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Nonlinear stationary vortices in gravitating fluid

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Abstract. The problem of nonlinear self-organization of perturbations in differentially rotating, nonuniform, gravitating systems is studied. Two types of stationary nonlinear solutions, in the form of tripolar vortices and vortex chains of gravitational potential and density, dependent on the spatial profiles of the basic state quantities, are found. They propagate with a constant velocity perpendicularly to the basic state gradients, and characteristic time scales for the creation of such vortical structures are shown to be much shorter than the corresponding intervals for gravitational contraction.

Key words: instabilities – turbulence – waves – ISM: clouds

1. Introduction

The problem of plasma and fluid self-organization into various structures of the form of dipole vortices (Hasegawa & Mima 1978, Shukla & Weiland 1989, Vranješ & Weiland 1992), and vortex chains (Vranješ 1998, 1999a), has attracted a lot of interest in the past two decades. Driven by nonlinear vector-product type terms, and possessing a certain number of integrals of motion, vortices are shown to be remarkably stable structures; in some numerical simulation experiments they survive direct, head-on collisions with other vortices (Makino et al. 1981). In our earlier paper (Jovanović & Vranješ 1990) it is shown that electromagnetic double vortices can appear in large astrophysical plasma clouds, where the term large means that the effects of self-gravitation are of importance and have to be taken into account. The formation of nonlinear vortex chains of gravitational potential in such large-scale systems has been shown recently by Shukla & Stenflo (1995).

New types of vortices of the form of tripoles have been found recently in a series of experimental works with rotating fluids (Van Heijst & Kloosterziel 1989, Van Heijst et al. 1991), observed in the seas of our planet (Pingree & Le Cann 1992), and obtained theoretically in the investigations of strongly nonlinear processes in magnetized, spatially nonuniform plasmas (Vranješ et al. 1998, Vranješ 1999b). Tripoles consist of three parts, of a rotating vortex core and two satellites of opposite vorticity and their origin is closely connected with the existence of some nonuniformity of the system.

On the other hand, the problem of gravitational instability of large astrophysical systems has been investigated throughout this century, starting with the early work of Jeans (1902). In this paper, using equations that describe a gravitating, differentially rotating nonuniform astrophysical gas cloud, we derive a nonlinear equation which is generic for vortex chains and tripolar vortices, and is similar to the plasma equations obtained in our previous papers (Vranješ et al. 1998, Vranješ 1999a). It is shown also that the eventual gravitational collapse in the Jeans' sence is a higher order effect occuring on much larger time scales. In this limit the nonlinear equation is integrated resulting in the equation which comprises an arbitrary functional form, and further particular solutions for the gravitational potential and density, dependent on the choice of the functional, are found in the form of a tripolar vortex (for a linear profile of the basic state flow), and a vortex chain (for the hyperbolic-tangent one).

The paper is organized in the following manner. In Sect. 2 basic equations are given, and derivation of the nonlinear equation which is in the basis of work performed in Sect. 3, where tripolar vortices are found, and in Sect. 4, presenting solutions of the form of vortex chains. At the end a Summary and conclusion are given.

2. Starting equations and derivations

We study perturbations propagating in differentially rotating, spatially nonuniform, self-gravitating fluid systems in the stage of highly developed nonlinearities when $\partial/\partial t \sim \boldsymbol{v}\cdot\nabla$, that can be described by the following set of equations:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla\right)\boldsymbol{v} = 2\boldsymbol{v}\times\boldsymbol{\Omega}_0 - \nabla\varphi - \frac{\nabla p}{\rho}, \tag{2}$$

$$\nabla^2\varphi = 4\pi G\rho. \tag{3}$$

Here $\Omega_0(x)\boldsymbol{e}_z$ is the vector of angular velocity, φ is the gravitational potential, $p = c_S^2\rho$, c_S is the isothermal sound speed, and other notation is standard. Assuming that all basic state quantities depend on the x-coordinate only, we may write the following equation describing the basic state in a local Cartesian coordinate system:

$$\frac{d}{dx}\left[v_0(x)\Omega_0(x)\right] - 2\pi G\rho_0(x) - \frac{c_s^2}{2}\frac{d^2}{dx^2}\log\rho_0(x) = 0. \quad (4)$$

Here $v_0(x)\boldsymbol{e}_y$ is the fluid velocity describing differential rotation in the basic state, and the subscript 0 is used to denote the basic state quantities. Further, we shall study low frequency, $\partial/\partial t \ll \Omega_0$, two dimensional perturbations propagating perpendicularly to $\boldsymbol{\Omega}_0$. According to Dolotin & Fridman (1991) and references cited therein, this approach is well satisfied for certain interstelar objects, for example molecular clouds with the period of rotation of approximately $7 \cdot 10^6$ years, which is only about 1/50 part of their average time of existence. In such circumstances slow modes investigated here have enough time to develop. From Eq. (2), taking the cross product by $\boldsymbol{e}_z$, we obtain the following expression for the perpendicular velocity:

$$\boldsymbol{v}_\perp = \frac{1}{2\Omega_0(x)}\boldsymbol{e}_z \times \nabla_\perp\left(\varphi + c_s^2\log\rho\right) + \frac{1}{2\Omega_0(x)}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\nabla\right)\boldsymbol{e}_z \times \boldsymbol{v}_\perp, \quad (5)$$

where $\nabla_\perp = \boldsymbol{e}_x(\partial/\partial x) + \boldsymbol{e}_y(\partial/\partial y)$. In accordance with the assumption of low-frequency limit, the first, linear term in Eq. (5), $\boldsymbol{v}_1 = [\boldsymbol{e}_z \times \nabla_\perp\left(\varphi + c_s^2\log\rho\right)]/(2\Omega_0)$, is the leading order one, and the perpendicular velocity $\boldsymbol{v}_\perp$ can be calculated recursively by putting it in the rest of equation. In order to demonstrate the instability in Jeans' sence, it turns out that third order small terms $(\partial/\partial t)^3/\Omega_0^3$ need to be kept. To show that we linearize the above expressions and neglect the effects of spatial nonuniformity, and the flow. Eq. (5) yields:

$$\boldsymbol{v}_\perp = \boldsymbol{v}_1 + \frac{1}{2\Omega_0}\boldsymbol{e}_z \times \frac{\partial \boldsymbol{v}_1}{\partial t} - \frac{1}{4\Omega_0^2}\frac{\partial^2\boldsymbol{v}_1}{\partial t^2} - \frac{1}{8\Omega_0^3}\boldsymbol{e}_z \times \frac{\partial^3\boldsymbol{v}_1}{\partial t^3}. \quad (6)$$

For perturbations of the form $\exp(-i\omega t + i\boldsymbol{k}\boldsymbol{r})$, $k = 2\pi/\lambda$, from Eqs. (1)-(3) we obtain the following condition for instability:

$$\lambda > \lambda_c \equiv \pi c_s(\pi G\rho_0 - \Omega^2)^{-1/2}. \quad (7)$$

This is the well known classical result describing the stabilizing effect of rotation on gas clouds (Fricke 1954). Further, it will be shown that for the purpose of finding nonlinear stationary vortex solutions of the basic Eqs. (1)-(3) it is enough to keep the second order linear and nonlinear small terms only. On that time scale the effects of eventual gravitational contraction will be assumed as small ones and will not be taken into account. In this sence also the pressure terms will be neglected in the future derivations, which, according to Eqs. (3), (5) is formally possible if $c_s/v_{ph} \ll \omega_0^2/2\Omega_0\omega$, and we shall concentrate on finding coherent vortex structures. Here $v_{ph} = \omega/k$, and $\omega_0^2 = 4\pi G\rho_{00}$, and ρ_{00} is some average density of the system. In the limit when the basic-state and perturbed velocities, $\boldsymbol{v}_0(x)$, $\delta\boldsymbol{v}$, are of the same order, from Eq. (5) we obtain:

$$\delta\boldsymbol{v}_\perp \approx \frac{1}{2\Omega_0}\boldsymbol{e}_z \times \nabla_\perp\varphi - \frac{1}{4\Omega_0^2}\left[\frac{\partial}{\partial t} + \frac{1}{2\Omega_0}\boldsymbol{e}_z \times \nabla_\perp(\varphi+\psi)\cdot\nabla_\perp\right]\nabla_\perp(\varphi+\psi), \quad (8)$$

where we introduced $\boldsymbol{v}_0(x) = (\boldsymbol{e}_z \times \nabla_\perp\psi)/2\Omega_0$. Using Eq. (8), and Eqs. (1), (3) we derive the nonlinear equation that will be basic for the further study:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{e}_z \times \nabla_\perp(\varphi+\psi)\cdot\nabla_\perp\right] \times \left(\log\rho_0 - \log\Omega_0 + \alpha\nabla_\perp^2\varphi - \alpha_0\nabla_\perp^2\psi\right) = 0. \quad (9)$$

Here the following normalization and notation is introduced:

$$\frac{\partial}{\partial t} \to \frac{l_0}{v_{00}}\frac{\partial}{\partial t}, \quad \nabla_\perp \to l_0\nabla_\perp, \quad (\varphi,\psi) \to \frac{1}{2\Omega_0 l_0 v_{00}}(\varphi,\psi),$$

$$\alpha = \alpha_0\left(\frac{4\Omega_0^2}{\omega_0^2} - 1\right), \quad \alpha_0 = \frac{v_{00}}{2l_0\Omega_0}, \quad (10)$$

and l_0, v_{00} are some characteristic dimension and velocity of the system that is investigated, respectively.

Eq. (9) is generic in problems dealing with nonlinear vortex structures driven by vector product type nonlinearities in spatially nonuniform systems, and it can be solved using a procedure developed in some previously published papers dealing with quite different problems in plasma physics theory (Vranješ 1999a, 1999b). We look for traveling solutions that are stationary in a reference frame moving with the velocity u in the direction of y-axis. In this case Eq. (9) can be integrated yielding:

$$\log\rho_0(x) - \log\Omega_0(x) + \alpha\nabla_\perp^2\varphi - \alpha_0\nabla_\perp^2\psi = F(\varphi + \psi - ux), \quad (11)$$

where $F(\xi)$ is an arbitrary function of the given argument.

3. Tripolar vortex solutions

We proceed by choosing the functional form $F(\xi)$ as the linear one, i.e., $F(\xi) = F_0 + F_1\cdot\xi$, and allowing for different values of the given constants $F_{0,1}$ inside and outside of an arbitrary circle of radius r_0. On condition when the basic state can be approximated by the following set of equations:

$$\psi(x) - ux = ax^2,$$
$$\frac{1}{\alpha}\log\rho_0 - \frac{1}{\alpha}\log\Omega_0 - \frac{\alpha_0}{\alpha}\nabla_\perp^2\psi = bx^2, \quad (12)$$

Eq. (11) can be rewritten as:

$$(\nabla_\perp^2 - F_1)\left[\varphi - \frac{b - aF_1}{F_1}x^2 + \frac{F_0}{F_1} - \frac{2(b - aF_1)}{F_1^2}\right] = 0. \quad (13)$$

According to the definition of $\psi(x)$, Eq. (12) yields a linear profile of the basic state shear flow. In polar coordinates (r,θ), Eq. (13) separates variables, and can be solved independently outside and inside of the circle of the radius r_0. On condition of vanishing perturbations at infinity we have $F_1^+ = b/a$ (here the superscript + denotes the outside values regarding the given circle), and the outside solution of the above equation can be written as:

$$\varphi^+(r,\theta) = b_0\,K_0(\lambda_1 r) + b_2\,K_2(\lambda_1 r)\cos 2\theta, \quad (14)$$

where $F_0^+ = 0$, and we introduced $F_1^+ \equiv b/a = \lambda_1^2$, and $K_{0,2}(\lambda_1 r)$ are modified Bessel functions of the given order. Similarly, the inside solution is:

$$\varphi^-(r,\theta) = a_0 J_0(\lambda_2 r) - \frac{C_1 r^2}{2} - C_2 + \left[a_2 J_2(\lambda_2 r) - \frac{C_1 r^2}{2}\right]\cos 2\theta. \quad (15)$$

Here $J_{0,2}(\lambda_2 r)$ are Bessel functions, and

$$\lambda_2^2 = -F_1^-, \quad C_1 = \frac{b + a\lambda_2^2}{\lambda_2^2}, \quad C_2 = -\frac{2}{\lambda_2^2}\left(\frac{F_0^-}{2} + C_1\right).$$

The unknown constants in the solutions (14), (15) are to be found using the following physically justified continuity conditions at $r = r_0$: continuity of the potential φ and its derivative with respect to the coordinate r; the continuity of the function $F(\xi)$ to avoid singularities of the basic nonlinear equation at the circle r_0, and we take the argument ξ as constant at $r = r_0$. These conditions yield the following set of equations:

$$C_2 = \alpha_0 J_0(\lambda_2 r_0) - \frac{C_1 r_0^2}{2} - \beta_0 K(\lambda_1 r_0), \quad (16)$$

$$\alpha_2 = \frac{(C_1 - a) r_0^2}{2} \frac{1}{J_2(\lambda_2 r_0)}, \quad (17)$$

$$\beta_2 = -\frac{a r_0^2}{2} \frac{1}{K_2(\lambda_1 r_0)}, \quad (18)$$

$$-2C_1 - \lambda_2^2 \left[\alpha_0 J_0(\lambda_2 r_0) + \frac{(a - C_1) r_0^2}{2}\right] = \lambda_1^2 \left[\beta_0 K_0(\lambda_1 r_0) + \frac{a r_0^2}{2}\right], \quad (19)$$

$$\beta_0 = \frac{\alpha_0 \lambda_2 J_1(\lambda_2 r_0)}{\lambda_1 K_1(\lambda_1 r_0)} + \frac{C_1 r_0}{\lambda_1 K(\lambda_1 r_0}, \quad (20)$$

$$r_0 (C_1 - a) \frac{\frac{\partial J_2(\lambda_2 r_0)}{\partial r_0}}{J_2(\lambda_2 r_0)} - 2C_1 = -a r_0 \frac{\frac{\partial K_2(\lambda_1 r_0)}{\partial r_0}}{K_2(\lambda_1 r_0)}. \quad (21)$$

In Eq. (21) we use the following identities $K_0'(x) = -K_1(x)$, $J_0'(x) = -J_1(x)$, $K_2'(x) = -(K_1(x) + K_3(x))/2$, $J_2'(x) = J_1(x) - 2J_2(x)/x$, and choosing $a = 1.5$, $b = 2.5$, $r_0 = 2$ we have $\lambda_1 = 1.29$, and solve Eq. (21) in the vicinity of the first zero of Bessel function $J_2(\lambda_2 r_0)$ yielding $\lambda_2 = 2.67$. Further, from Eqs. (16), (20) we find:

$$\alpha_0 = 5.3, \quad \beta_0 = -13.85, \quad \alpha_2 = -10.6, \quad \beta_2 = -27.7,$$
$$C_1 = 1.85, \quad C_2 = -3.26.$$

The contour plot of the gravitational potential expressed by Eqs. (14), (15) with the above given constants is represented in Fig. 1. The central part is positive and rotates in the direction which is opposite to the direction of rotation of its satellites. The three dimensional view of the tripole is given in Fig. 2.

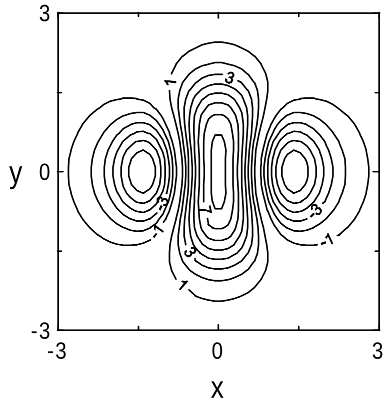

Fig. 1. Contour plot of the gravitational potential $\varphi(r,\theta)$ given by Eqs. (14), (15). Spatial variables are in units of l_0 and the potential normalized according to Eq. (10). The contour step is 1.

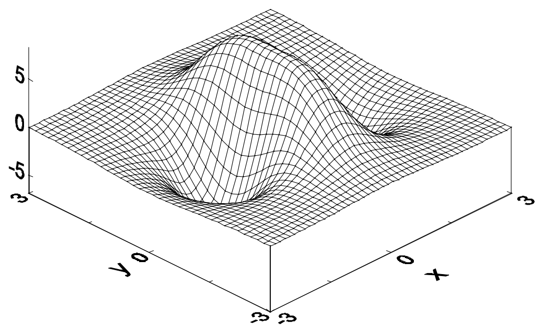

Fig. 2. Three dimensional view of $\varphi(r,\theta)$ from Fig. 1. The structure moves along the y-axis with the velocity u.

Now, from Eqs. (3), (14), (15), and having in mind normalization (10) we may calculate the density ρ (in units of ρ_{00}):

$$\rho^+(r,\theta) = \frac{b}{a}\left[\beta_0 K_0(\lambda_1 r) + \beta_2 K_2(\lambda_1 r)\cos 2\theta\right], \quad r > r_0, \quad (22)$$

$$\rho^-(r,\theta) = -(a\lambda_2^2 + b)\frac{r^2}{2} - 2C_1 + \lambda_2^2\left[\frac{C_1 r^2}{2} - \alpha_0 J_0(\lambda_2 r)\right] - \left[(a\lambda_2^2 + b)\frac{r^2}{2} + \lambda_2^2\left[\alpha_2 J_2(\lambda_2 r) - \frac{C_1 r^2}{2}\right]\right] \times \cos 2\theta, \quad r < r_0. \quad (23)$$

The contour plot and 3D-view of $\rho(r,\theta)$ are given in Figs. 3, and 4, respectively. The central part is negative and represents a rarefication of the background cloud density, while two lateral parts are positive, rotate in the same direction and can be subject to eventual further contraction in the Jeans' sence.

4. Vortex chain solution

The function $F(\xi)$ in Eq. (11) can be chosen in another way, yielding another type of nonlinear stationary solution; let $F(\xi) = \xi + A\kappa^2 \exp(-2\xi/A)$, where A and κ are some constants, and assume a tanh profile of the fluid flow, i. e.,

$$\psi(x) = ux + A \log\cosh \kappa x.$$

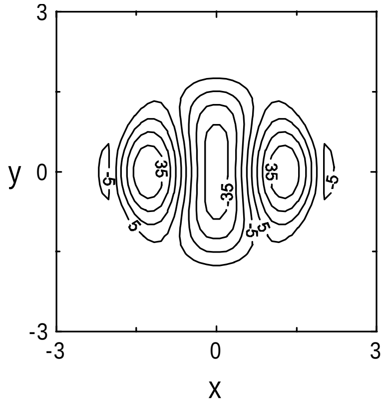

Fig. 3. Density contour plot of Eqs. (22), (23). The step between contours is 10. Moving is in the y-direction.

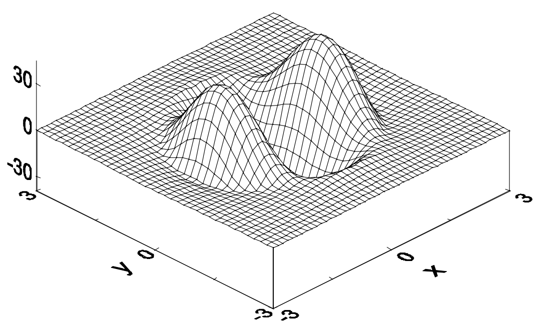

Fig. 4. 3D view of the density distribution from Fig. 3. The lateral positive humps represent local condensations rotating in the same direction and may become subject to gravitational contraction.

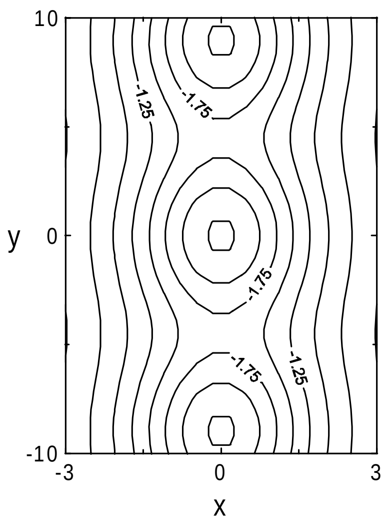

Fig. 5. Vortex chain of gravitational potential as a solution of Eqs. (25), (26) obtained for $\kappa \simeq 0.4$, $k \simeq 0.7$ and $\alpha_0/\alpha = 1.5$. The contour step is 0.25.

For the density $\rho_0(x)$ and angular velocity given by

$$\frac{\rho_0(x)}{\Omega_0(x)} = (\cosh \kappa x)^{A\alpha},$$

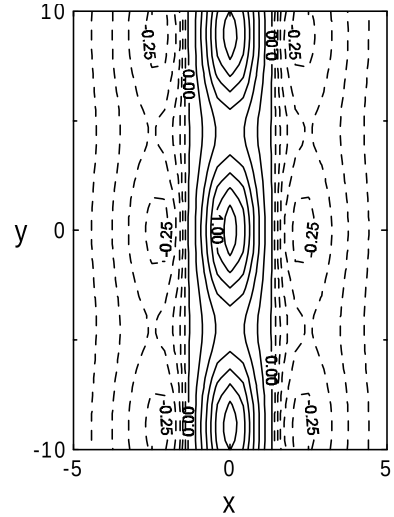

Fig. 6. Chains of the perturbed density ρ corresponding to the chain of φ from Fig. 5. The dashed lines represent negative, rarefacted part; the step between negative contours is 0.05, and 0.2 between positive ones.

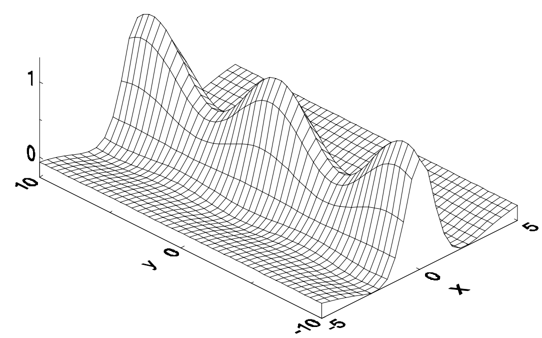

Fig. 7. 3D view of the density chain from Fig. 6.

after some algebra, Eq. (11) can be written as:

$$(\nabla_\perp^2 - 1)\widehat{\varphi} - f(x)\left[\exp(-\widehat{\varphi}) + \frac{\alpha_0}{\alpha}\right] = 0, \quad \text{where}$$

$$f(x) = \frac{2\kappa^2}{\cosh^2 \kappa x}, \quad \widehat{\varphi} = \frac{2\varphi}{A}. \tag{24}$$

Eq. (24) can be solved numerically in the following manner. We look for solutions consisting of a nonlinearly generated potential $\widehat{\varphi}_0(x)$ localized in the x-direction, and a wave-like perturbation periodic in the y-direction:

$$\widehat{\varphi}(x, y) = \widehat{\varphi}_0(x) + \widehat{\delta\varphi}(x) \cos ky,$$

where $|\widehat{\delta\varphi}(x)| \ll |\widehat{\varphi}_0(x)|$. Similar to the procedure from Vranješ (1999a), we obtain the following set of equations for $\widehat{\varphi}_0(x)$, and $\widehat{\delta\varphi}(x)$:

$$\left(\frac{d^2}{dx^2} - 1\right)\widehat{\varphi}_0(x) - f(x)\left[\exp[-\widehat{\varphi}_0(x)] + \frac{\alpha_0}{\alpha}\right] = 0, \tag{25}$$

$$\left(\frac{d^2}{dx^2} - k^2 - 1\right)\widehat{\delta\varphi}(x) + f(x)\exp[-\widehat{\varphi}_0(x)]\,\widehat{\delta\varphi}(x) = 0. \tag{26}$$

By varying parameters κ, k, and looking for localized solutions in the x-direction, Eqs. (25), (26) are solved numerically for

$\alpha_0/\alpha = 1.5$, yielding $\Omega_0/\omega_0 = \sqrt{5/12}$. The contour plot of one particular solution in the form of a single vortex chain of the perturbed gravitational potential, symmetric around the x-coordinate is presented in Fig. 5. Here $\kappa \approx 0.4$ and $k \approx 0.7$, and Eqs. (25), (26) are solved from the point $x = 0$, for $\widehat{\varphi}_0(0) = -2$, $\widehat{\varphi}_0'(0) = 0$, and $\widehat{\delta\varphi}_0(0) = -0.3$, $\widehat{\delta\varphi}_0'(0) = 0$. The corresponding expression for the perturbed density $\rho(x, y)$ can be readily written as:

$$\rho(x,y) = \widehat{\varphi}_0(x) + \left[\exp[-\widehat{\varphi}_0(x)] + \frac{\alpha_0}{\alpha}\right] f(x) + [1 - f(x)\exp[-\widehat{\varphi}_0(x)]]\,\widehat{\delta\varphi}_0(x)\cos ky. \quad (27)$$

The contour, and 3D plots of the perturbed density in the form of a triple chain of traveling vortices are given in Figs. 6, and 7, respectively. The lateral chains in Fig. 6, presented by dashed lines, have much less amplitude of the central hump and represent rarefactions of the basic state density.

5. Summary and conclusion

In this paper, starting from standard equations describing perturbations in a differentially rotating nonuniform gas cloud in a Cartesian geometry, is derived a nonlinear equation comprising the vector-product type nonlinear term which is known to be responsible for creation of stationary vortex structures, both in plasmas and ordinary fluids. It is shown that, depending on the spatial profile of the fluid flow and on other variables describing the basic state, two qualitatively different types of solutions are possible.

For the flow that can be locally approximated by a linear (along the x-axis) function, described by Eq. (12), we obtain the tripolar vortex, a structure consisting of three rotating parts, moving with a constant velocity that equals the flow velocity at the center of the tripol, i.e., it is carried by the flow in the y-direction. The signs of the three parts of tripoles for the gravitational potential and density are reversed; for the density we obtain two lateral vortices with the positive sign, representing local whirling condenstations which can further eventually collapse due to gravitational contraction, and the central rarefaction that should in principle disappear, due to dissipative processes which are not taken into account in this study.

On the other hand, for a $\tanh(x)$ profile of the flow, the basic nonlinear Eq. (9) is solved numerically using the procedure developed in our previous publication (Vranješ 1999a), and a vortex chain solution is obtained. For the density perturbation it represents a triple, even, and localized with respect to x-axis, structure carried by the flow, consisting of three parts, the positive central wave-like (along the y-axis) one, and two lateral chains representing rarefications of the basic state density.

The structures obtained here are formed on time scales that are much shorter than the corresponding scales for gravitational contraction in Jeans' sence, and on arbitrary spatial scales which may, without loss of generality, be taken equal to local Jeans' length. Consequently, these higher order effects leading to gravitational instability should be worth investigating, especially numerically, but this is beyond the scope of this paper.

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