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Coupling of solar p modes: quasi-degenerate perturbation theory

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Abstract. The interaction of a large-scale stationary poloidal velocity field (as a simple model for possibly existing giant cells) with solar p-mode oscillations is described by quasi-degenerate perturbation theory, as proposed by Lavely & Ritzwoller (1992). For oscillations of low degree ($l \leq 12$) and sectoral velocity fields we obtain numerical solutions of the eigenvalue problem, and derive an approximate formula for the ensuing frequency splitting. The coupling of the oscillation modes leads to asymmetric frequency multiplets, with splittings of up to $\approx 100\,\mathrm{nHz}$ for a velocity with amplitude $v = 100\,\mathrm{m\,s^{-1}}$. The splitting scales with v^2, with the oscillation frequency itself, and with the inverse difference of the squared frequencies of the coupling partners. Possible observable effects are briefly discussed.

Key words: convection – Sun: interior – Sun: oscillations

1. Introduction

Recent observations of solar oscillations with networks of ground-based telescopes and from space yield time sequences of long duration, extending over more than a year. Such observations permit a frequency resolution of 30 nHz or better. Until now not all the features of the power spectra, esp., the "fine structure" deduced from those time sequences have found an explanation, but may have their origin in the dynamic processes within the solar convection zone. It therefore appears worth while to investigate theoretically the small effects that a slow global convective motion might have on the frequencies of oscillation.

It is well-known that a non-radial linear oscillation mode of degree l has a $(2l+1)$-fold degeneracy if the equilibrium model is spherically symmetric, and that the degeneracy of the Sun's p modes is lifted into $(2l+1)$-fold multiplets as a consequence of the symmetry breaking by rotation. Mathematically, this lift is described by degenerate perturbation theory as far as one considers only real degeneracy. But, in addition to real degeneracy, *quasi-degeneracy* occurs if two eigenvalues that belong to modes of different degrees l and l' accidentally coincide, or are very close to each other.

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Lavely & Ritzwoller (1992) have proposed to study the coupling of solar oscillations by global-scale convection with a quasi-degenerate perturbation theory. In this theory the mode coupling is not only controlled by the selection rules that follow from the spherical harmonic patterns of the oscillation and convection, but, in addition, is limited to modes with frequencies in close vicinity to each other. As these authors have shown, the additional effects of quasi-degenerate perturbation theory are small in comparison to the effects of degenerate perturbation theory. Hence, in a situation where both effects are present, e.g., for the toroidal motions used to describe the solar differential rotation, it is sufficient to deal with the usual degenerate theory (Lavely & Ritzwoller 1993).

In the present contribution we apply the scheme of Lavely and Ritzwoller to the case of a pure poloidal convective motion. In this case, which was not treated by Lavely & Ritzwoller, there is no effect from degenerate perturbation theory, and only the small effect resulting from the quasi-degenerate perturbation theory remains. Furthermore, a pure spherical harmonic poloidal velocity field is a useful first approach for a stationary cellular convective pattern. The sectoral convection inferred by Wagner & Gilliam (1976) from Hα filaments is an example of such a pattern; another example has been reported recently by Beck et al. (1998). Theoretically, non-axisymmetric global convection has been postulated as an ingredient to models of the Sun's differential rotation (e.g., Gilman & Miller 1986). In Sects. 2 and 3 we briefly outline the theory as far as required for the present paper. In Sect. 4 we present examples of calculated frequency splittings for a steady non-axisymmetric poloidal motion in the solar convection zone. For a better understanding of the numerical results we derive an approximate analytic expression in Sect. 5, which in many cases gives a frequency splitting that deviates from the exact result by only 10% or less.

2. The eigenvalue problem

We consider a spherically symmetric equilibrium model of the Sun, and adiabatic oscillations of small amplitude around that equilibrium. These adiabatic oscillations are eigenmodes $\boldsymbol{\xi}_k$, where k stands for three indices (l, m, n): the harmonic degree l, and the azimuthal and radial orders m and n. The eigenmodes obey the equation

$$\mathcal{L}_0 \boldsymbol{\xi}_k = -\rho_0 \omega_k^2 \boldsymbol{\xi}_k \; , \tag{1}$$

where ρ_0 is the density, ω_k the frequency of oscillation, and

$$\mathcal{L}_0 \boldsymbol{\xi} = -\nabla P' + \rho_0 \boldsymbol{g'} + \rho' \boldsymbol{g_0} \; ; \tag{2}$$

the primed quantities are the Eulerian oscillatory variations of the pressure, the gravitational acceleration, and of the density associated with the mode (e.g., Stix 1989, Unno et al. 1989).

If a perturbation is present we replace Eq. (1) by

$$\tilde{\mathcal{L}} \tilde{\boldsymbol{\xi}}_j = -\rho_0 \tilde{\omega}_j^2 \tilde{\boldsymbol{\xi}}_j \; , \tag{3}$$

with $\tilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_1$; we do not perturb the density ρ_0. According to Rayleigh's principle the eigenfunctions of the perturbed system can be represented by a linear combination of the original functions $\boldsymbol{\xi}_k$,

$$\tilde{\boldsymbol{\xi}}_j = \sum_{k \in \mathcal{K}} a_k^j \boldsymbol{\xi}_k \; . \tag{4}$$

Here $\mathcal{K}$ denotes the subspace of those eigenfunctions $\boldsymbol{\xi}_k$ which we shall admit for the coupling. The subscript j varies between 1 and dim($\mathcal{K}$); we shall arbitrarily identify j with that k for which a_k^j has its maximum magnitude. Following Lavely and Ritzwoller we write for the frequency of the perturbed system

$$\tilde{\omega}^2 = \omega_{\rm ref}^2 + \lambda \; . \tag{5}$$

The reference frequency $\omega_{\rm ref}$ is arbitrary, but should be chosen in the vicinity of the eigenfrequencies of the modes admitted for the coupling. Below we shall substitute $\omega_{\rm ref}$ by the frequencies ω_k, $\omega_{k'}$ of the coupling partners.

We substitute (4) and (5) into the perturbed eigenvalue problem (3), multiply with a complex conjugate eigenfunction $\boldsymbol{\xi}_{k'}^*$ and integrate over the volume of the sphere. The result is the algebraic eigenvalue problem

$$\sum_{k \in \mathcal{K}} a_k^j Z_{k'k} = \sum_{k \in \mathcal{K}} a_k^j \lambda_j \delta_{k'k} \qquad \text{for } k' \in \mathcal{K} \; , \tag{6}$$

where the elements of the *supermatrix* **Z** are given by

$$Z_{k'k} = \frac{1}{N_{k'}} \left\{ H_{n'n,l'l}^{m'm} - (\omega_{\rm ref}^2 - \omega_k^2) N_k \delta_{k'k} \right\} \; , \tag{7}$$

and

$$H_{n'n,l'l}^{m'm} = -\int \boldsymbol{\xi}_{k'}^* \cdot \mathcal{L}_1(\boldsymbol{\xi}_k) \, d^3r \; . \tag{8}$$

The eigenfunctions are normalized according to

$$N_k = \int_0^{r_\odot} \rho_0 [\xi_r^2 + l(l+1)\xi_h^2] r^2 \, dr \; . \tag{9}$$

The matrix element (7) slightly differs from that given by Lavely & Ritzwoller (1992): In the denominator we have $N_{k'}$ instead of their N_k.

3. The poloidal velocity field

The perturbation operator $\mathcal{L}_1$ is defined in terms of the velocity field $\boldsymbol{u}_0$ that constitutes the deviation of the equilibrium model from spherical symmetry (e.g., Christensen-Dalsgaard 1998, Unno et al. 1989),

$$\mathcal{L}_1(\boldsymbol{\xi}_k) = -2{\rm i}\omega_{\rm ref} \rho_0 (\boldsymbol{u}_0 \cdot \nabla) \boldsymbol{\xi}_k \; . \tag{10}$$

For the present study $\boldsymbol{u}_0$ is a pure poloidal velocity field, represented by a single spherical surface harmonic $Y_s^t(\theta, \phi)$:

$$\boldsymbol{u}_0 = u_s^t(r) Y_s^t(\theta,\phi) \boldsymbol{e}_r + v_s^t(r) \nabla_h Y_s^t(\theta,\phi) \; , \tag{11}$$

where ∇_h is a horizontal gradient. We assume that the time scale of the motion is long in comparison with the oscillation time scale, so that $\nabla \cdot (\rho_0 \boldsymbol{u}_0) = 0$. This implies that v_s^t can be expressed in terms of u_s^t,

$$\rho_0 r s(s+1) v_s^t = \partial_r (r^2 \rho_0 u_s^t) \; . \tag{12}$$

For our numerical examples below we have specified the function u_s^t in such a way that the cells of global convection fill the solar convection zone, with only a single cell in the depth range between $r_{\rm conv}$, the base of the convection zone, and $r_\odot$, the surface:

$$u_s^t(r) = u_{\max} \frac{4(r_\odot - r)(r - r_{\rm conv})}{(r_\odot - r_{\rm conv})^2} \quad \text{for} \quad r_{\rm conv} \le r \le r_\odot \; , \tag{13}$$

and $u_s^t = 0$ outside this interval. In addition, we shall restrict the numerical examples to sectoral cells, aligned in the North-South direction, i.e., to the special case $t = s$.

Once the various surface harmonics are substituted, the angular integration of (8) can be evaluated. The result is

$$\begin{aligned} H_{n'n,l'l}^{m'm} = \; & 8{\rm i}\omega_{\rm ref}\pi\gamma_s\gamma_l\gamma_{l'}(-1)^{m'} \int_0^{r_\odot} \rho_0 r^2 \\ & \times \Bigg\{ u_s^t \Bigg[(\partial_r \xi_r)\xi_r' \begin{pmatrix} s & l & l' \\ 0 & 0 & 0 \end{pmatrix} \\ & -2(\partial_r \xi_h)\xi_h' \begin{pmatrix} s & l & l' \\ 0 & -1 & 1 \end{pmatrix} \Bigg] \\ & -\frac{v_s^t}{r} 2\Omega_0^s \Bigg[\Omega_0^l \xi_r \xi_r' \begin{pmatrix} s & l & l' \\ 1 & -1 & 0 \end{pmatrix} \\ & +\Omega_0^{l'} \xi_r \xi_h' \begin{pmatrix} s & l & l' \\ 1 & 0 & -1 \end{pmatrix} \\ & -\Omega_0^l \Omega_0^{l'} \xi_h \xi_h' \Bigg(\Omega_2^l \begin{pmatrix} s & l & l' \\ 1 & -2 & 1 \end{pmatrix} \\ & +2\Omega_0^l \begin{pmatrix} s & l & l' \\ 1 & 0 & -1 \end{pmatrix} \Bigg) \\ & +2\Omega_0^l \xi_h \xi_h' \begin{pmatrix} s & l & l' \\ 1 & -1 & 0 \end{pmatrix} \Bigg] \Bigg\} \\ & \times dr \begin{pmatrix} s & l & l' \\ t & m & -m' \end{pmatrix} \; , \end{aligned} \tag{14}$$

where $\gamma_l = \sqrt{(2l+1)/4\pi}$ and $\Omega_N^l = \sqrt{(l+N)(l-N+1)/2}$. The 3×2 matrices are Wigner $3j$ symbols that result from the integration of three surface harmonics over the full sphere (e.g., Edmonds 1960). The Wigner $3j$ symbol

$$\begin{pmatrix} s & l & l' \\ t & m & -m' \end{pmatrix}$$

differs from zero only if the following three *selection rules* are satisfied:
The harmonic degrees s, l, l' must form a triangle, i.e.,

$$|l - l'| \leq s \,, \qquad |l' - s| \leq l \,, \qquad |s - l| \leq l' \,; \tag{15}$$

The sum of the azimuthal orders must vanish,

$$t + m - m' = 0 \,; \tag{16}$$

the sum of the degrees must be even ($\boldsymbol{u}_0$ is a poloidal velocity),

$$s + l + l' \equiv 0 \mod 2 \,. \tag{17}$$

If the matrix element $H^{m'm}_{n'n,l'l}$ is non-zero, we speak of *coupling* of two oscillations. Within quasi-degenerate perturbation theory the coupling of oscillations with different harmonic degrees l and l' is possible, while degenerate perturbation theory allows only coupling of oscillations with the same harmonic degrees.

4. Numerical results

For the velocity field given by Eqs. (11), (12), and (13), with $t = s$ (sectoral pattern) and with $u_{\max} = 100\,\mathrm{m\,s^{-1}}$, we have solved the eigenvalue problem (6) in a few special cases. These cases have been chosen with the help of a table of eigenfrequencies calculated for a standard solar model. From this table we have selected a frequency ω_k that has a number of other frequencies $\omega_{k'}$ in its close vicinity. Generally, every selected p mode is allowed to couple with all the other modes of any frequency as long as the selection rules are satisfied. However, for the actual calculation we have restricted the coupling to a frequency range $|\omega_k^2 - \omega_{k'}^2| \leq \tau^2$, where τ is a parameter with the dimensions of an angular frequency. In the subsequent section we shall give a justification for this restriction and show that τ can be chosen small indeed.

Fig. 1 shows the resulting frequency splittings for a velocity with $s = 3$, and the p mode $l = 7$, $n = 18$, with frequency $\nu = 3008.05\,\mu$Hz. As the parameter τ is increased, more modes are included, and the number of possible coupling partners is increased; but – due to the selection rules – only 4 of the finally admitted 11 modes are true coupling partners. This example demonstrates that the coupling of modes with more distant frequencies has a smaller effect than the coupling of modes with closer frequencies. In Fig. 1 only the nearest two coupling partners cause a discernable splitting.

As a consequence of the selection rule (16), not all of the $2l + 1$ components of a degenerate multiplet become nondegenerate, because the values of m and m' belong to different intervals. In the example of Fig. 1 only 9 out of 15 components of a $l = 7$ mode are shifted by the nearest partner (the $l' = 4$ mode is ninefold degenerate); only the second and further coupling partners lift the degeneracy of all components.

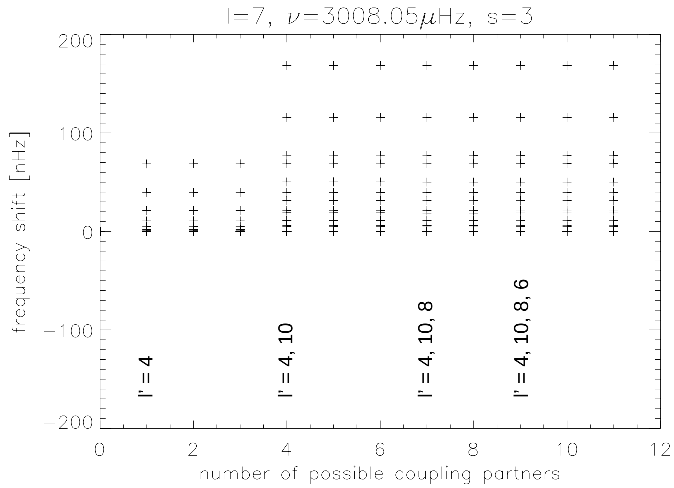

Fig. 1. Frequency splitting for the mode $l = 7$, $n = 18$, $\nu = 3008.05\,\mu$Hz as a function of the number of possible coupling partners. The modes are ordered according to their (absolute) frequency distance from ν (*left to right*). For a sectoral velocity with $s = 3$ only 4 out of 11 possible partners actually couple, as indicated by their degrees l'.

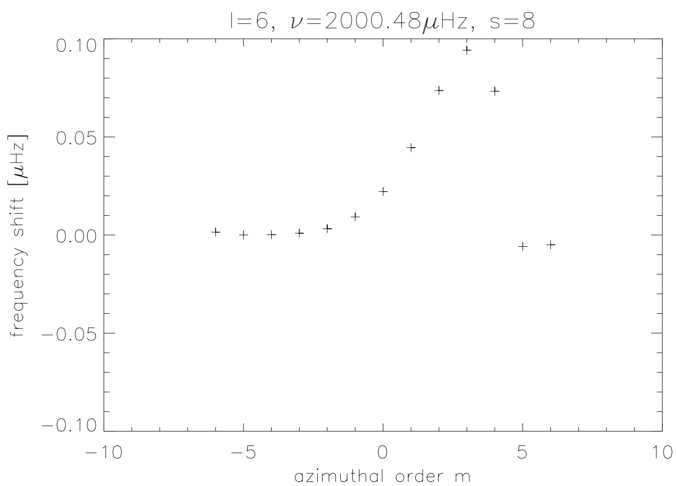

Fig. 2. Frequency shift as a function of the azimuthal order m for the multiplet $l = 6$, $n = 11$, $\nu = 2000.48\,\mu$Hz and a sectoral velocity with $s = 8$

In addition, we see that the first coupling shifts all multiplet components that are affected into the *same direction*; the shift is positive – since $\nu' < \nu$ this will appear natural from the approximate result derived below. The second coupling acts in the same direction while the further couplings make no significant contribution. Another example is illustrated in Fig. 2, where the frequency shift of an $l = 6$ multiplet is shown as a function of the azimuthal order m. In this case the shift results from the coupling with four partners; but the predominant contribution comes from the mode $l' = 12$ with the smaller frequency $\nu' = 1946.05\,\mu$Hz and is, therefore, positive. We notice that, in contrast to the rotational splitting which is antisymmetric with respect to m, poloidal fields cause an *asymmetric* splitting. As equations (24) and (25) will show, each of the two coupling modes splits into such an asymmetric multiplet. The two asymmetric multiplets are antisymmetric to each other; thus the whole system of the two coupling oscillations has again antisymmetry. We shall return to the asymmetric frequency splitting in the last section.

5. Approximations

The supermatrix **Z** is Hermitian and therefore has real eigenvalues λ_j. It is composed of a real diagonal matrix and an imaginary matrix that is antisymmetric with respect to the diagonal. As a consequence of the selection rules many zeros appear in the off-diagonal part. The non-zero off-diagonal elements are all proportional to the amplitude of the perturbation velocity. The eigenvalues are of second order in this velocity.

In the following we shall derive an approximate expression for the special case where only two oscillations (of the unperturbed system) couple. We shall then demonstrate that the frequency changes arising from the coupling of more than two oscillations can approximately be obtained by adding the effects of the coupling of pairs.

Let us especially consider one oscillation out of a multiplet with eigenfrequency ω_k, and three oscillations of a multiplet with eigenfrequency $\omega_{k'}$. Let the spherical harmonic indices be such that the selection rules allow only one of those three oscillations to couple. Then the supermatrix has the form

$$\mathbf{Z} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ c^* & 0 & 0 & b \end{pmatrix} , \tag{18}$$

where we have introduced the abbreviations

$$a = \omega_k^2 - \omega_{\rm ref}^2 , \qquad b = \omega_{k'}^2 - \omega_{\rm ref}^2 , \tag{19}$$

and where $c = H^{m'm}_{n'n,l'l}/N_{k'}$. Two eigenvalues of this supermatrix are equal to b which, by (5), means that the frequencies remain unchanged. The other two eigenvalues are

$$\lambda_{k,k'} = \frac{a+b}{2} \pm \frac{\sqrt{(a-b)^2 + 4|c|^2}}{2} . \tag{20}$$

From (5) and (19) we obtain the new squared frequencies:

$$\tilde\omega_{k,k'}^2 = \frac{\omega_k^2 + \omega_{k'}^2}{2} \pm \frac{\sqrt{(\omega_k^2 - \omega_{k'}^2)^2 + 4|c|^2}}{2} . \tag{21}$$

The further discussion depends on the relative magnitudes of the two terms in the square root of (21). We pay particular attention to the case where the distance of the squared frequencies is the larger term. An expansion of the square root then yields, to first order,

$$\tilde\omega_k^2 = \omega_k^2 + \frac{|c|^2}{\omega_k^2 - \omega_{k'}^2} , \tag{22}$$

$$\tilde\omega_{k'}^2 = \omega_{k'}^2 - \frac{|c|^2}{\omega_k^2 - \omega_{k'}^2} . \tag{23}$$

The second term in each of these equations represents the correction $\delta(\omega^2)$ to the squared frequency. To first order $\delta(\omega^2) \approx 2\omega\delta\omega$; with c substituted, this yields

$$\delta\omega_k = \frac{|H^{m'm}_{n'n,l'l}|^2}{2\omega_k N_{k'}^2 (\omega_k^2 - \omega_{k'}^2)} , \tag{24}$$

$$\delta\omega_{k'} = -\frac{|H^{m'm}_{n'n,l'l}|^2}{2\omega_{k'} N_{k'}^2 (\omega_k^2 - \omega_{k'}^2)} . \tag{25}$$

We recognize that the frequency shift is proportional to the squared element of the supermatrix, i.e., to the square of the perturbing velocity. Since the matrix elements are proportional to the frequency, the shift increases with the frequency itself. It also *increases with the inverse of the difference of the squared frequencies* of the coupling partners. The latter result is characteristic for quasi-degenerate perturbation theory; it is for this reason that only modes with frequencies in close vicinity must be taken into account for the coupling.

The matrix element occurring in (24) and (25) is given by (14). For a crude estimate we may further simplify that expression. Considering only modes with moderate degree l, we may cancel all contributions from the horizontal components of the oscillation vector (those involving ξ_h), and at the same time use $\Omega_0^l \approx l/\sqrt{2}$. Hence

$$\begin{aligned} \frac{H^{m'm}_{n'n,l'l}}{N_{k'}} &= 8{\rm i}\omega_{\rm ref}\pi\gamma_s\gamma_l\gamma_{l'}(-1)^{m'} \int_0^{r_\odot} \rho_0 r^2 \\ &\times \Big\{ u_s^t (\partial_r \xi_r)\xi_r' \begin{pmatrix} s & l & l' \\ 0 & 0 & 0 \end{pmatrix} \\ &- \frac{v_s^t}{r} s l \xi_r \xi_r' \begin{pmatrix} s & l & l' \\ 1 & -1 & 0 \end{pmatrix} \Big\} dr \\ &\times \begin{pmatrix} s & l & l' \\ t & m & -m' \end{pmatrix} \left(\int_0^{r_\odot} \rho_0 r^2 \xi_r \xi_r' \, dr \right)^{-1} . \end{aligned} \tag{26}$$

In the present paper we consider a global sectoral velocity field with a horizontal scale that is of the order of the depth of the convection zone. The giant cells that have been inferred by Wagner & Gilliam (1976) from a rather regular distribution of Hα filaments are of this type. We may expect that a velocity field with such a sectoral geometry affects the p-mode frequencies in a manner that significantly differs from the rotational splitting caused by a toroidal motion. For $s \approx 7\ldots9$, then, $\max(v_s^t) \approx \max(u_s^t) \equiv v$, which we shall use to evaluate (26). Further, we approximate the radial derivative of ξ_r by

$$\frac{d\xi_r}{dr} \approx \frac{n\pi}{r_\odot}\xi_r , \tag{27}$$

and replace v_s^t/r by $v_s^t/r_\odot$. With these estimates

$$\begin{aligned} \frac{H^{m'm}_{n'n,l'l}}{N_{k'}} &\approx 8{\rm i}\omega_{\rm ref}\pi\gamma_s\gamma_l\gamma_{l'}(-1)^{m'} \frac{v(n\pi - sl\gamma_{\rm w})}{r_\odot} \\ &\times \begin{pmatrix} s & l & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & l & l' \\ t & m & -m' \end{pmatrix} . \end{aligned} \tag{28}$$

where

$$\gamma_{\rm w} = \begin{pmatrix} s & l & l' \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} s & l & l' \\ 0 & 0 & 0 \end{pmatrix}^{-1} . \tag{29}$$

Table 1. Comparison of exact and approximate frequency shifts, for the multiplet $l = 6$, $n = 22$, $\nu = 3516.76\,\mu$Hz, caused by coupling with the mode $l' = 12$, $n = 20$, $\nu' = 3500.12\,\mu$Hz through a velocity with $s = 8$. The components $m = 5$, 6 do not couple because the selection rule (16) is not satisfied.

m	Exact [μHz]	Approx. [μHz]
−6	0.00000070	0.00000063
−5	0.00000873	0.00000786
−4	0.00005716	0.00005144
−3	0.00025914	0.00023320
−2	0.00090695	0.00081619
−1	0.00258453	0.00232613
0	0.00615233	0.00553840
1	0.01230009	0.01107681
2	0.02028537	0.01827673
3	0.02591141	0.02335361
4	0.02073560	0.01868288
5	0.00000000	0.00000000
6	0.00000000	0.00000000

Hence the frequency shifts (24) and (25) become independent of the special form of the eigenfunction ξ_r. This is the main achievement of the diverse simplification made. Using $\omega_{\rm ref} \approx \omega_k$ and substituting the γ factors we finally obtain

$$\delta\omega_k \approx \frac{\omega_k v^2 (n\pi - sl\gamma_{\rm w})^2}{(\omega_k^2 - \omega_{k'}^2) 2\pi r_\odot^2} (2s+1)(2l+1)(2l'+1) \times \left[\begin{pmatrix} s & l & l' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & l & l' \\ t & m & -m' \end{pmatrix} \right]^2 . \quad (30)$$

In a number of cases we have checked this approximate frequency shift with the exact numerical result. In view of the approximations made the agreement appears satisfactory. Table 1 shows an example, with relative deviations of the approximate estimate from the exact result of less than 10%; other examples show deviations of similar size.

The approximate formula confirms the asymmetric behavior seen in the numerical results: For any given pair of coupling modes the *sign* of the frequency shift depends only on the sign of the difference of the squared degenerate frequencies, and not on the numbers s, n, and l, or the indices m and m' within the multiplets. Moreover, the approximate formula illustrates which modes of the multiplet become non-degenerate, and that the coupling of oscillations with closer frequencies has a stronger effect, so that the choice of a small parameter τ is justified.

6. Discussion

Our approximate formula (30) fails at very small distance of the squared frequencies of the two coupling partners. In the limiting case we could again expand the square root of (21). However, we may as well discuss that result as it stands: Eq. (21) states that the coupling of two oscillations generates two multiplets of oscillation frequencies. The squared frequencies of these multiplets are located symmetrically with respect to the mean $(\omega_k^2 + \omega_{k'}^2)/2$.

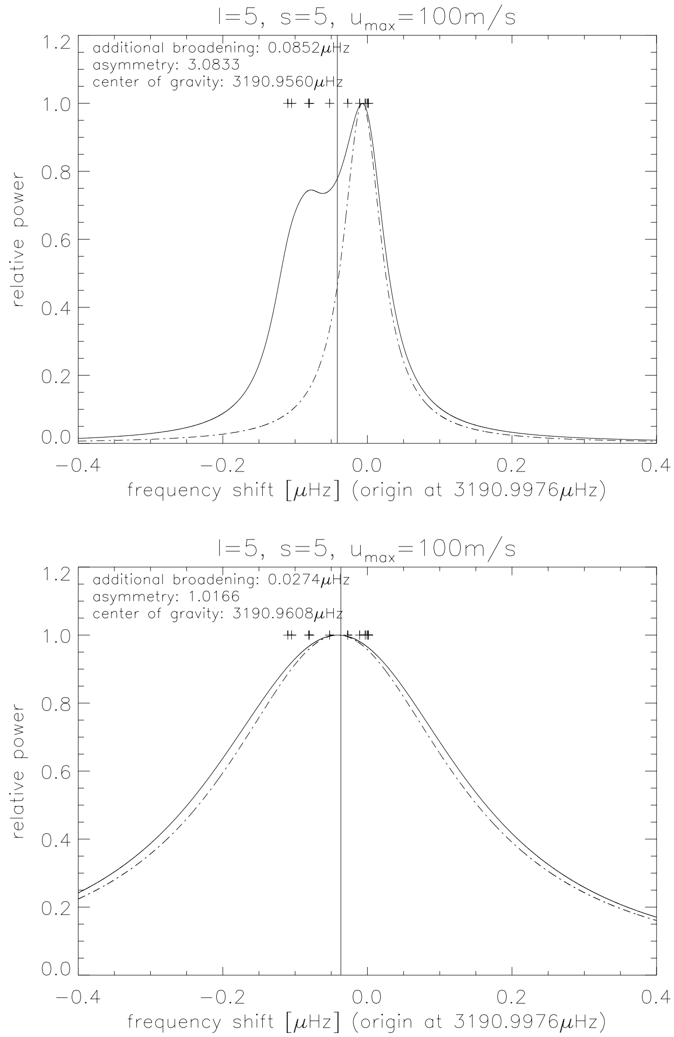

Fig. 3. The multiplet $l = 5$, $n = 20$ obtained from a sectoral velocity with $s = 5$ through coupling with the mode $l' = 8$, $n' = 19$ (*crosses*). The *solid* line profiles result from the folding with Lorentzian profiles (*dashed*) corresponding to mode lifetimes of 1 year (*upper panel*) and 2 months (*lower panel*)

On the other hand, each of the two multiplets is itself *asymmetric*, as demonstrated above by means of numerical results and the approximation (30).

Is there an observable effect of the asymmetric multiplet structure described in this paper? In principle, the answer to this question is yes, but there are several difficulties. The first is that the effect described here is *small*. Even if there were no noise and no contributions to the velocity other than a sectoral field with a single value of s, one would probably not see the individual components of the multiplet, because of their finite line width. Only their net effect in form of a line asymmetry in the power spectrum might be present. The lines are broadened by the finite length of the data set used but, given the long observational periods now available, the predominant line broadening comes from the finite lifetime of the modes.

Two examples of asymmetric line profiles are shown in Fig. 3: Of the eigenfunctions of the perturbed system, 11 have

Table 2. Frequency shifts (crosses in Fig. 3), and the largest one or two squared coefficients $|a_k^j|^2$ of the normalized eigenfunctions of the perturbed system.

$\delta\omega$ [μHz]	l	m	l'	m'	$|a_k^j|^2$
0.0013	5	−5			0.999969
0.0011	5	−4			1.000000
−0.0801	5	−3			0.787039
			8	−8	0.212968
−0.1100	5	−2			0.749010
			8	−7	0.251003
−0.1049	5	−1			0.756410
			8	−6	0.243602
−0.0813	5	0			0.789465
			8	−5	0.210539
−0.0523	5	1			0.842020
			8	−4	0.157975
−0.0272	5	2			0.904490
			8	−3	0.095503
−0.0108	5	3			0.958224
			8	−2	0.041778
−0.0030	5	4			0.988339
			8	−1	0.011675
−0.0004	5	5			0.998386
			8	0	0.001636

frequencies in close neighborhood to the $l = 5, n = 20$ mode of the unperturbed Sun. We have normalized those eigenfuctions so that the sum of their $|a_k^j|^2$ is unity (equivalent to assuming equal energies of all modes), and have made a convolution with Lorentzian profiles corresponding to mode lifetimes of one year and two months, respectively. The asymmetry of the profiles arises from the fact that most of the 11 modes have a very small frequency shift and hence contribute to the main peak, while the rest has frequencies distributed over a small range on *one side* of that peak. It is evident from these examples that the line asymmetry can only be identified for modes with a sufficiently long lifetime. Moreover, the calculated frequency splitting scales with v^2. Hence, from the absence of a line asymmetry one might derive an upper limit for the magnitude of the velocity of the poloidal motions.

Table 2 gives a list of the frequency shifts of the modes contributing to the profiles shown in Fig. 3. For each mode we also list the largest one or two of the squared expansion coefficients. These numbers illustrate that the perturbed eigenmodes with very small frequency shifts virtually coincide with an eigenmode of the unperturbed system, whereas those with larger frequency shifts have noticeable contributions from components of the two multiplets involved in the coupling.

Real observational data are not free of noise. This results in errors when the oscillation frequency is determined, in particular at low and high frequencies where the amplitude of oscillation is small; the errors may entirely cover up the small effect. Fortunately, the frequency resolution and the signal-to-noise ratio become better with longer periods of observation.

A second complication arises from the fact that, in addition to the finite mode lifetime, the large-scale velocity field in the solar convection zone itself might be time-dependent, which certainly would affect the line profiles. An additional time-dependence arrises from the solar cycle. The cycle causes p-mode frequency shifts of order 0.1 μHz per year; this necessitates a restriction of the length of the observational series and of the frequency resolution.

A third difficulty is the presence of other contributions to the velocity field that cause line splitting, line broadening, and asymmetric line profiles. The rotational splitting is the largest of these effects and leads to a symmetric lifting of the p-mode degeneracy. It dominates the $\delta\omega(m)$ profile and produces the well-known S-shape. The main difficulty in the analysis is to disentangle the contribution of the poloidal velocity field considered in the present work from the rotational effect. This problem can only be solved with new techniques of data analysis that allow for asymmetric line profiles.

According to the selection rule (15) small-scale poloidal velocity fields (high s) affect p modes of high and low degree. Velocity fields with smaller scales than considered in the present paper have larger amplitudes; therefore we may expect larger effects on the lines of the p-mode power spectrum. On the basis of the present work the profiles of those lines are expected to be asymmetric, although a time-dependent theory would be even more necessary than for a large-scale velocity. Indeed, the power spectra of high-l p modes show line asymmetries, an observational fact that requires explanation. Our results suggest that pairs of p modes with different l and n, but nearly the same frequency should be looked for, in particular.

With the availability of ever better and longer sets of data from space and from ground-based networks, helioseismology will improve its ability to probe, to reveal and to investigate the effects of convection on the solar oscillations.

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