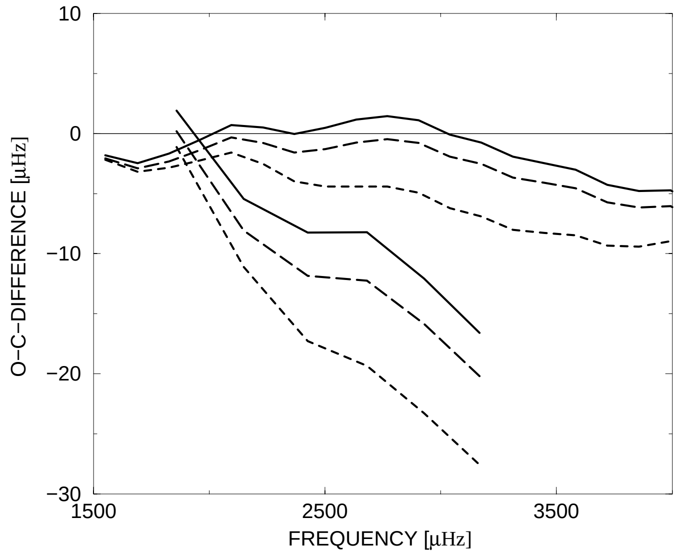


**Fig. 5.** Observed minus computed frequency-differences for $l = 0$ (upper curves; observed frequencies from Lazrek et al. 1998) and for $l = 100$ (lower curves; observed frequencies from Bachmann et al. 1995). With computed frequencies from the reference model (dashed), from a model with turbulent corrections due to the smoothed velocity field (long dashed) and with corrections due to the original velocity field (solid)

## 5. Summary and outlook

Turbulence, although not the only explanation, provides an important contribution to the difference between the observed and calculated solar p-modes. Here we show, for a total anisotropic turbulence, that the corrections arising from the Reynolds stress are of the right order of magnitude and have the right sign to reduce this difference. Moreover, we find that the correction terms increase with increasing harmonic degree $l$ of the modes – the same result that is found comparing calculated with observed data.

Still some refinements can be taken into account. The approximation of an anisotropic turbulence may be appropriate in the dominant part of the convection zone, but, especially in the outermost region, it is no longer valid. Including horizontal velocities would be of special interest for the nonradial oscillations. Additionally, the radial velocity field emerging from the MLT increases in an unrealistic way. Therefore, a more elaborated velocity field, including nonradial parts, would lead improved correction terms. This sensitivity of the correction terms to the form of the velocity field, on the other hand, makes helioseismology an important tool to refine our knowledge about the properties of turbulence in the solar convection zone.

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