

Deformation of relativistic magnetized stars

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Received 9 August 1999 / Accepted 4 October 1999

Abstract. We formulate deformation of relativistic stars due to the magnetic stress, considering the magnetic fields to be perturbations from spherical stars. The ellipticity for the dipole magnetic field is calculated for some stellar models. We have found that the ellipticity becomes large with increase of a relativistic factor for the models with the same energy ratio of the magnetic energy to the gravitational energy.

Key words: relativity – stars: magnetic fields – stars: neutron – methods: analytical

1. Introduction

There is a growing interest in new classes of objects: soft-gamma repeaters (SGRs) and anomalous X-ray pulsars (AXPs). Until now, four identified SGRs are reported observationally. Their associations with supernova remnants strongly suggest that the SGRs are young neutron stars (see e.g. Kulkarni & Frail 1993; Murakami et al. 1994). Furthermore, recent measurements (see e.g. Kouveliotou et al. 1998, 1999) of the period and period derivative, yield evidence for these pulsars to be ultramagnetized neutron stars with field strength ($\sim 10^{15}$ G) in excess of $B_{\text{cr}} \sim 10^{13}$ G, i.e., *magnetars* (Duncan & Thompson 1992; Thompson & Duncan 1993, 1995, 1996). Some classes of X-ray pulsars also suggest the magnetic fields with 10^{14} – 10^{15} G (see e.g. Mereghetti & Stella 1995). Such magnetic fields are much stronger than those of known pulsars (10^8 – 10^{13} G; see e.g. Taylor et al. 1993). Though the relation between the SGRs and the AXPs is not yet clear, there exist neutron stars with very strong magnetic fields. In these ultramagnetized stars, the magnetic influence becomes important along with the relativistic effects. If we assume that a long-lived electric current flows in highly conductive neutron-star matter, the magnetic pressure corresponding to the Lorentz force comes into play. Hence, it induces deformation of stars. In this paper, we study such deformation from spherical stars within a general relativistic framework.

The quadrupole deformation of magnetized Newtonian stars was discussed by Chandrasekhar & Fermi (1953) and Ferraro

(1954), in which incompressible fluid body with a dipole magnetic field is assumed. This deformation has been discussed also in relation to the gravitational radiation (Gal'tsov et al. 1984; Gal'tsov & Tsetkov 1984). The general relativistic approach by Bocquet et al. (1995) and Bonazzola & Gourgoulhon (1996) has appeared recently. However, their approach is fully numerical. In this paper, we develop almost an analytical treatment by assuming weak magnetic fields compared with gravity. This assumption is valid even in the magnetars. Our formulation is regarded as a general relativistic version of Chandrasekhar & Fermi (1953) and Ferraro (1954). In our method, we can easily include realistic equations of state (EOS) and construct relativistic magnetized stars. Furthermore, this method gives simple calculations of ellipticity of deformed stars etc.

Since the observed ultramagnetized neutron stars have long periods ($T \sim$ several sec), we may neglect the rotation of the magnetized stars, that is, we discuss static cases. We take non-rotating, spherical relativistic stars as backgrounds, and consider the magnetic fields as the perturbation. In particular, we consider only axisymmetric, poloidal magnetic fields produced by long-lived (toroidal) electric currents, because toroidal magnetic fields would break the symmetric property (see also Bocquet et al. 1995 and reference therein). Furthermore, we assume a perfectly conducting interior. Since we now consider non-rotating configurations, this implies that the electric field inside the stars must be zero. Hence, there is no electric charge inside the stars. From this, we can write the 4-current as $J_\mu = (0, 0, 0, J_\phi)$ (Bocquet et al. 1995). Furthermore, the surface charge should be absent, since the total charge should vanish in astrophysical situations. Otherwise, the electromagnetic field itself would have the angular momentum due to the non-vanishing electric field produced by the charge (Feynman et al. 1964; Ma 1986; de Castro 1991). This is not the purely static case.

The current distribution is introduced as the first-order quantity with respect to the perturbation. The corresponding magnetic field is solved by the Maxwell equation. We shall investigate deformation of stars due to the resulting magnetic stress, which arises as the second-order correction to the background field. This perturbation method is very similar to that of slowly rotating stars developed by Hartle (1967), in which the rotation is regarded as a small parameter. Our formalism can be applied to any configurations of the magnetic fields. However, we

restrict ourselves to dipole magnetic fields because the dipole fields are important in many astrophysical situations.

The plan of this paper is as follows. In Sect. 2, the magnetic fields are investigated in the background space-time. The effect arising from the magnetic stress on equilibrium of stars is considered in Sect. 3. The solution corresponding to the quadrupole deformation of the stars will be given. To evaluate the deformation quantitatively, the ellipticity is discussed in Sect. 4. Finally, Sect. 5 is devoted to the discussion. Throughout this paper, we use the units in which $c = G = 1$, and the Gaussian system of units for electromagnetic fields.

2. Magnetic fields in spherically symmetric space-time

We now consider an axisymmetric, poloidal magnetic field created by a 4-current $J_\mu = (0, 0, 0, J_\phi)$ in a non-rotating, spherical star. We suppose that the magnetic field is weak, i.e., $B_\mu \sim O(\varepsilon)$. The line element of the spherically symmetric space-time is given by

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (1)$$

where ν and λ are functions of the radial coordinate r only. Since we do not consider electric fields, we can assume that the electromagnetic 4-potential A_μ has only the ϕ -component, i.e., $A_\mu = (0, 0, 0, A_\phi)$ (see also Bocquet et al. 1995). In this case, the Maxwell equation is reduced to

$$e^{-\lambda} \frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{2} (\nu' - \lambda') e^{-\lambda} \frac{\partial A_\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_\phi}{\partial \theta^2} - \frac{1}{r^2} \cot \theta \frac{\partial A_\phi}{\partial \theta} = -4\pi J_\phi, \quad (2)$$

where the prime denotes the differentiation with respect to r .

We expand the potential A_ϕ and the current J_ϕ as follows (Regge & Wheeler 1957):

$$A_\phi = \sum_{l=1}^{\infty} a_l(r) \sin \theta \frac{dP_l(\cos \theta)}{d\theta}, \quad (3)$$

$$J_\phi = \sum_{l=1}^{\infty} j_l(r) \sin \theta \frac{dP_l(\cos \theta)}{d\theta}, \quad (4)$$

where P_l is the Legendre's polynomial of degree l . Substituting these forms into Eq. (2), we have

$$e^{-\lambda} \frac{d^2 a_l}{dr^2} + \frac{1}{2} (\nu' - \lambda') e^{-\lambda} \frac{da_l}{dr} - \frac{l(l+1)}{r^2} a_l = -4\pi j_l. \quad (5)$$

For a given current j_l , we can obtain the potential a_l and, therefore, the magnetic field. From now on, we only consider a dipole magnetic field, i.e., $l = 1$. The potential outside the star is easily solved (Ginzburg & Ozernoi 1965; Petterson 1974; Wasserman & Shapiro 1983) in the form

$$a_1 = -\frac{3\mu}{8M^3} r^2 \left[\ln \left(1 - \frac{2M}{r} \right) + \frac{2M}{r} + \frac{2M^2}{r^2} \right], \quad (6)$$

where μ is a constant corresponding to the magnetic dipole moment with respect to an observer at infinity, and M is the total

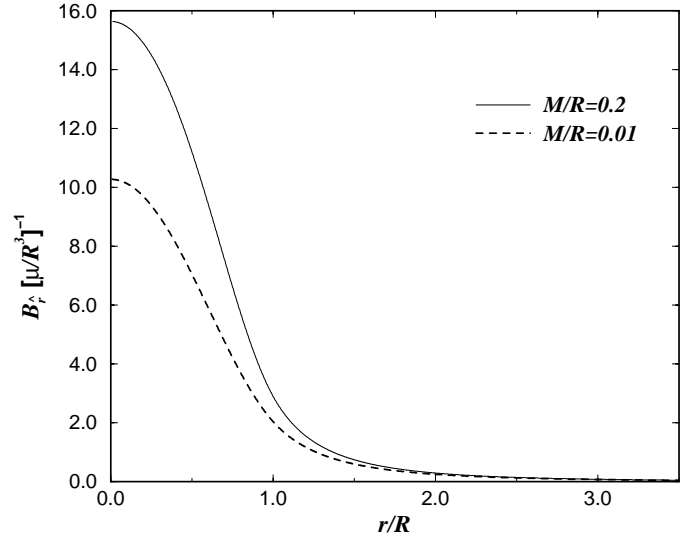


Fig. 1. The tetrad component of the magnetic fields, $B_{\hat{r}}$ on the symmetry axis ($\theta = 0$), plotted against r/R . The solid line denotes a relativistic case ($M/R = 0.2$), while the dashed line corresponds to a Newtonian case ($M/R = 0.01$). The magnetic fields are normalized by the typical field strength μ/R^3 .

mass of the background star. In order to describe the magnetic field inside the star, we require the current distribution j_1 . The current j_1 is not arbitrary but subject to an integrability condition (Ferraro 1954; Chandrasekhar & Prendergast 1956; Bonazzola et al. 1993). As will be shown in Eq. (25), this current is given, up to the first order in ε , by

$$j_1(r) = c_0 r^2 (\rho_0(r) + p_0(r)), \quad (7)$$

where c_0 is an arbitrary constant, and ρ_0 and p_0 denote the density and pressure, respectively, of the background star. By requiring that a_1 behaves as a regular function at the center of the star, we now obtain the potential a_1 in the vicinity of the center:

$$a_1 \simeq \alpha_0 r^2 + O(r^4), \quad (8)$$

where α_0 is a constant, which is fixed by the boundary condition at the surface. In this way, we can construct the magnetic field in the whole space-time.

Fig. 1 displays the tetrad component of the magnetic field,

$$B_{\hat{r}} = -\frac{1}{r^2 \sin \theta} \partial_\theta A_\phi = \frac{2 \cos \theta}{r^2} a_1 \quad (9)$$

on the symmetry axis (i.e., $\theta = 0$), and Fig. 2 displays the tetrad component

$$B_{\hat{\theta}} = \frac{e^{-\frac{\lambda}{2}}}{r \sin \theta} \partial_r A_\phi = -\frac{e^{-\frac{\lambda}{2}} \sin \theta}{r} a_1' \quad (10)$$

on the equatorial plane (i.e., $\theta = \pi/2$) with respect to the radial coordinate r . We have normalized them by the typical magnetic field strength μ/R^3 , where R is the radius of the star. The solid lines denote a relativistic case, whereas the dashed lines correspond to a Newtonian case. In these calculations, we have used

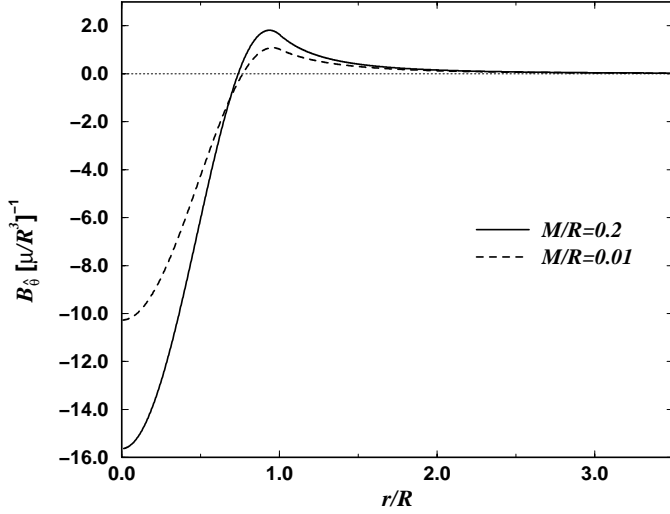


Fig. 2. The tetrad component of the magnetic fields, $B_{\hat{\theta}}$ on the equatorial plane ($\theta = \pi/2$), plotted against r/R . The solid line denotes a relativistic case ($M/R = 0.2$), while the dashed line corresponds to a Newtonian case ($M/R = 0.01$). The magnetic fields are normalized by the typical field strength μ/R^3 .

the polytropic EOS: $p_0 = \kappa \rho_0^\gamma$ ($\gamma = 2$). From these figures, we see that the intensity of the magnetic field increases as r becomes closer to the center. Furthermore, these two figures show that despite the same magnetic moment with respect to an observer at infinity, the central magnetic field of the relativistic star is stronger than that of the Newtonian star by about 50% of the Newtonian case. Therefore, it follows that the relativistic effect strengthens the internal magnetic fields.

3. Equilibrium configurations of magnetized stars

Next, we consider deformation of magnetized stars due to the magnetic stress, which is regarded as the second-order effect. We formulate the deformation of the star and space-time, following Hartle (1967).

3.1. Equations of equilibrium

The metric can be expanded in multipoles around the spherically symmetric space-time. In particular, when we deal only with a dipole field, i.e., $l = 1$ in Eqs. (3) and (4), the metric can be written in the form (see also Hartle 1967; Chandrasekhar & Miller 1974)

$$ds^2 = -e^{\nu(r)} [1 + 2(h_0(r) + h_2(r)P_2(\cos\theta))] dt^2 + e^{\lambda(r)} \left[1 + \frac{2e^{\lambda(r)}}{r} (m_0(r) + m_2(r)P_2(\cos\theta)) \right] dr^2 + r^2 [1 + 2k_2(r)P_2(\cos\theta)] (d\theta^2 + r^2 \sin^2\theta d\phi^2), \quad (11)$$

where h_0 , h_2 , m_0 , m_2 and k_2 are the corrections of the second order in ε .

The total energy-momentum tensor is the sum of the perfect-fluid part $T_{(m)}^\mu{}_\nu$ and the electromagnetic part $T_{(em)}^\mu{}_\nu$:

$$T^\mu{}_\nu = T_{(m)}^\mu{}_\nu + T_{(em)}^\mu{}_\nu, \quad (12)$$

where

$$T_{(m)}^\mu{}_\nu = (\rho + p) u^\mu u_\nu + p \delta^\mu{}_\nu, \quad (13)$$

$$T_{(em)}^\mu{}_\nu = \frac{1}{4\pi} \left(F^{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} F_{\sigma\lambda} F^{\sigma\lambda} \delta^\mu{}_\nu \right). \quad (14)$$

In Eq. (14), $F_{\mu\nu}$ is the Faraday tensor. The pressure p and the energy density ρ can also be expanded in multipoles as

$$p(r, \theta) = p_0 + (\delta p_{(l=0)} + \delta p_{(l=2)} P_2), \quad (15)$$

$$\rho(r, \theta) = \rho_0 + \frac{\rho'_0}{p'_0} (\delta p_{(l=0)} + \delta p_{(l=2)} P_2), \quad (16)$$

where $\delta p_{(l=0)}$ and $\delta p_{(l=2)}$ depend on r only, and we have assumed a barotropic case.

From the Einstein equation, we can obtain

$$m'_0 = 4\pi r^2 \frac{\rho'_0}{p'_0} \delta p_{(l=0)} + \frac{1}{3} \left[e^{-\lambda} (a'_1)^2 + \frac{2}{r^2} a_1^2 \right], \quad (17)$$

$$h'_0 = 4\pi r e^\lambda \delta p_{(l=0)} + \frac{1}{r} \nu' e^\lambda m_0 + \frac{1}{r^2} e^\lambda m_0 + \frac{e^\lambda}{3r} \left[e^{-\lambda} (a'_1)^2 - \frac{2}{r^2} a_1^2 \right], \quad (18)$$

$$h'_2 + k'_2 = h_2 \left(\frac{1}{r} - \frac{\nu'}{2} \right) + \frac{e^\lambda}{r} m_2 \left(\frac{1}{r} + \frac{\nu'}{2} \right) + \frac{4}{3r^2} a_1 a'_1, \quad (19)$$

$$h_2 + \frac{e^\lambda}{r} m_2 = \frac{2}{3} e^{-\lambda} (a'_1)^2, \quad (20)$$

$$\begin{aligned} h'_2 + k'_2 + \frac{1}{2} r \nu' k'_2 &= 4\pi r e^\lambda \delta p_{(l=2)} + \frac{1}{r^2} e^\lambda m_2 + \frac{1}{r} \nu' e^\lambda m_2 \\ &+ \frac{3}{r} e^\lambda h_2 + \frac{2}{r} e^\lambda k_2 - \frac{1}{3r} e^\lambda \left[e^{-\lambda} (a'_1)^2 + \frac{4}{r^2} a_1^2 \right]. \end{aligned} \quad (21)$$

Furthermore, from the conservation law of the total energy-momentum tensor, we obtain

$$\begin{aligned} \delta p'_{(l=0)} &= -\frac{1}{2} \nu' \left(\frac{\rho'_0}{p'_0} + 1 \right) \delta p_{(l=0)} \\ &- (\rho_0 + p_0) h'_0 + \frac{2}{3r^2} a_1 j_1, \end{aligned} \quad (22)$$

$$\delta p_{(l=2)} = -(\rho_0 + p_0) h_2 - \frac{2}{3r^2} a_1 j_1, \quad (23)$$

$$\begin{aligned} \delta p'_{(l=2)} &= -\frac{1}{2} \nu' \left(\frac{\rho'_0}{p'_0} + 1 \right) \delta p_{(l=2)} \\ &- (\rho_0 + p_0) h'_2 - \frac{2}{3r^2} a_1 j_1. \end{aligned} \quad (24)$$

The integrability condition for Eqs. (22) and (24) leads to

$$\frac{j_1}{r^2 (\rho_0 + p_0)} = \text{const.} (\equiv c_0). \quad (25)$$

This is consistent with Eq. (5.29) of Bonazzola et al. (1993) up to the first order in ε . Using this current distribution, Eq. (22) can also be integrated as

$$\delta p_{(l=0)} = -(\rho_0 + p_0) h_0 + \frac{2}{3r^2} a_1 j_1 + c_1 (\rho_0 + p_0), \quad (26)$$

where c_1 is a constant of integration.

Consequently, we have two sets of differential equations,

$$m'_0 = -4\pi r^2 \frac{\rho'_0}{p'_0} (\rho_0 + p_0) (h_0 - c_1) + \frac{1}{3} e^{-\lambda} (a'_1)^2 + \frac{2}{3r^2} a_1^2 + \frac{8\pi}{3} \frac{\rho'_0}{p'_0} a_1 j_1, \quad (27)$$

$$h'_0 = \left(\frac{1}{r^2} + \frac{\nu'}{r} \right) e^\lambda m_0 - 4\pi r e^\lambda (\rho_0 + p_0) (h_0 - c_1) + \frac{1}{3r} (a'_1)^2 - \frac{2}{3r^3} e^\lambda a_1^2 + \frac{8\pi}{3r} e^\lambda a_1 j_1, \quad (28)$$

and

$$v'_2 = -\nu' h_2 + \frac{2}{3} e^{-\lambda} \left(\frac{1}{r} + \frac{\nu'}{2} \right) (a'_1)^2 + \frac{4}{3r^2} a_1 a'_1, \quad (29)$$

$$h'_2 = -\frac{4e^\lambda}{r^2 \nu'} v_2 + \left[8\pi \frac{e^\lambda}{\nu'} (\rho_0 + p_0) + \frac{2}{r^2 \nu'} (1 - e^\lambda) - \nu' \right] h_2 + \frac{8}{3r^4 \nu'} e^\lambda a_1^2 + \frac{8}{3r^3 \nu'} \left(1 + \frac{r\nu'}{2} \right) a_1 a'_1 + \left(\frac{1}{3} \nu' e^{-\lambda} + \frac{2}{3r^2 \nu'} \right) (a'_1)^2 + \frac{16\pi}{3r^2 \nu'} e^\lambda a_1 j_1, \quad (30)$$

where $v_2 \equiv h_2 + k_2$. These equations govern the relativistic magnetized star. We have to solve four differential equations (27)–(30) and one algebraic equation (20) for the unknown functions (m_0, m_2, h_0, h_2, v_2). Furthermore, we can derive $\delta p_{(l=0)}$ and $\delta p_{(l=2)}$ by substituting the solution of h_0 and h_2 into Eqs. (26) and (23).

In order to solve Eqs. (27) and (28) inside the star, it is also convenient to introduce a quantity

$$\delta P_0 \equiv \frac{\delta p_{(l=0)}}{\rho_0 + p_0}. \quad (31)$$

From Eq. (26), we have

$$\delta P_0 + h_0 - \frac{2}{3} \frac{j_1}{r^2 (\rho_0 + p_0)} a_1 = c_1. \quad (32)$$

Moreover, Eqs. (27) and (28) are rewritten as

$$m'_0 = 4\pi r^2 \frac{\rho'_0}{p'_0} (\rho_0 + p_0) \delta P_0 + \frac{1}{3} e^{-\lambda} (a'_1)^2 + \frac{2}{3r^2} a_1^2, \quad (33)$$

$$\delta P'_0 = - \left(8\pi p_0 + \frac{1}{r^2} \right) e^{2\lambda} m_0 - 4\pi r e^\lambda (\rho_0 + p_0) \delta P_0 - \frac{1}{3r} (a'_1)^2 + \frac{2}{3r^3} e^\lambda a_1^2 + \frac{2}{3} \frac{j_1}{r^2 (\rho_0 + p_0)} a'_1. \quad (34)$$

In the next subsection, we solve these differential equations for the metric functions.

3.2. The exterior solution and boundary condition

First, we consider the solution outside of the star, in which $\rho_0 = p_0 = 0$ and $j_1 = 0$.

The solution of m_0 and h_0 is given by

$$m_0 = \frac{3\mu^2}{8M^4 r} (r^2 - M^2) + \frac{3\mu^2}{8M^5} (r^2 - Mr - M^2) \ln \left(1 - \frac{2M}{r} \right) + \frac{3\mu^2}{32M^6} r^2 (r - 2M) \left[\ln \left(1 - \frac{2M}{r} \right) \right]^2 + c_2, \quad (35)$$

$$h_0 = -\frac{3\mu^2}{8M^3} \frac{4r - M}{r(r - 2M)} + \frac{3\mu^2}{8M^5} \frac{(r - M)(r - 3M)}{r - 2M} \ln \left(1 - \frac{2M}{r} \right) + \frac{3\mu^2}{32M^6} r^2 \left[\ln \left(1 - \frac{2M}{r} \right) \right]^2 - \frac{c_2}{r - 2M} + c_3, \quad (36)$$

where c_2 and c_3 are constants of integration. At large r , m_0 and h_0 behave as

$$m_0 \simeq c_2 - \frac{\mu^2}{3r^3} + O\left(\frac{1}{r^4}\right), \quad (37)$$

$$h_0 \simeq c_3 - \frac{3\mu^2}{8M^4} - \frac{c_2}{r} + O\left(\frac{1}{r^2}\right). \quad (38)$$

Since h_0 must vanish at infinity, we obtain

$$c_3 = \frac{3\mu^2}{8M^4}. \quad (39)$$

The constant c_2 corresponds to the mass shift, which is fixed by matching with the interior solution at the surface.

On the other hand, the differential equations for v_2 and h_2 is rather complicated, but analytically solved. The solution of v_2 is

$$v_2(z) = \frac{2K}{\sqrt{z^2 - 1}} Q_2^1(z) - \frac{3\mu^2}{4M^4 \sqrt{z^2 - 1}} P_2^1(z) + v_{2p}(z), \quad (40)$$

where z is defined as $z \equiv r/M - 1$, K is a constant of integration, P_2^1 and Q_2^1 are the associated Legendre functions, and v_{2p} is

$$v_{2p} = \frac{9\mu^2}{4M^4} z + \frac{3\mu^2}{8M^4} \frac{7z^2 - 4}{z^2 - 1} + \frac{3\mu^2}{16M^4} \frac{z(11z^2 - 7)}{z^2 - 1} \ln \frac{z - 1}{z + 1} + \frac{3\mu^2}{16M^4} (2z^2 + 1) \left(\ln \frac{z - 1}{z + 1} \right)^2. \quad (41)$$

Furthermore, h_2 is given by

$$h_2(z) = K Q_2^2(z) - \frac{3\mu^2}{8M^4} P_2^2(z) + h_{2p}(z), \quad (42)$$

where P_2^2 and Q_2^2 are the associated Legendre functions, and h_{2p} is written as

$$h_{2p} = -\frac{3\mu^2}{16M^4} \left[6z^2 + 3z - 6 - \frac{4z^2 + 2z}{z^2 - 1} \right]$$

$$\begin{aligned}
& -\frac{3\mu^2}{32M^4} \left[3z^2 - 8z - 3 - \frac{8}{z^2 - 1} \right] \ln \frac{z-1}{z+1} \\
& + \frac{3\mu^2}{16M^4} (z^2 - 1) \left(\ln \frac{z-1}{z+1} \right)^2. \quad (43)
\end{aligned}$$

In Eqs. (40) and (42), we have used the boundary condition at infinity. The remaining constant K will be fixed by the boundary condition at the surface. Thus we have obtained the analytical solution outside the star.

We turn our attention to the interior solution. For a given EOS, we can obtain the solution numerically. For the actual numerical work, we investigate the behavior of the metric functions in the vicinity of the center.

First, we consider the metric functions m_0 and δP_0 . The solution in which both m_0 and δP_0 vanish at the center (see also Chandrasekhar & Miller 1974) is given by

$$m_0 \simeq \frac{2}{3} \alpha_0^2 r^3 + \dots, \quad (44)$$

$$\delta P_0 \simeq -\frac{2}{3} (\alpha_0^2 - c_0 \alpha_0) r^2 + \dots, \quad (45)$$

where α_0 is defined in Eq. (8).

Next, we consider v_2 and h_2 . The regular solution at the center is

$$\begin{aligned}
v_2 & \simeq \beta_1 r^4 + \dots, \\
h_2 & \simeq \beta_2 r^2 + \dots, \quad (46)
\end{aligned}$$

where constants β_1 and β_2 are not independent by the regularity condition at the center.

Finally, we can obtain the metric functions by imposing the junction conditions (O'Brien & Sygne 1952) at the surface:

$$g_{\mu\nu}|_{+R} = g_{\mu\nu}|_{-R} \quad (\mu, \nu = t, r, \theta, \phi), \quad (47)$$

$$g_{ij,r}|_{+R} = g_{ij,r}|_{-R} \quad (i, j = t, \theta, \phi), \quad (48)$$

where $g_{\mu\nu}$ denotes the metric components. From these conditions, the integration constants c_2 , K , β_1 and β_2 are fixed.

4. Ellipticity of magnetized stars

We consider the magnetic field on the stellar shape of the equilibrium. The additional Lorentz force $\mathbf{J} \times \mathbf{B}$ mainly acts on it in the perpendicular direction to the symmetry axis ($\theta = \pi/2$), that is, flattens the star. The flattening effect is also recognized by considering the (r, r) component of the magnetic stress tensor,

$$T_{(\text{em})r}^r = \frac{1}{8\pi} (B_\theta B^\theta - B_r B^r). \quad (49)$$

Along the symmetry axis, B_θ must vanish owing to the axisymmetry. Hence, the stress $T_{(\text{em})r}^r$ is negative on this axis. On the other hand, on the equatorial plane, since B_r is zero at any r , the stress $T_{(\text{em})r}^r$ has the opposite sign. This indicates that the spherical star is shrunk in the parallel direction to the symmetry axis ($\theta = 0$) and expanded in the perpendicular direction ($\theta = \pi/2$) by the magnetic effect. Thus we can see the flattening effect.

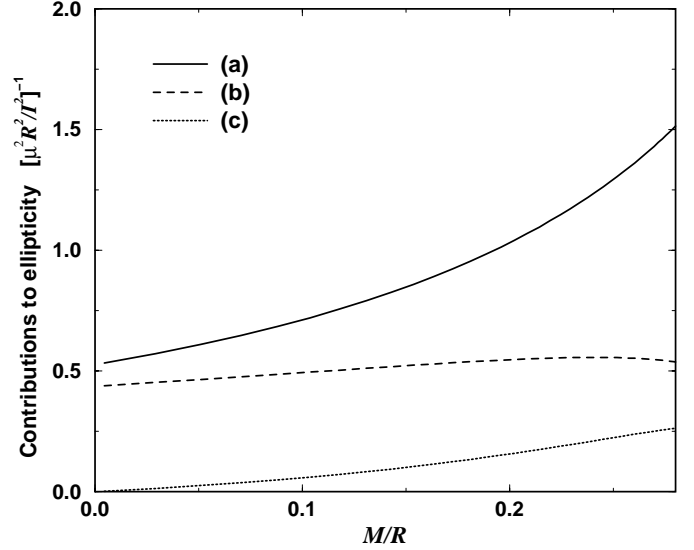


Fig. 3. The three contributions to ellipticity: (a) the effect of the ‘Lorentz force’, (b) the effect of the perturbation of the ‘gravitational potential’, and (c) the ‘purely relativistic effect’ (see text). These are plotted against M/R . We have used the polytropic EOS ($\gamma = 2$).

Next, in order to evaluate the deformation quantitatively, let us introduce the ellipticity, which is defined as

$$\text{ellipticity} \equiv \frac{(\text{equatorial radius}) - (\text{polar radius})}{(\text{mean radius})}, \quad (50)$$

where these radii denote the circumferential radii under general relativistic situations. From this definition, ellipticity is given by (Chandrasekhar & Miller 1974)

$$\text{ellipticity} = \frac{2c_0}{r\nu'} a_1 + \frac{3h_2}{r\nu'} - \frac{3}{2} k_2. \quad (51)$$

The first term of the right-hand side in Eq. (51) corresponds to, in a sense, the effect of the ‘Lorentz force’, the second term represents the effect of the perturbation of the ‘gravitational potential’ induced by the magnetic effect, and the third term is a ‘purely relativistic term’ which arises from a definition of the radius, that is, the circumferential radius. These contributions to the ellipticity are shown in Fig. 3 as a function of the relativistic factor M/R . We have normalized them by $\mu^2 R^2 / I^2$ (I is the moment of inertia, which is well defined also in the relativistic case), and we have used a polytropic EOS ($\gamma = 2$). From this figure, we can see that the term due to the ‘gravitational potential’ does not change significantly, while the term concerning the ‘Lorentz force’ increases with the relativistic factor M/R as known from Figs. 1 and 2.

Fig. 4 displays the (total) ellipticity for different polytropic models ($\gamma = 5/3$ and $\gamma = 2$). From this figure, we find that the ellipticity becomes large as the relativistic factor M/R increases, in each case of $\gamma = 5/3$ and $\gamma = 2$. The common feature of the monotonic increase shows the effect of the ‘Lorentz force term’ to be effective. An important thing is that the relativistic calculation leads to much larger ellipticity for a fixed value of $\mu^2 R^2 / I^2$. Finally, we give a comment concerning the previous general-relativistic studies. The quantity plotted in Fig. 4 is

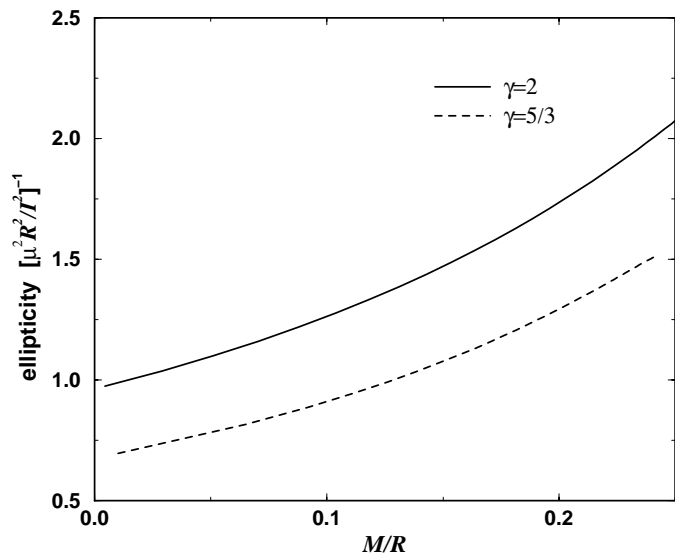


Fig. 4. The variation of the ellipticity with respect to M/R . The solid line denotes the case of the polytropic EOS with $\gamma = 2$, and the dashed line corresponds to the case of $\gamma = 5/3$.

exactly the *magnetic distortion factor* β introduced by Bonazzola & Gourgoulhon (1996). It is noticed that although the EOS which we have used is different from that of Bonazzola & Gourgoulhon (1996), both calculations give the same results (see Fig. 2 of their paper), that is, β takes the values very close to 1.

5. Discussion

Recent observations suggest that some classes of neutron stars have strong magnetic fields. These may promote a new branch with magnetized stars. As a simple approach to the models, we have formulated the structure of the magnetized stars within a general relativistic framework, considering the perturbation from a spherical star. In particular, a dipole magnetic field has been dealt with. We have showed the current distribution which yields equilibrium configurations up to the second order in ε . Furthermore, the ellipticity of the star has been estimated as a simple example. We have found that the ellipticity becomes large as the relativistic factor M/R increases, for the same energy ratio of the magnetic energy to the gravitational energy. Our analytical approach have made the calculations much simpler than that of the previous work. This method can be extended to more general cases of realistic EOS and general current distribution, in which the current exists in some domains of the star. Therefore, this can be applied to wider range of astrophysical situations.

Another extension of this work is to incorporate rotation of stars. The stationary configurations, in which the rotation axis is aligned with the magnetic axis, make the calculations complex because of appearance of non-vanishing electric fields. However, this can be managed. Since the rotational effect deforms the star as well as the magnetic effect, we are also interested in seeing which of them is effective. This will be the subject of further investigations.

Acknowledgements. We would like to thank M. Hosonuma for providing us with a numerical code for rotating stars and useful discussions.

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